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# SOME SUPERVALUATION-BASED CONSEQUENCE RELATIONS

ABSTRACT. In this paper, we define some consequence relations based on supervaluation semantics for partial models, and we investigate their properties. For our main consequence relation, we show that natural versions of the following fail: upwards and downwards Lowenheim–Skolem, axiomatizability, and compactness. We also consider an alternate version for supervaluation semantics, and show both axiomatizability and compactness for the resulting consequence relation.

KEY WORDS: consequence relations, partial models, supervaluations, three-valued logic

## 1. PARTIAL MODELS AND SUPERVALUATION SEMANTICS

In this paper, we define some consequence relations based on supervaluation semantics for partial models, and we investigate their properties.

Given a first order language $L$, let a *partial model for $L$* be an ordered pair $M = \langle D, I \rangle$, where $D$, the *domain of discourse*, is a non-empty set, and $I$ is a function assigning

- to each name of $L$ a member of $D$;
- to each $n$-place function symbol of $L$ an $n$-place function on $D$;
- to some, all or none of the propositional variables of $L$ a member of $\{\mathbf{t}, \mathbf{f}\}$ so that $I$ restricted to the propositional variables is a partial function; and
- to each $n$-place relation symbol $R$ of $L$ an ordered pair $I(R) = \langle I^+(R), I^-(R) \rangle$ with $I^+(R) \subseteq D^n$, $I^-(R) \subseteq D^n$ and $I^+(R) \cap I^-(R) = \varnothing$. $I^+(R)$ and $I^-(R)$ are the *extension* and the *antiextension* of $R$.

Partial models have been used to represent a number of linguistic phenomena, such as vagueness and truth-theoretic paradoxes.

A partial model $M' = \langle D', I' \rangle$ is a *precisification* of $M = \langle D, I \rangle$ (in symbols, $M \leq M'$) iff

- $D' = D$;
- $I'(c) = I(c)$ for each name or function symbol $c$;

- for each propositional variable $p$, if $I(p)$ is defined then $I'(p) = I(p)$; and
- $I^+(R) \subseteq I'^+(R)$ and $I^-(R) \subseteq I'^-(R)$ for each relation symbol $R$.

A partial model $M = \langle D, I \rangle$ is *classical* iff both $I(p)$ is defined for each propositional variable and $I^+(R) \cup I^-(R) = D^n$ for each $n$-place relation symbol $R$.

Given a model $M = \langle D, I \rangle$, an *assignment of values to the variables*, or *assignment*, is a function $s : vbles \to D$, where *vbles* is the set of variables in the language. If $M$ is classical, $s$ is an assignment, and $A$ is a formula with or without free variables, the truth value $V_{M,s}(A) \in \{\mathbf{t}, \mathbf{f}\}$ is defined in the standard way.

Given a partial model $M$, assignment $s$, and sentence $A$, define

$$V_{M,s}(A) = \mathbf{t} \text{ iff } V_{M',s}(A) = \mathbf{t} \text{ for all classical precisifications } M' \text{ of } M,$$

$$V_{M,s}(A) = \mathbf{f} \text{ iff } V_{M',s}(A) = \mathbf{f} \text{ for all classical precisifications } M' \text{ of } M.$$

If $A$ has no free individual variables, we just write $V_M(A)$ for $V_{M,s}(A)$, since $s$ drops out as irrelevant. The *valuation*, i.e. the partial assignment of truth values to sentences, thus defined is a *supervaluation* determined by the partial model $M$. Note that the sentence $A$ is a theorem of classical logic iff $V_M(A) = \mathbf{t}$ for every partial model $M$ iff $V_M(A) \neq \mathbf{f}$ for every partial model $M$.

Formal supervaluation semantics are first introduced by van Fraassen (1966), but in a context different from that of our partial models. Van Fraassen is interested in languages with truth value gaps generated by nondenoting names, rather than by partial predicates: thus his notion of a partial model is quite different to ours. Supervaluation semantics for languages with nondenoting names are studied in-depth by Woodruff (1984). The supervaluation idea occurs as early as Mehlberg's (1958) treatment of vagueness, though Mehlberg's presentation is informal. Fine (1975) presents formal supervaluation semantics for vagueness. See also Lewis (1970) and Dummett (1970). Supervaluations for partial models in which "true" is a partial predicate were implicit in van Fraassen's (1968) discussion of the truth-theoretic paradoxes, but this was first fully developed by Kripke (1975). See McGee (1991) for a supervaluation treatment, using partial models, of "true" as a vague predicate.

We define three *superconsequence* relations (adopting terminology from Woodruff (1984)). Here we assume that $\Gamma$ and $\Delta$ are sets of sentences.

$$\Gamma \vDash_{\mathbf{t}} \Delta \quad \text{iff} \quad \text{for every partial model } M, \text{ if } V_M(A) = \mathbf{t}$$
$$\text{for every } A \in \Gamma \text{ then } V_M(B) = \mathbf{t} \text{ for some}$$
$$B \in \Delta.$$
$$\Gamma \vDash_{\mathbf{f}} \quad \text{iff} \quad \text{for every partial model } M, \text{ if } V_M(B) = \mathbf{f}$$
$$\text{for every } B \in \Delta \text{ then } V_M(A) = \mathbf{f} \text{ for some}$$
$$A \in \Gamma.$$
$$\Gamma \vDash \Delta \quad \text{iff} \quad \Gamma \vDash_{\mathbf{t}} \Delta \text{ and } \Gamma \vDash_{\mathbf{f}} \Delta.$$

The idea of adding backwards falsehood preservation to the more standard truth preservation, and the idea of allowing both multiple antecedents and multiple consequents à la Gentzen, are due to Scott (1975), though he was working with the strong Kleene scheme (see below) for evaluating sentences in partial models, rather than with the supervaluation scheme. Motivations for adopting these ideas are discussed in M. Kremer (1986) (68–75) and (1988) (see Section 5, below). Note:

(1)    $\Gamma \vDash_{\mathbf{t}} B$ iff $\Gamma$ classically entails $B$,
(2)    $A \vDash_{\mathbf{f}} \Delta$ iff $A$ classically entails $\Delta$ (in the sense that every classical model that makes $A$ true makes some member of $\Delta$ true), and
(3)    $A \vDash B$ iff $A$ classically entails $B$.

Our main results concerning $\vDash$, $\vDash_{\mathbf{t}}$ and $\vDash_{\mathbf{f}}$ are that their finitary fragments are not axiomatizable, and that the relations themselves are not compact (see Section 3, below). Thus, although supervaluation semantics have been touted as preserving classical logic – mostly on account of (1), above – any such claim must be muted by our results.

Implicit in Mehlberg and explicit in Fine and others is an idea not yet mentioned: that the truth value of $A$ in a partial model $M$ is determined not by its truth value in *all* classical precisifications, but by its truth value in all *admissible* classical priecisifications. *Scarlet* and *red* are probably both vague predicates, and the partial model $M$ representing the use of these predicates will contain some item in neither the extension nor the antiextension of either predicate. Yet we might insist that in every *admissible* classical precisification – every classical precisification relevant to the truth values of the sentences of the language – if an item is in the extension of *scarlet* then it must also be in the extension of *red*. This would ensure, for example, that $\forall x(x \text{ is scarlet} \supset x \text{ is red})$ is true. In Section 4, below, we consider superconsequence relations based on this additional idea.

The main rivals to supervaluations are the valuations determined by the weak and strong Kleene schemes. The Kleene schemes agree with the supervaluation scheme for atomic sentences, for example, they agree that the atomic sentence $Rc$ is $\mathbf{t}$ if $I(c) \in I^+(R)$ and $\mathbf{f}$ if $I(c) \in I^-(R)$. If

we add a third "truth value" $\mathbf{n}$ for "neither true nor false", then the weak and strong Kleene schemes can be characterized as follows, with $\forall$ and $\exists$ treated analogously to & and $\vee$. For both Kleene schemes, $\neg\mathbf{t} = \mathbf{f}$ and $\neg\mathbf{f} = \mathbf{t}$ and $\neg\mathbf{n} = \mathbf{n}$. For both Kleene schemes, if $\mathbf{x}, \mathbf{y} \in \{\mathbf{t}, \mathbf{f}\}$ then $(\mathbf{x} \& \mathbf{y})$ and $(\mathbf{x} \vee \mathbf{y})$ are as in the classical scheme, e.g., $(\mathbf{t} \& \mathbf{f}) = \mathbf{f}$. For the strong Kleene scheme, $(\mathbf{t} \& \mathbf{n}) = (\mathbf{n} \& \mathbf{t}) = (\mathbf{f} \vee \mathbf{n}) = (\mathbf{n} \vee \mathbf{f}) = (\mathbf{n} \& \mathbf{n}) = (\mathbf{n} \vee \mathbf{n}) = \mathbf{n}$ and $(\mathbf{f} \& \mathbf{n}) = (\mathbf{n} \& \mathbf{f}) = \mathbf{f}$ and $(\mathbf{t} \vee \mathbf{n}) = (\mathbf{n} \vee \mathbf{t}) = \mathbf{t}$. For the weak Kleene scheme, if either $\mathbf{x} = \mathbf{n}$ or $\mathbf{y} = \mathbf{n}$ then $(\mathbf{x} \vee \mathbf{y}) = (\mathbf{x} \& \mathbf{y}) = \mathbf{n}$. Thus, on both Kleene schemes, if $A$ is $\mathbf{n}$, then so is $(A \vee \neg A)$. The method of supervaluations, by contrast, always assigns the value $\mathbf{t}$ to $(A \vee \neg A)$ and to every other classical first order theorem.

