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# The Incompleteness of S4 $\oplus$ S4 for the Product Space $\mathbb{R} \times \mathbb{R}$

**Abstract.** Shehtman introduced bimodal logics of the products of Kripke frames, thereby introducing *frame products* of unimodal logics. Van Benthem, Bezhanishvili, ten Cate and Sarenac generalize this idea to the bimodal logics of the products of topological spaces, thereby introducing *topological products* of unimodal logics. In particular, they show that the topological product of S4 and S4 is S4 $\oplus$ S4, i.e., the fusion of S4 and S4: this logic is strictly weaker than the frame product S4 $\times$ S4. Indeed, van Benthem *et al.* show that S4 $\oplus$ S4 is the bimodal logic of the particular product space $\mathbb{Q} \times \mathbb{Q}$, leaving open the question of whether S4 $\oplus$ S4 is also complete for the product space $\mathbb{R} \times \mathbb{R}$. We answer this question in the negative.

*Keywords*: Bimodal logic, Multimodal logic, Topological semantics, Topological product, Product space.

Let $\mathcal{L}$ be a propositional language with a set $PV$ of propositional variables; standard Boolean connectives &, $\vee$ and $\neg$; and two modal operators, $\Box_1$ and $\Box_2$. We define the Boolean connectives $\supset$ and $\equiv$ as usual and the modal operators $\Diamond_1$ and $\Diamond_2$ in the obvious way. Let S4 $\oplus$ S4 be the *fusion* of S4 and S4: i.e., the bimodal logic axiomatized by S4-axioms for both modal operators $\Box_1$ and $\Box_2$ as well as the rules of Modus Ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution.[1]

A *unirelational* (*Kripke*) *frame* is a pair $\mathcal{U} = \langle W, R\rangle$, where $W$ is a nonempty set and $R$ is a reflexive transitive relation on $W$. A *birelational* (*Kripke*) *frame* is a triple $\mathcal{B} = \langle W, R_1, R_2\rangle$, where $W$ is a nonempty set and $R_1$ and $R_2$ are reflexive transitive relations on $W$. A *birelational model* is a quartuple $\mathcal{M} = \langle W, R_1, R_2, V\rangle$, where $\langle W, R_1, R_2\rangle$ is a birelational frame and $V : PV \to \mathcal{P}(W)$. $V$ is extended to all formulas as follows:

$$\begin{array}{rcl} V(\neg A) & = & W - V(A) \\ V(A \,\&\, B) & = & V(A) \cap V(B) \\ V(A \vee B) & = & V(A) \cup V(B) \\ V(\Box_1 A) & = & \{w \in W : \forall v \in W(wR_1v \Rightarrow v \in V(A))\} \\ V(\Box_2 A) & = & \{w \in W : \forall v \in W(wR_2v \Rightarrow v \in V(A))\} \end{array}$$

[1] We are following the notation in [5] and [6] here, though other notation is used for S4 $\oplus$ S4: [1] and others use S4 $\otimes$ S4 and [2] uses S4 $*$ S4.

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We say that $\mathcal{M} \models A$ iff $V(A) = W$. Given a birelational frame $\mathcal{B} = \langle W, R_1, R_2 \rangle$, we say that $\mathcal{B} \models A$ iff $\mathcal{M} \models A$ for every birelational model $\mathcal{M} = \langle W, R_1, R_2, V \rangle$. The proof of the following theorem is a straightforward generalization of the unimodal case for S4.

THEOREM 1. *$A \in$ S4 $\oplus$ S4 iff $\mathcal{B} \models A$ for every birelational frame $\mathcal{B}$.*

Van Benthem, Bezhanishvili, ten Cate and Sarenac note, in [6] (a slightly updated version of [5]), that Shehtman [4] initiated the study of a particular class of birelational frames: those that are the *products* of unirelational frames.[2] Given two unirelational frames $\mathcal{U}_1 = \langle W_1, R_1 \rangle$ and $\mathcal{U}_2 = \langle W_2, R_2 \rangle$, define the birelational frame $\mathcal{U}_1 \times \mathcal{U}_2 =_{\mathrm{df}} \langle W_1 \times W_2, R_1', R_2' \rangle$ where

$$\langle w, v \rangle R_1' \langle x, y \rangle \quad \text{iff} \quad wR_1x \text{ and } v = y\text{; and}$$
$$\langle w, v \rangle R_2' \langle x, y \rangle \quad \text{iff} \quad w = x \text{ and } vR_2y.$$