## 2. ANALOGUES TO CLASSICAL METATHEOREMS

Say that a set $\Gamma$ of sentences is *satisfiable* [*classically satisfiable*] iff there is a partial model [classical model] $M$ such that $V_M(A) = \mathbf{t}$ for each $A \in \Gamma$. Note that satisfiability is equivalent to classical satisfiability. Thus, by classical compactness, if every finite subset $\Gamma'$ of the set $\Gamma$ of sentences is satisfiable, then so is $\Gamma$. Thus the supervaluation semantics (trivially) satisfies compactness in one sense. In Section 3 we will show that a different kind of compactness, *superconsequence compactness* as opposed to the current *sentence-set compactness*, fails for the superconsequence relations $\vDash_{\mathbf{t}}$, $\vDash_{\mathbf{f}}$ and $\vDash$. In Section 4, we will show that superconsequence compactness succeeds on the definition of superconsequence based on the supervaluation approach that evaluates a sentence in a partial model by considering all *admissible* precisifications.

The supervaluation semantics also satisfies weak versions of the downwards and upwards Lowenheim–Skolem Theorems. Thus:

DOWNWARDS AND UPWARDS LOWENHEIM–SKOLEM THEO-REM. *Suppose that the cardinality of the language is $\kappa$, that $\Gamma$ is a set of closed sentences, and that there is a model $M$ of cardinality $\lambda \geq \kappa$, such that $V_M(A) = \mathbf{t}$ for every $A \in \Gamma$. Then for any $\mu \geq \kappa$ there is a model $M'$ of cardinality $\mu$ such that $V_{M'}(A) = \mathbf{t}$ for every $A \in \Gamma$.*

*Proof.* Given $M$ as in the hypothesis of the theorem, let $M^*$ be any classical precisification of $M$; then $V_{M^*}(A) = \mathbf{t}$ for every $A \in \Gamma$, and the cardinality of $M^*$ is $\lambda$. By the classical Lowenheim–Skolem theorem, for any $\mu \geq \kappa$ there is a classical model $M'$ of cardinality $\mu$ which is elementarily equivalent to $M^*$; then $M'$ is also a partial model, and $V_{M'}(A) = \mathbf{t}$ for every $A \in \Gamma$. $\qquad\qquad\square$

Nonetheless, stronger versions of the Lowenheim–Skolem theorem fail. For example, *neither the upwards nor the downwards parts of the following holds*:

*Suppose that the cardinality of the language is $\kappa$, that $\Gamma$ and $\Delta$ are sets of closed sentences, and that there is a model M of cardinality $\lambda \geq \kappa$, which is a counterexample to the consequence claim that $\Gamma \vDash_t \Delta$, that is $V_M(A) = t$ for every $A \in \Gamma$ and $V_M(A) \neq t$ for every $A \in \Delta$. Then for any $\mu \geq \kappa$ there is a counterexample $M'$ of cardinality $\mu$ such that $V_{M'}(A) = t$ for every $A \in \Gamma$ and $V_{M'}(A) \neq t$ for every $A \in \Delta$. (Similarly for $\vDash_f$, and $\vDash$.)*

*Proof.* For the downwards part, consider a countable language with identity, and with one relational predicate $L$ and one unary predicate $F$. Let $M$ be the following partial model: $D = \mathbb{R}$, $I^+(F) = I^-(F) = \varnothing$, $I^+(L) = \{\langle r, s \rangle: r \leq s\}$, and $I^-(L) = \{\langle r, s \rangle: r > s\}$. Let $\Gamma$ be the set of sentences true in $M$, and let $\Delta = \{\exists x Fx, \exists x \neg Fx\}$. Note that $V_M((\exists x Fx \,\&\, \exists y \forall x(Fx \supset Lxy)) \supset \exists z(\forall x(Fx \supset Lxz) \,\&\, \forall y(\forall x(Fx \supset Lxy) \supset Lzy)) = t$. Note also that $V_M(\exists x Fx) = V_M(\exists x \neg Fx) = n$; so that $M$ is a counterexample to the consequence $\Gamma \vDash_t \Delta$. Finally, note that for any sentence $A$ in the $F$-free fragment of the language, $V_M(A) = t$ or $V_M(A) = f$.

Now suppose that $M' = \langle D', I' \rangle$ is a countable model which makes all of $\Gamma$ true. Note first that for any sentence $A$ in the $F$-free fragment of the language, we will have that $V_{M'}(A) = V_M(A)$. (Either $V_M(A) = t$ or $V_M(A) = f$; in the first case, $A \in \Gamma$ and $V_{M'}(A) = t$; in the second case, $V_M(\neg A) = t$, $\neg A \in \Gamma$, and $V_{M'}(A) = f$.) Furthermore, the interpretation of $L$ in $I'$ is total in that $I'^+(L) \cup I'^-(L) = D' \times D'$, since $V_{M'}(\forall x \forall y(Lxy \supset \neg Lyx)) = t$. (Suppose $\langle d, d' \rangle \in D' \times D'$ and $\langle d, d' \rangle \notin I'^+(L) \cup I'^-(L)$. Then, whether $\langle d', d \rangle \in I'^+(L)$, $I'^-(L)$, or neither, it is easy to construct a precisification $M^*$ of $M'$ with $V_{M^*}(\forall x \forall y(Lxy \supset \neg Lyx)) = f$.) Hence the structure of $M'$ under $I'^+(L)$ is isomorphic to that of the rational numbers under $\leq$, and there is some non-empty $X \subseteq D'$ which has an upper bound under $I'(L)$ but has no least upper bound under $I'(L)$.

Consider the classical model $M^*$ with domain $D'$ such that $I^{*+}(L) = I'^+(L)$ and $I^{*+}(F) = X$. $V_{M^*}((\exists x Fx \,\&\, \exists y \forall x(Fx \supset Lxy)) \supset \exists z(\forall x(Fx \supset Lxz) \,\&\, \forall y(\forall x(Fx \supset Lxy) \supset Lzy)) = f$. It follows that $M^*$ is *not* a precisification of $M'$. Hence either for some $d \in X$, $d \in I'^-(F)$, or for some $d \notin X$, $d \in I'^+(F)$. In the first case, $V_{M'}(\exists x \neg Fx) = t$, and in the second case $V_{M'}(\exists x Fx) = t$, so $M'$ is not a counterexample to the consequence $\Gamma \vDash_t \Delta$.

For the upwards part, consider a countable language with identity, and with one relational predicate $S$ and one unary predicate $F$. Let $M$ be the following partial model: $D = \mathbb{N}$, $I^+(F) = I^-(F) = \varnothing$, $I^+(S) = \{\langle n, m \rangle : m = n + 1\}$, $I^-(S) = \{\langle n, m \rangle : m \neq n + 1\}$. Let $\Gamma$ be the set of sentences true in $M$, and let $\Delta = \{\exists x Fx, \exists x \neg Fx\}$. Note that $V_M((\forall y(\neg \exists x Syx \supset Fx) \,\&\, \forall x \forall y((Fx \,\&\, Syx) \supset Fy)) \supset \forall x Fx) = \mathbf{t}$. As above, $V_M(\exists x Fx) = V_M(\exists x \neg Fx) = \mathbf{n}$, so $M$ is a counterexample to the consequence $\Gamma \vDash_{\mathbf{t}} \Delta$; and for any sentence $A$ in the $F$-free fragment of the language, $V_M(A) = \mathbf{t}$ or $V_M(A) = \mathbf{f}$.

Now suppose that $M' = \langle D', I' \rangle$ is an uncountable model which makes all of $\Gamma$ true. As above, for any sentence $A$ in the $F$-free fragment of the language, $V_{M'}(A) = V_M(A)$; and $I'^+(S) \cup I'^-(S) = D' \times D'$. Hence the structure of $M'$ under $I'^+(S)$ is a non-standard uncountable model of the theory of the natural numbers under successor; and so there is a set $X = \{d_0, d_1, d_2, \ldots\} \subseteq D'$ which behaves like the natural numbers under $I'^+(S)$, and there is an element $d \in D' - X$.

Consider the classical model $M^*$ with domain $D'$ such that $I^{*+}(S) = I'^+(S)$ and $I^{*+}(F) = X$. $V_{M^*}((\forall y(\neg \exists x Syx \supset Fx) \,\&\, \forall x \forall y((Fx \,\&\, Syx) \supset Fy)) \supset \forall x Fx) = \mathbf{f}$. As above, $M^*$ is *not* a precisification of $M'$, so either for some $d \in X$, $d \in I'^-(F)$, or for some $d \notin X$, $d \in I'^+(F)$. In the first case, $V_{M'}(\exists x \neg Fx) = \mathbf{t}$, and in the second case $V_{M'}(\exists x Fx) = \mathbf{t}$, and $M'$ is not a counterexample to the consequence $\Gamma \vDash_{\mathbf{t}} \Delta$.                          $\square$

The above proofs can easily be adapted to show that *none of the following hold*:

(1) *If the cardinality of M is greater than the cardinality of the language, then there is a model M' with cardinality equal to that of the language which is elementarily equivalent to M.*

(2) *If the cardinality of M is the cardinality of the language, and if $\kappa$ is some greater cardinality, then there is a model M' of cardinality $\kappa$, such that M is elementarily equivalent to M'.*

(3) *If the cardinality of M is greater than the cardinality of the language, then M has an elementary submodel whose cardinality is that of the language.*

Here, partial models $M$ and $M'$ are elementarily equivalent iff for every closed sentence $A$, $V_M(A) = V_{M'}(A)$; a partial model $M' = \langle D', I' \rangle$ is a *submodel* of $M = \langle D, I \rangle$ iff $D' \subseteq D$, $I'(c) = I(c)$ for every name $c$, $I'(f)$ is $I(f)$ restricted to $D^n$, for every $n$-place function symbol $f$, and $I'^+(R) = I^+(R) \cap D^n$ and $I'^-(R) = I^-(R) \cap D^n$ for each $n$-place predicate $R$; and $M'$ is an *elementary submodel* of $M$ iff $M'$ is a submodel of $M$,

and for every formula $A$ and assignment $s$ of values *in $D'$* to the variables, $V_{M,s}(A) = V_{M',s}(A)$.

## 3. Superconsequence Relations

Our main theorems concerning $\vDash_t$, $\vDash_f$ and $\vDash$ are the Superconsequence Nonaxiomatizability Theorem and the Superconsequence Noncompactness Theorem, below. First, we should point out that $\vDash$ is a mildly strange consequence relation: it is not closed under substitution. For example, suppose that $p$ and $q$ are propositional variables. Note that $p \vee q \vDash p, q$, but that $p \vee \neg p \nvDash p, \neg p$. For more about this, see the Substitution Theorem, in Section 4, below.

We say that $\Gamma \vDash B$ is *axiomatizable* iff the set $\{\langle \Gamma, B \rangle : \Gamma$ is a finite set of sentences and $B$ is a sentence and $\Gamma \vDash B\}$ is recursively enumerable; and that $\Gamma \vDash \Delta$ is *axiomatizable* iff $\{\langle \Gamma, \Delta \rangle : \Gamma$ and $\Delta$ are finite sets of sentences and $\Gamma \vDash \Delta\}$ is recursively enumerable. Similarly for "$A \vDash \Delta$ is axiomatizable," and similarly with $\vDash$ replaced by $\vDash_t$ or $\vDash_f$.

SUPERCONSEQUENCE NONAXIOMATIZABILITY THEOREM. *Suppose that the language has two two-place relation symbols, $R$ and $S$. Then the following are nonaxiomatizable:* (1) $\Gamma \vDash_f B$, (2) $\Gamma \vDash B$, (3) $A \vDash_t \Delta$, (4) $A \vDash \Delta$, (5) $\Gamma \vDash_t \Delta$, (6) $\Gamma \vDash_f \Delta$, *and* (7) $\Gamma \vDash \Delta$.

*Proof.* We will simultaneously show (1) and (2). (3) follows from (1) since $\Gamma \vDash_f B$ iff $\neg B \vDash_t \{\neg A : A \in \Gamma\}$. Similarly, (4) follows from (2). (5) follows from (3), (6) from (1), and (7) from (2). Let an *R-sentence* be any sentence $A$ in which the identity sign does not occur, and such that $R$ is the only relation symbol, name or function symbol occurring in $A$. For (1) and (2), it will suffice to show that if either $\Gamma \vDash_f B$ is axiomatizable or $\Gamma \vDash B$ is axiomatizable, then there is a positive test for the classical consistency of $R$-sentences $B$.

So assume that either $\Gamma \vDash_f B$ is axiomatizable or $\Gamma \vDash B$ is axiomatizable. Shortly we will define a sentence $C$, and we will define, for every $R$-sentence $B$, a finite set $\Gamma_B$ such that

$$(*) \quad \Gamma_B \vDash B \vee C \quad \text{iff} \quad \Gamma_B \vDash_f B \vee C \quad \text{iff} \quad B \text{ is classically consistent.}$$

Moreover our definition will imply that the function from $B$ to $\Gamma_B$ is recursive. $(*)$ will suffice to give us a positive test for the classical consistency of $R$-sentences $B$, since (1) our function from $B$ to $\Gamma_B$ is recursive and (2) by our assumption of the axiomatizability either of $\Gamma \vDash_f B$ or of $\Gamma \vDash B$, we have a positive test either of whether $\Gamma_B \vDash_f B \vee C$ or of whether $\Gamma_B \vDash B \vee C$.

Let $C$ be $(\forall x \exists y S x y \,\&\, \forall x \forall y \forall z (S x y \,\&\, S y z \supset S x z)) \supset \exists x S x x$. The important thing about $C$ is that $C$ is true in every finite classical model

and thus in every finite partial model. For every $R$-sentence $B$, let $\Gamma_B = \{B \lor \forall x \forall y Rxy,\ B \lor \neg \forall x \forall y Rxy,\ B \lor \exists x \exists y Rxy,\ B \lor \neg \exists x \exists y Rxy\}$. Note that $\Gamma_B \vDash_{\mathbf{t}} B \lor C$ since $\Gamma_B$ classically entails $B$. So $\Gamma_B \vDash B \lor C$ iff $\Gamma_B \vDash_{\mathbf{f}} B \lor C$. So for (∗) it suffices to show that $\Gamma_B \vDash_{\mathbf{f}} B \lor C$ iff $B$ is classically consistent.

($\Rightarrow$) Assume that $B$ is not classically consistent. Then every partial model falsifies $B$. Let $M = \langle D, I \rangle$ where $D = \mathbb{N}$ (the set of natural numbers), and where $I$ assigns $\langle \varnothing, \varnothing \rangle$ to $R$ and interprets $S$ as classical $<$ (less than). Note that $V_M(B) = \mathbf{f}$ and $V_M(C) = \mathbf{f}$ so that $V_M(B \lor C) = \mathbf{f}$. On the other hand, $V_M(A)$ is neither $\mathbf{t}$ nor $\mathbf{f}$ for any sentence $A \in \Gamma_B$. Thus $\Gamma_B \nvDash_{\mathbf{f}} B \lor C$.

($\Leftarrow$) Assume that $B$ is classically consistent. Let $M = \langle D, I \rangle$ be any partial model such that $V_M(B \lor C) = \mathbf{f}$, in which case $V_M(B) = V_M(C) = \mathbf{f}$. To show that $\Gamma_B \vDash_{\mathbf{f}} B \lor C$, we want $V_M(A) = \mathbf{f}$ for some $A \in \Gamma_B$. For this it suffices to show that $I(R) \neq \langle \varnothing, \varnothing \rangle$. So suppose that $I(R) = \langle \varnothing, \varnothing \rangle$. First, $D$ is infinite since $V_M(C) = \mathbf{f}$. Since $D$ is infinite and $B$ is classically consistent and does not contain the identity sign, $B$ is true in some classical model with domain $D$. And since $S$ does not occur in $B$, $B$ is true in some classical model $M' = \langle D, I' \rangle$ with $I^+(S) \subseteq I'^+(S)$ and $I^-(S) \subseteq I'^-(S)$. Note that $M \leq M'$, since $I(R) = \langle \varnothing, \varnothing \rangle$ and $I^+(S) \subseteq I'^+(S)$ and $I^-(S) \subseteq I'^-(S)$. But this contradicts the fact that $V_{M'}(B) = \mathbf{t}$ and $V_M(B) = \mathbf{f}$.                                          □

*Remark.* With some coding, the Nonaxiomatizability Theorem can be strengthened to languages with only one two-place relation symbol, no other relation symbol, no identity sign, no names and no function symbols.

SUPERCONSEQUENCE NONCOMPACTNESS THEOREM.    $\vDash$ *is not compact: there are sets $\Gamma$ and $\Delta$ of sentences such that $\Gamma \vDash \Delta$ but $\Gamma' \nvDash \Delta'$ for every finite $\Gamma' \subseteq \Gamma$ and every finite $\Delta' \subseteq \Delta$. Similarly, $\vDash_{\mathbf{t}}$ and $\vDash_{\mathbf{f}}$ are not compact.*

*Proof.* Let $A$ be the sentence $\exists x Rxx \lor \exists x \forall y \neg Rxy \lor \exists x \exists y \exists z (Rxy\ \&\ Ryz\ \&\ \neg Rxz)$. For $n \geq 1$, let $B_n$ be the sentence $\forall x_1 \ldots \forall x_{n+1}((Rx_1x_2\ \&\ \cdots\ \&\ Rx_nx_{n+1}) \supset (Rx_1x_1 \lor Rx_2x_1 \lor \cdots \lor Rx_nx_1 \lor Rx_2x_2 \lor \cdots \lor Rx_nx_2 \lor \cdots \lor Rx_nx_n))$. (For example, $B_3$ is $\forall x_1 \forall x_2 \forall x_3 \forall x_4((Rx_1x_2\ \&\ Rx_2x_3\ \&\ Rx_3x_4) \supset (Rx_1x_1 \lor Rx_2x_1 \lor Rx_3x_1 \lor Rx_2x_2 \lor Rx_3x_2 \lor Rx_3x_3))$.) Note that $V_M(A) = \mathbf{t}$ in every classical model with finite domain; and $V_M(B_n) = \mathbf{t}$ in every classical model whose domain has cardinality $\leq n$.

Let $\Delta =_{\text{df}} \{\exists x \exists y Rxy, \exists x \exists y \neg Rxy, B_1, B_2, B_3, \ldots\}$. Shortly, we will show that

(1)  $A \nvDash_t \Delta'$ for every finite $\Delta' \subseteq \Delta$,

(2)  $A \vDash_t \Delta$,

(3)  $A \nvDash \Delta'$ for every finite $\Delta' \subseteq \Delta$, and

(4)  $A \vDash \Delta$.

(1) and (2) imply that $\vDash_t$ is not compact. (3) and (4) imply not only that $\vDash$ is not compact, but that $\vDash$ is not even "compact on the right" in the obvious sense, even when there is only one sentence on the left. A dual argument shows that $\vDash_f$ is not compact and that $\vDash$ is not "compact on the left", even when there is only one sentence on the right. Note that $A$ classically implies $\Delta$. So $A \vDash_f \Delta$. Thus, for (4) it suffices to show (2). Also, (3) follows from (1). So it suffices to show (1) and (2).

First we prove (1). It suffices to show that $A \nvDash_t \Delta_n$ where $\Delta_n = \{\exists x \exists y Rxy, \exists x \exists y \neg Rxy, B_1, B_2, \ldots, B_n\}$, for each $n \geq 1$. For this, let $M = \langle D, I \rangle$ where $D = \{1, 2, 3, \ldots, n+1\}$ and where $I^+(R) = I^-(R) = \varnothing$. Note:

(1.1)  $V_M(\exists x \exists y Rxy) = V_M(\exists x \exists y \neg Rxy) = \mathbf{n}$.

(1.2)  For $i = 1, \ldots, n$, $V_M(B_i) \neq \mathbf{t}$. To see this, let $M' = \langle D, I' \rangle$, where $I'^+(R) = \{\langle i, i+1 \rangle : i = 1, 2, \ldots, n\}$; and $I'^-(R) = D^2 - I'^+(R)$. $M'$ is a classical precisification of $M$. Furthermore $V_{M'}(B_i) = \mathbf{f}$, for $i = l, \ldots, n$.

(1.3)  $V_M(A) = \mathbf{t}$. This is because $D$ is finite, so that in every precisification of $M$, $A$ is true.

(1) follows from (1.1), (1.2), and (1.3).

Next we prove (2). So suppose that $A \nvDash_t \Delta$. Let $M = \langle D, I \rangle$ be a partial model such that $V_M(\exists x Rxx \lor \exists x \forall y \neg Rxy \lor \exists x \exists y \exists z (Rxy \& Ryz \& \neg Rxz)) = \mathbf{t}$ and such that, for every $B \in \Delta$, $V_M(B) \neq \mathbf{t}$. Since $V_M(\exists x \exists y Rxy) \neq \mathbf{t}$, $I^+(R) = \varnothing$, and similarly, since $V_M(\exists x \exists y \neg Rxy) \neq \mathbf{t}$, $I^-(R) = \varnothing$.

Since $V_M(\exists x Rxx \lor \exists x \forall y \neg Rxy \lor \exists x \exists y \exists z (Rxy \& Ryz \& \neg Rxz)) = \mathbf{t}$, $D$ must be finite. For suppose that $D$ is infinite; then $D$ has a subset $D' = \{d_1 d_2, \ldots\}$ with the $d_i$ all distinct. Let $M' = \langle D, I' \rangle$, where $I'^+(R) = \{\langle d_m, d_n \rangle : m < n\} \cup \{\langle d, d_n \rangle : d \notin D' \text{ and } n \in \mathbb{N}\}$ and $I'^-(R) = D^2 - I'^+(R)$. $M'$ is a classical precisification of $M$, and $V_{M'}(A) = \mathbf{f}$, contradicting the assumption that $V_M(A) = \mathbf{t}$.

Now since $D$ is finite, suppose that the cardinality of $D$ is $n$. It follows that $V_M(B_n) = \mathbf{t}$. This contradicts the assumption that for every $B \in \Delta$, $V_M(B) \neq \mathbf{t}$. This completes the proof of (2). $\qquad\square$

*Remark.* The present proof uses an example due to an anonymous referee and replaces a more complex example and proof in the original version of the paper.

## 4. Truth in all Admissible Precisifications

As pointed out in Section 1, above, given a partial model $M$, we might want to evaluate sentences not by looking at all precisifications, but rather by looking only at the *admissible* precisifications. The idea is that the admissible precisifications are the precisifications that satisfy certain constraints. One way to implement this idea is to define a *supermodel* to be a partial model $M$ together with some set of classical precisifications of $M$. Truth in a supermodel would then be truth in all the specified precisifications. But notice that the partial model $M$ drops out of the picture: any two supermodels, thus defined, with the same set of classical precisifications will make the same sentences true and the same sentences false, regardless of whether the underlying partial models are distinct.

So we define a *supermodel* to be a nonempty set $\mathfrak{M}$ of classical models such that, for any $M = \langle D, I \rangle \in \mathfrak{M}$ and $M' = \langle D', I' \rangle \in \mathfrak{M}$, we have

- $D = D'$;
- $I(c) = I'(c)$ for every constant $c$; and
- $I(f) = I'(f)$ for every function symbol $f$.

Notice that for any supermodel $\mathfrak{M}$, there is a partial model of which every $M \in \mathfrak{M}$ is a classical precisification. In particular, choose any $M = \langle D, I \rangle \in \mathfrak{M}$, let $I'(c) = I(c)$ for every constant $c$; let $I'(f) = I(f)$ for every function symbol $f$; let $I'(K) = \langle \varnothing, \varnothing \rangle$ for every relation symbol $R$; and let $I'(p)$ be undefined for every propositional variable $p$. Note that every $M \in \mathfrak{M}$ is a classical precisification of $M' = \langle D, I' \rangle$.

Given a supermodel $\mathfrak{M}$, an assignment $s$ of values to the variables in the common domain $D$ of the models in $\mathfrak{M}$, and a formula $A$, define

$$V_{\mathfrak{M},s}(A) = \mathbf{t} \quad \text{iff} \quad V_{M,s}(A) = \mathbf{t} \quad \text{for all } M \in \mathfrak{M},$$
$$V_{\mathfrak{M},s}(A) = \mathbf{f} \quad \text{iff} \quad V_{M,s}(A) = \mathbf{f} \quad \text{for all } M \in \mathfrak{M}.$$

If $A$ has no free individual variables, we again just write $V_{\mathfrak{M}}(A)$ for $V_{\mathfrak{M},s}(A)$.

We define three new superconsequence relations. Here we assume that $\Gamma$ and $\Delta$ are sets of formulas.

$\Gamma \vDash^*_\mathbf{t} \Delta$   iff   for every supermodel $\mathfrak{M}$ and every assignment $s$
of values to the variables,
if $V_{\mathfrak{M},s}(A) = \mathbf{t}$ for every $A \in \Gamma$ then $V_{\mathfrak{M},s}(B) = \mathbf{t}$
for some $B \in \Delta$.

$\Gamma \vDash^*_\mathbf{f} \Delta$   iff   for every supermodel $\mathfrak{M}$ and every assignment $s$
of values to the variables,
if $V_{\mathfrak{M},s}(B) = \mathbf{f}$ for every $B \in \Delta$ then $V_{\mathfrak{M},s}(A) = \mathbf{f}$
for some $A \in \Gamma$.

$\Gamma \vDash^* \Delta$   iff   $\Gamma \vDash_\mathbf{t} \Delta$ and $\Gamma \vDash_\mathbf{f} \Delta$.

Unlike $\vDash$ from Section 3, the superconsequence relation $\vDash^*$ is closed under substitution. For example, as with $\vDash$, we have $p \vee \neg p \nvDash^* p, \neg p$. But we also have $p \vee q \nvDash^* p, q$, in contrast to the fact that $p \vee q \vDash p, q$. In fact, for the propositional fragment of the language, we have the Substitution Theorem, below, relating $\vDash^*_\mathbf{t}$, $\vDash^*_\mathbf{f}$ and $\vDash^*$ to $\vDash_\mathbf{t}$, $\vDash_\mathbf{f}$ and $\vDash$. Before we state this theorem, some preliminary definitions are in order. If $S$ is a sentence in the propositional fragment of the language, $p_1, \ldots, p_n$ are distinct propositional variables, and $A_1, \ldots, A_n$ are sentences in the propositional fragment, let $S[A_1, \ldots, A_n/p_1, \ldots, p_n]$ be the result of simultaneously replacing $A_i$ for $p_i$ in $S$. If $\Gamma$ is a set of sentences in the propositional fragment define $\Gamma[A_1, \ldots, A_n/p_1, \ldots, p_n]$ similarly. Finally, if $\Gamma$ and $\Delta$ are sets of sentences in the propositional fragment of the language, then we say that $\langle \Gamma', \Delta' \rangle$ is a *propositional substitution instance* of $\langle \Gamma, \Delta \rangle$ iff, for some sentences $A_1, \ldots, A_n, \ldots$ in the propositional fragment, and for some distinct propositional variables $p_1, \ldots, p_n$, $\Gamma' = \Gamma[A_1, \ldots, A_n/p_1, \ldots, p_n]$ and likewise for $\Delta'$ and $\Delta$.

SUBSTITUTION THEOREM. *Suppose that $\Gamma$ and $\Delta$ are finite sets of sentences in the propositional fragment of the language (or, more generally, sets of sentences in which only a finite number of propositional variables occur). Then $\Gamma \vDash^*_\mathbf{t} \Delta$ iff $\Gamma' \vDash^*_\mathbf{t} \Delta'$ for every propositional substitution instance $\langle \Gamma', \Delta' \rangle$ of $\langle \Gamma, \Delta \rangle$. Similarly for $\vDash^*_\mathbf{f}$ and $\vDash_\mathbf{f}$, and for $\vDash^*$ and $\vDash$.*

Prior to proving this theorem, we introduce some terminology and a lemma. First the terminology. Let $P = \{p_1, \ldots, p_n\}$ be a set of distinct propositional variables. A *P-sentence* is a sentence in the propositional fragment of the language all of whose propositional variables are in $P$. If $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \{\mathbf{t}, \mathbf{f}\}$, then we will use the notation $\mathbf{x}_1 \ldots \mathbf{x}_n$ for the classical model $M = \langle D, I \rangle$ such that

- $D = \{0\}$;
- $I(b) = 0$ for every name $b$;
- $I(f)(0, \ldots, 0) = 0$ for every $n$-ary function symbol $f$;

– $I^+(R) = D^n$ and $I^-(R) = \varnothing$ for every nonlogical $n$-ary relation symbol $R$; and
– $I(p_i) = \mathbf{x}_i$ for $i = 1, \ldots, n$; and $I(p) = \mathbf{t}$ for $p \notin P$.

We say that a model $M = \mathbf{x}_1 \ldots \mathbf{x}_n$ is a *P-model*. We say that a super-model $\mathfrak{M}$ is a *P-supermodel* iff every $M \in \mathfrak{M}$ is a $P$-model.

Let $M_0 = \langle D, I \rangle$ be the following partial model: $D = \{0\}$; $I(b) = 0$ for every name $b$; $I(f)(0, \ldots, 0) = 0$ for every $n$-ary function symbol $f$; $I^+(R) = D^n$ and $I^-(R) = \varnothing$ for every nonlogical $n$-ary relation symbol $R$; $I(p_i) = \mathbf{n}$ for $i = 1, \ldots, n$; and $I(p) = \mathbf{t}$ for $p \notin P$. Then the classical precisifications of $M_0$ are the $P$-models. We now have the following lemma.

LEMMA. *For any finite set $P = \{p_1, \ldots, p_n\}$ of propositional variables, and any P-supermodel $\mathfrak{M}$ there are P-sentences $A_1, \ldots, A_n$ such that for every P-sentence $S$, $V_{M_0}(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = V_{\mathfrak{M}}(S)$.*

*Proof.* Choose $M^* = \bar{\mathbf{m}} = \mathbf{m}_1 \ldots \mathbf{m}_n \in \mathfrak{M}$. For each $P$-model $M = \bar{\mathbf{x}} = \mathbf{x}_1 \ldots \mathbf{x}_n$, let $\alpha_i(M) = \mathbf{x}_i$ if $M \in \mathfrak{M}$, and let $\alpha_i(M) = \mathbf{m}_i$ if $M \notin \mathfrak{M}$. The $\alpha_i$ so defined are essentially $n$-ary truth-functions, so that by the expressive completeness of the propositional calculus, there are $P$-sentences $A_1, \ldots, A_n$ such that $V_M(A_i) = \alpha_i(M)$ for every $P$-model $M$. A trivial induction on the complexity of $S$ shows that for every $P$-sentence $S$ and $P$-model $M = \bar{\mathbf{x}} = \mathbf{x}_1 \ldots \mathbf{x}_n$, $V_M(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = V_M(S)$ if $M \in \mathfrak{M}$, and $V_M(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = V_{M^*}(S)$ if $M \notin \mathfrak{M}$.

Now we claim that $A_1, \ldots, A_n$ satisfy the requirements of the lemma.

Consider three cases. (1) $V_{\mathfrak{M}}(S) = \mathbf{t}$. Then $V_{\bar{\mathbf{x}}}(S) = \mathbf{t}$ for every $\bar{\mathbf{x}} \in \mathfrak{M}$, and in particular $V_{\bar{\mathbf{m}}}(S) = \mathbf{t}$. Consequently, $V_{\bar{\mathbf{x}}}(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = \mathbf{t}$ for *every* $\bar{\mathbf{x}}$, that is, for all the classical precisifications of $M_0$, and so $V_{M_0}(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = \mathbf{t}$. (2) $V_{\mathfrak{M}}(S) = \mathbf{f}$. This is similar to the first case. (3) $V_{\mathfrak{M}}(S) = \mathbf{n}$. Then $V_{\bar{\mathbf{x}}}(S) = \mathbf{t}$ for some $\bar{\mathbf{x}} \in \mathfrak{M}$, and $V_{\bar{\mathbf{y}}}(S) = \mathbf{f}$ for some $\bar{\mathbf{y}} \in \mathfrak{M}$. Thus $V_{\bar{\mathbf{x}}}(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = \mathbf{t}$ and $V_{\bar{\mathbf{y}}}(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = \mathbf{f}$. And, as $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are classical precisifications of $M_0$, $V_{M_0}(S[A_1, \ldots, A_n/p_1, \ldots, p_n]) = \mathbf{n}$.  □

With this lemma in hand, we can now prove the substitution theorem.

*Proof.* We only prove the case for $\vDash^*_{\mathbf{t}}$ and $\vDash_{\mathbf{t}^*}$.

($\Rightarrow$) Suppose that $\Gamma \vDash^*_{\mathbf{t}} \Delta$ and that $\langle \Gamma', \Delta' \rangle$ is a propositional substitution instance of $\langle \Gamma, \Delta \rangle$. By the first Axiomatizability Theorem (immediately below), $\vDash^*_{\mathbf{t}}$ is closed under substitution. So $\Gamma' \vDash^*_{\mathbf{t}} \Delta'$. It can be shown from the definitions that if $\Gamma' \vDash^*_{\mathbf{t}} \Delta'$ then $\Gamma' \vDash_{\mathbf{t}} \Delta'$. So $\Gamma' \vDash_{\mathbf{t}} \Delta'$, as desired.

($\Leftarrow$) Assume $\Gamma \nvDash_{\mathbf{t}}^* \Delta$. Let $P = \{p_1, \ldots, p_n\}$ be the set of propositional variables occurring in $\Gamma \cup \Delta$, with the $p_i$ distinct. Since $\Gamma \nvDash_{\mathbf{t}}^* \Delta$, there is a supermodel $\mathfrak{M}$ such that $V_{\mathfrak{M}}(A) = \mathbf{t}$ for every $A \in \Gamma$ and $V_{\mathfrak{M}}(B) \neq \mathbf{t}$ for any $B \in \Delta$. Since only the interpretations of the propositional variables in $P$ are relevant to the truth values of the sentences in $\Gamma$ and $\Delta$, there is a $P$-supermodel $\mathfrak{M}$ such that $V_{\mathfrak{M}}(A) = \mathbf{t}$ for every $A \in \Gamma$ and $V_{\mathfrak{M}}(B) \neq \mathbf{t}$ for any $B \in \Delta$. Let $A_1, \ldots, A_n$ be as in the lemma. Let $\Gamma' = \Gamma[A_1, \ldots, A_n / p_1, \ldots, p_n]$ and $\Delta' = \Delta[A_1, \ldots, A_n / p_1, \ldots, p_n]$. It follows that $V_{M_0}(A') = \mathbf{t}$ for every $A' \in \Gamma'$, and $V_{M_0}(B') \neq \mathbf{t}$ for every $B' \in \Delta'$, and so $\Gamma' \nvDash_{\mathbf{t}} \Delta'$, as desired.          $\square$

*Remark.* The present proof of the substitution theorem is adapted from an argument of Thomason (1973). It replaces a much more cumbersome proof in an earlier version of this paper. We thank an anonymous referee for pointing out to us the similarity of our substitution theorem to Thomason's results.

The connection of our topic to modal logic bears spelling out: in the propositional context, a supermodel is effectively an S5 possible worlds model, and our definition of truth in a supermodel for a sentence $A$ corresponds to the usual definition of truth in S5 for $\square A$. Therefore, in the propositional context, $\Gamma \vDash_{\mathbf{t}}^* \Delta$ iff $\square\Gamma \vDash_{S5} \square\Delta$, where $\square\Gamma = \{\square A : A \in \Gamma\}$. Similarly, our definition of falsehood in a supermodel for a sentence $A$ corresponds to the usual definition of truth in S5 for $\square\neg A$, and so $\Gamma \vDash_{\mathbf{f}}^* \Delta$ iff $\square\neg\Delta \vDash_{S5} \square\neg\Gamma$ iff $\Diamond\Gamma \vDash_{S5} \Diamond\Delta$. Hence, $\Gamma \vDash^* \Delta$ iff $\square\Gamma \vDash_{S5} \square\Delta$ and $\Diamond\Gamma \vDash_{S5} \Diamond\Delta$. Similarly, in the full quantificational context, a supermodel is effectively a QS5 model, where QS5 models have a constant domain of objects shared by all possible worlds, and the Barcan and converse Barcan formulas are valid. Thus we will have, quite generally, $\Gamma \vDash_{\mathbf{t}}^* \Delta$ iff $\square\Gamma \vDash_{QS5} \square\Delta$, $\Gamma \vDash_{\mathbf{f}}^* \Delta$ iff $\Diamond\Gamma \vDash_{QS5} \Diamond\Delta$, and $\Gamma \vDash^* \Delta$ iff $\square\Gamma \vDash_{QS5} \square\Delta$ and $\Diamond\Gamma \vDash_{QS5} \Diamond\Delta$. Finally, if we add the identity sign, the above equivalences continue to hold if we posit in QS5, as is usual, that true identities are necessarily true and false identities are necessarily false.

PROBLEM. Can the Substitution Theorem be extended to the first-order case, and/or to $\Gamma$ and $\Delta$ in which an infinite number of propositional variables occur?

Without the identity sign, all three of our new consequence relations are somewhat trivial.

AXIOMATIZABILITY THEOREM 1. *If the identity sign does not occur in any sentence in $\Gamma \cup \Delta$ then*

(1) $\Gamma \vDash^*_{\mathbf{t}} \Delta$ *iff either $\Gamma$ is classically inconsistent or $\Gamma$ classically entails $B$ for some $B \in \Delta$;*

(2) $\Gamma \vDash^*_{\mathbf{f}} \Delta$ *iff either $\varnothing$ classically entails $\Delta$ or $A$ classically entails $\Delta$ for some $A \in \Gamma$; and*

(3) $\Gamma \vDash^* \Delta$ *iff both $\Gamma$ classically entails $B$ for some $B \in \Delta$ and $A$ classically entails $\Delta$ for some $A \in \Gamma$.*

*Proof.* (3) follows from (1) and (2), and the proof of (2) is dual to the following proof of (1).

($\Leftarrow$) Straight from the definitions.

($\Rightarrow$) Suppose that $\Gamma$ is classically consistent and that $\Gamma$ does not classically entail $B$ for any $B \in \Delta$. If $\Delta$ is empty, then choose some classical model $M = \langle D, I \rangle$ and some assignment $s$ such that $V_{M,s}(A) = \mathbf{t}$ for every $A \in \Gamma$. Let $\mathfrak{M} = \{M\}$. Then $\Gamma \nvDash^*_{\mathbf{t}} \Delta$ since $V_{\mathfrak{M},s}(A) = \mathbf{t}$ for every $A \in \Gamma$, but $V_{\mathfrak{M},s}(A) = \mathbf{f}$ for no $B \in \Delta$.

So assume that $\Delta$ is nonempty. Add constants $k_0, k_1, \ldots, k_n, \ldots$ to the language. Let $D$ be the set of terms of the expanded language, and let $s(x) = x$, for each variable $x$. List the *existentially quantified* formulas of the expanded language as $\exists x_0 A_0, \exists x_1 A_1, \ldots, \exists x_n A_n, \ldots$ in such a way that $k_n$ does not occur in $\exists x_m A_m$ when $m \leq n$. (For simplicity, we are making an inessential assumption that the original language is countable.) Let $\Gamma' = \Gamma \cup \{\exists x_n A_n \supset A_n[k_n/x_n] : n \in \mathbb{N}\}$. Note that $\Gamma' \nvDash^*_{\mathbf{t}} B$ for each $B \in \Delta$. For each formula $B \in \Delta$, let $\Gamma_B$ be a complete consistent theory containing $\Gamma'$ such that $\Gamma_B \nvDash^*_{\mathbf{t}} B$. For each $B \in \Delta$, define the model $M_B = \langle D, I_B \rangle$ as follows. $I_B(c) = c$ for each constant $c$. $I_B(f)(t_1, \ldots, t_n) = ft_1, \ldots t_n$, for each $n$-place function symbol $f$ an terms $t_1, \ldots, t_n$. $\langle t_1, \ldots, t_n \rangle \in I_B(R)$ iff $Rt_1, \ldots, t_n \in \Gamma_B$, for each $n$-place relation symbol $R$ and terms $t_1, \ldots, t_n$. Note that $V_{M_B,s}(A) = \mathbf{t}$ for every $A \in \Gamma_B$ and that $V_{M_B,s}(B) = \mathbf{f}$. Finally, let $\mathfrak{M} = \{M_B : B \in \Delta\}$. Note that $\mathfrak{M}$ is a supermodel since all the $M_B$'s have the same domain, and the $I_B$'s agree on the interpretation of all constants and function symbols. Also note that $V_{\mathfrak{M},s}(A) = \mathbf{t}$ for every $A \in \Gamma$ and that $V_{\mathfrak{M},s}(B) \neq \mathbf{t}$ for every $B \in \Delta$. $\qquad \square$

*Remark.* (1) delivers an axiomatization of $\vDash^*_{\mathbf{t}}$ and (2) of $\vDash^*_{\mathbf{f}}$ for languages without the identity sign. For $\vDash^*_{\mathbf{t}}$, just take any axiomatization of classical consequence with multiple antecedents and a single or empty consequent, and add the rule of weakening, usable on the right only as the last step in a derivation. Dually, for $\vDash^*_{\mathbf{f}}$, just take any axiomatization of classical consequence with a single or empty antecedent and multiple consequents, and add the rule of weakening, usable on the left only as the last step in a derivation. These give an axiomatization of $\vDash^*$ for languages

without the identity sign. Take the axiomatization of $\vDash_{\mathbf{t}}^*$ and the axiomatization of $\vDash_{\mathbf{f}}^*$ and add the following rule of inference, only to be used as the last step in a derivation: from $\Gamma \vDash_{\mathbf{t}}^* \Delta$ and $\Gamma \vDash_{\mathbf{f}}^* \Delta$ to infer $\Gamma \vDash^* \Delta$. We do not know whether there is any elegant axiomatization of $\vDash^*$ that does thus not piggyback on axiomatizations of $\vDash_{\mathbf{t}}^*$ and of $\vDash_{\mathbf{f}}^*$.

If the identity sign occurs in some sentence in either $\Gamma$ or $\Delta$, things are not quite so simple. The main problem is that the identity sign is *classical* in the sense that either $V_{\mathfrak{M},s}(x = y) = \mathbf{t}$ or $V_{\mathfrak{M},s}(x = y) = \mathbf{f}$, for every supermodel $\mathfrak{M}$, for every assignment $s$ of values to the variables and for every formula of the form $x = y$. Thus, for example, we have $\vDash^* x = y$, $x \neq y$.

For our second axiomatizability theorem, we will generalize by considering uninterpreted languages with two classes of relation symbols: (1) *classical* relation symbols, including the identity sign but possibly including other symbols; (2) *nonclassical* relation symbols. We add one clause to the definition of a supermodel $\mathfrak{M}$ for any $M = \langle D, I \rangle \in \mathfrak{M}$ and $M' = \langle D', I' \rangle \in \mathfrak{M}$, (4) $I(R) = I'(R)$ for every nonlogical classical relation symbol $R$. We say that a formula $A$ is *classical* iff no nonclassical relation symbols occur in it.

We define a relation $\vdash_{\mathbf{t}}^*$ between finite sets $\Gamma$ and $\Delta$ of formulas by the following axiomatization:

Axioms:   (1) $\Gamma \vdash_{\mathbf{t}}^* \varnothing$ when $\Gamma$ is classically inconsistent.

(2) $\Gamma \vdash_{\mathbf{t}}^* B$ when $\Gamma$ classically entails $B$.

(3) $\varnothing \vdash_{\mathbf{t}}^* B, \neg B$ when $B$ is a classical formula.

Rules:    (1) Weakening: from $\Gamma \vdash_{\mathbf{t}}^* \Delta$ to infer $\Gamma' \vdash_{\mathbf{t}}^* \Delta'$, where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

(2) Cut: from $\Gamma \vdash_{\mathbf{t}}^* \Delta$, $C$ and $C$, $\Gamma \vdash_{\mathbf{t}}^* \Delta$ to infer $\Gamma \vdash_{\mathbf{t}}^* \Delta$.

(3) $\neg$intro, when $C$ is a classical formula: from $C$, $\Gamma \vdash_{\mathbf{t}}^* \Delta$ to infer $\Gamma \vdash_{\mathbf{t}}^* \Delta, \neg C$.

For any sets $\Gamma$ and $\Delta$ of formulas, we say that $\Gamma \vdash_{\mathbf{t}}^* \Delta$ iff $\Gamma' \vDash_{\mathbf{t}}^* \Delta'$ for some finite $\Gamma' \subseteq \Gamma$ and some finite $\Delta' \subseteq \Delta$. In the presence of Cut, we could derive Axiom (3) from Rule (3) and vice versa.

Dually, we define a relation $\vdash_{\mathbf{f}}^*$ between finite sets $\Gamma$ and $\Delta$ of formulas:

Axioms:   (1) $\varnothing \vdash_{\mathbf{f}}^* \Delta$ when $\varnothing$ classically entails $\Delta$.

(2) $A \vdash_{\mathbf{f}}^* \Delta$ when $A$ classically entails $\Delta$.

(3) $A, \neg A \vdash_{\mathbf{f}}^* \varnothing$ when $A$ is a classical formula.

Rules:   (1) Weakening: from $\Gamma \vdash_{\mathbf{f}}^* \Delta$ to infer $\Gamma' \vdash_{\mathbf{f}}^* \Delta'$, where $\Gamma \subseteq \Gamma'$
and $\Delta \subseteq \Delta'$.
(2) Cut: from $\Gamma \vdash_{\mathbf{f}}^* \Delta$, $C$ and $C$, $\Gamma \vdash_{\mathbf{f}}^* \Delta$ to infer $\Gamma \vdash_{\mathbf{f}}^* \Delta$.
(3) $\neg$intro, when $C$ is a classical formula: from $\Gamma \vdash_{\mathbf{f}}^* \Delta$, $C$ to
infer $\neg C$, $\Gamma \vdash_{\mathbf{f}}^* \Delta$.

For any sets $\Gamma$ and $\Delta$ of formulas, we say that $\Gamma \vdash_{\mathbf{f}}^* \Delta$, iff $\Gamma' \vdash_{\mathbf{f}}^* \Delta'$ for
some finite $\Gamma' \subseteq \Gamma$ and some finite $\Delta' \subseteq \Delta$. In the presence of Cut, we
could derive Axiom (3) from Rule (3) and vice versa.

AXIOMATIZABILITY THEOREM 2.   (1) $\Gamma \vdash_{\mathbf{t}}^* \Delta$ *iff* $\Gamma \vDash_{\mathbf{t}}^* \Delta$.
(2) $\Gamma \vdash_{\mathbf{f}}^* \Delta$ *iff* $\Gamma \vDash_{\mathbf{f}}^* \Delta$.
*Proof.* The proof of (2) is dual to the following proof of (1).
($\Rightarrow$) Straightforward.
($\Leftarrow$) First, we state and prove two claims.

CLAIM 1.   $\Gamma \vdash_{\mathbf{t}}^* B, \neg B$ *when B is a classical formula.*
*Proof.* By Axiom (2) and Rule (3), or Axiom (3) and Rule (1).

CLAIM 2.   *If $\Gamma \nvdash_{\mathbf{t}}^* \Delta$ and if the constant c does not occur in $\Gamma$, $\Delta$ or A,*
*then $\exists x A \supset A[c/x]$, $\Gamma \nvdash_{\mathbf{t}}^* \Delta$.*
*Proof.* An induction on the complexity of proof shows that if $\exists x A \supset$
$A[c/x]$, $\Gamma \vdash_{\mathbf{t}}^* \Delta$ then $\Gamma \vdash_{\mathbf{t}}^* \Delta$, when $c$ does not occur in $\Gamma$, $\Delta$ or $A$.
The basis steps rely on facts of classical logic, and the induction steps are
straightforward.

To continue with our proof that if $\Gamma \vDash_{\mathbf{t}}^* \Delta$ then $\Gamma \vdash_{\mathbf{t}}^* \Delta$, we consider two
cases.
*Case 1*: $\Delta = \varnothing$. Assume that $\Gamma \vDash_{\mathbf{t}}^* \Delta$. Then there is no supermodel in
which all members of $\Gamma$ are true. So there is no classical model in which
all members of $\Gamma$ are true, in which case $\Gamma$ has a classically inconsistent
finite subset $\Gamma'$. Thus $\Gamma' \vdash_{\mathbf{t}}^* \Delta$, by axiom 1. So $\Gamma \vdash_{\mathbf{t}}^* \Delta$.
*Case 2*: $\Delta \neq \varnothing$. Assume that $\Gamma \nvdash_{\mathbf{t}}^* \Delta$. Add constants $k_0, k_1, \ldots, k_n, \ldots$
to the language. List the *existentially quantified* formulas of the expanded
language as $\exists x_0 A_0, \exists x_1 A_1, \ldots, \exists x_n A_n, \ldots$ in such a way that $k_n$ does not
occur in $\exists x_m A_m$ when $m \leq n$. (For simplicity, we are making an inessential
assumption that the original language is countable.) Well-order the formu-

las of the expanded language as $C_0, C_1, \ldots, C_n, \ldots$ and define a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_n, \ldots$ as follows:

$$\Gamma_0 \quad = \quad \Gamma \cup \{\exists x_n A_n \supset A_n[k_n/x_n] : n \in \mathbb{N}\},$$
$$\Gamma_{n+1} = \quad \Gamma_n \cup \{C_n\} \quad \text{if } \Gamma_n, \, C_n \nvdash^*_{\mathbf{t}} \Delta; \text{ and } \Gamma_n \text{ otherwise.}$$

Let $\Gamma_\omega = \bigcup_n \Gamma_n$. Note that $\Gamma_0 \nvdash^*_{\mathbf{t}} \Delta$, by Claim 2, so that $\Gamma_n \nvdash^*_{\mathbf{t}} \Delta$, by induction on $n$. Thus $\Gamma_\omega \nvdash^*_{\mathbf{t}} \Delta$.

CLAIM 3. *If $\Gamma_\omega \vdash^*_{\mathbf{t}} \Delta, C$ then $C \in \Gamma_\omega$.*
   *Proof.* Choose finite $\Gamma' \subseteq \Gamma_\omega$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash^*_{\mathbf{t}} \Delta', C$. If $C = C_n \notin \Gamma_\omega$, then $\Gamma_n, C_n \vdash^*_{\mathbf{t}} \Delta$. Also, $\Gamma' \subseteq \Gamma_m$ for some $m$. If $k = \max(m, n)$ then $\Gamma_k \vdash^*_{\mathbf{t}} \Delta$ by weakening and cut.

So if $\Gamma_\omega \vdash^*_{\mathbf{t}} C$ then $C \in \Gamma_\omega$. So if $\Gamma_\omega$ classically entails $C$ then $C \in \Gamma_\omega$, by compactness of classical implication and Axiom (2). So $\Gamma_\omega$ does not classically entail $B$ for any $B \in \Delta$. So $\Gamma_\omega$ is classically consistent since $\Delta$ is nonempty. Furthermore, by Claim 1, above, either $C \in \Gamma_\omega$ or $\neg C \in \Gamma_\omega$ for any classical formula $C$.

For each $B \in \Delta$, extend $\Gamma_\omega$ to a complete consistent classical theory $\Gamma_B$ which does not classically entail $B$. $\Gamma_B$ is closed under classical implication, and so satisfies the *witnessing* condition: If $\exists x A \in \Gamma_B$ then $A[t/x] \in \Gamma_B$ for some term $t$. Moreover, if $C$ is classical then $C \in \Gamma_\omega$ iff $C \in \Gamma_B$. So, if $C$ is classical then $C \in \Gamma_B$ iff $C \in \Gamma_{B'}$ for any $B, B' \in \Delta$. This applies, in particular, to formulas $t = t'$ and $t \neq t'$, and to $Rt_1 \ldots t_n$, and $\neg Rt_1 \ldots t_n$ where $R$ is a classical relation symbol.

Say that two terms $t$ and $t'$ of the expanded language are *equivalent* iff the formula $t = t' \in \Gamma_\omega$. Let $[t]$ be the equivalence class determined by $t$. Let $D = \{[t] : t \text{ is a term}\}$, and define $s : \text{vbles} \to D$ by $s(x) = [x]$. For each $B \in \Delta$, define the model $M_B = \langle D, I_B \rangle$ as follows. $I_B(c) = [c]$ for each constant $c$. $I_B(f)([t_1], \ldots, [t_n]) = [ft_1, \ldots, t_n]$, for each $n$-place function symbol $f$ and terms $t_1, \ldots, t_n$. $\langle [t_1], \ldots, [t_n] \rangle \in I_B(R)$ iff $Rt_1, \ldots, t_n \in \Gamma_B$, for each $n$-place relation symbol $R$ and terms $t_1, \ldots, t_n$. Note that $V_{M_B,s}(A) = \mathbf{t}$ for every $A \in \Gamma_B$ and that $V_{M_B,s}(B) = \mathbf{f}$. Finally, let $\mathfrak{M} = \{M_B : B \in \Delta\}$. Note that $\mathfrak{M}$ is a supermodel since all the $M_B$'s have the same domain, and the $I_B$'s agree on the interpretation of all constants and function symbols and classical Relation symbols. Also note that $V_{\mathfrak{M},s}(A) = \mathbf{t}$ for every $A \in \Gamma$ and that $V_{\mathfrak{M},s}(B) \neq \mathbf{t}$ for every $B \in \Delta$.

*Remark.* (1) and (2) deliver an axiomatization of $\vDash^*$.

PROBLEM. Give a cut-free Gentzen-style axiomatization of $\vDash^*$ without piggybacking on axiomatizations of $\vDash^*_{\mathbf{t}}$ and $\vDash^*_{\mathbf{f}}$.

COROLLARY (SUPERCONSEQUENCE COMPACTNESS THEOREM). *If $\Gamma \vDash_{\mathbf{t}}^{*} \Delta$ then $\Gamma' \vDash_{\mathbf{t}}^{*} \Delta'$ for some finite $\Gamma' \subseteq \Gamma$ and some finite $\Delta' \subseteq \Delta$. Similarly for $\vDash_{\mathbf{f}}^{*}$ and $\vDash^{*}$.*

## 5. RELATED WORK AND FURTHER OPEN QUESTIONS

### 5.1. *Fixed Point Logics of Truth*

The current work was prompted by P. Kremer's reading of M. Kremer (1988). M. Kremer defines logics motivated by the *fixed point* semantics, of Kripke (1975) and of Martin and Woodruff (1975), for languages expressing their own truth concepts. These semantics use partial models assigning a partial interpretation to the special predicate $T$ representing "___ is true". Variations on the semantics depend on the scheme of evaluation used, whether the strong Kleene scheme, the weak Kleene scheme, the supervaluation scheme, or some other scheme. M. Kremer works with the strong Kleene scheme, and defines three consequence relations analogous to $\vDash_{\mathbf{t}}$, $\vDash_{\mathbf{f}}$ and $\vDash$ of Section 1, above, with truth preservation, backwards falsehood preservation, and multiple antecedents and consequents. He then provides a complete cut-free Gentzen-style axiomatization of the analogue of $\vDash$, thereby axiomatizing the *strong Kleene fixed point logic of truth*. The question that motivated P. Kremer to begin the current project is whether the *supervaluation* fixed point logic of truth is axiomatizable. That question remains open.

One reconstruction of M. Kremer's method is as follows.

*Step 1.* Begin with a complete cut-free Gentzen-style axiomatization of the strong Kleene logic of partial models: see M. Kremer's rules (1)–(3), (5) and (7)–(19). Scott (1975) also provides an axiomatization of strong Kleene $\vDash$, although it is not cut-free.

*Step 2.* Add special rules arising from the fixed point setting. M. Kremer points out that, by modifying Step 1, the same procedure allows us to base an axiomatization of the *weak* Kleene fixed point logic of truth on the weak Kleene logic of partial models. Unfortunately, this procedure is not available for an axiomatization of the supervaluation logic of truth: Step 1 is impossible because the supervaluation logic of partial models in nonaxiomatizable (see the Superconsequence Nonaxiomatizability Theorem in Section 3, above). If the supervaluation fixed point logic of truth is indeed axiomatizable, some other method will be needed to axiomatize it.

### 5.2. *Supervaluations with Nondenoting Singular Terms*

Woodruff (1984) investigates a supervaluation semantics and a superconsequence relation, but for languages with nondenoting singular terms rather than nonclassical predicates. In his semantics, the upwards and downwards Lowenheim–Skolem theorems fail, as does the analogue of the sehtence-set compactness of Section 2. Woodruff defines a superconsequence relation $\vDash_s$ based on the preservation of truth-in-a-partial-model, allowing multiple antecedents but only one consequent. Our analogue to $\vDash_s$ is $\vDash_{\mathbf{t}}$ restricted to one consequent. Our analogue to $\vDash_s$ is equivalent to classical consequence, while Woodruff's $\vDash_s$ is $\Pi_1^1$-complete. His techniques do not seem helpful in establishing the exact complexity, beyond the nonaxiomatizability, of our $\Gamma \vDash \Delta$. We leave this as another open question.

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