A birelational frame of the form $\mathcal{U}_1 \times \mathcal{U}_2$ is a *product frame*. The logic of product frames turns out to be the *product logic* S4 $\times$ S4, defined by adding the following two axiom schemes to the fusion S4 $\oplus$ S4:

$$\textit{com} \text{ (commutativity)} \quad \Box_1\Box_2 A \equiv \Box_2\Box_1 A$$
$$\textit{chr} \text{ (Church-Rosser)} \quad \Diamond_1\Box_2 A \supset \Box_2\Diamond_1 A$$

As noted in [6], the following theorem is an immediate corollary of a more general theorem of [2]:

THEOREM 2. *$A \in$ S4 $\times$ S4 iff $\mathcal{B} \models A$ for every product frame $\mathcal{B}$.*

The topological semantics for S4 generalizes the unirelational Kripke frame semantics for S4. [6] generalizes the above *bi*relational frame semantics for S4 $\oplus$ S4 to a *bi*topological semantics. A *bitopological space* is a triple $\mathcal{X} = \langle X, \tau_1, \tau_2 \rangle$, where $X$ is a nonempty set and each of $\tau_1$ and $\tau_2$ is a topology on $X$. Given any $S \subseteq X$, we can consider two interiors of $S$, $Int_1(S)$ and $Int_2(S)$, associated with the topologies $\tau_1$ and $\tau_2$ respectively. A *bitopological model* is a quartuple $\mathcal{M} = \langle X, \tau_1, \tau_2, V \rangle$, where $\langle X, \tau_1, \tau_2 \rangle$ is a bitopological space and $V : PV \to \mathcal{P}(X)$. $V$ is extended to all formulas as follows:

$$\begin{aligned} V(\neg A) &= X - V(A) \\ V(A \;\&\; B) &= V(A) \cap V(B) \\ V(A \vee B) &= V(A) \cup V(B) \\ V(\Box_1 A) &= Int_1(V(A)) \\ V(\Box_2 A) &= Int_2(V(A)) \end{aligned}$$

[2]As noted in [6], a systematic study of multi-dimensional modal logics of products of Kripke frames can be found in [2], and an up-to-date account of the most important results in the field can be found in [1].

We sometimes write $x \Vdash A$ instead of $x \in V(A)$. We say that $M \vDash A$ iff $V(A) = X$. Given a bitopological space $\mathcal{X} = \langle X, \tau_1, \tau_2 \rangle$, we say that $\mathcal{X} \vDash A$ iff $\mathcal{M} \vDash A$ for every bitopological model $\mathcal{M} = \langle X, \tau_1, \tau_2, V \rangle$. The following theorem is an immediate consequence of Theorem 1, above:

THEOREM 3. *$A \in$ S4 $\oplus$ S4 iff $\mathcal{X} \vDash A$ for every bitopological space $\mathcal{X}$.*

[6] defines *product spaces* analogously to the product frames defined above. Given two topological spaces $\mathcal{X}_1 = \langle X_1, \tau_1 \rangle$ and $\mathcal{X}_2 = \langle X_2, \tau_2 \rangle$, define the bitopological space $\mathcal{X}_1 \times \mathcal{X}_2 =_{\mathrm{df}} \langle X_1 \times X_2, \tau_1', \tau_2' \rangle$ where the following two families of subsets of $X_1 \times X_2$ form bases for the topologies $\tau_1'$ and $\tau_2'$, respectively:

$$\begin{array}{ll} \text{Basis for } \tau_1': & \{O \times \{x\} : O \in \tau_1 \ \& \ x \in X_2\} \\ \text{Basis for } \tau_2': & \{\{x\} \times O : x \in X_1 \ \& \ O \in \tau_2\} \end{array}$$

A bitopological space of the form $\mathcal{X}_1 \times \mathcal{X}_2$ is a *product space.*[3] [6] refers to the induced topologies $\tau_1'$ and $\tau_2'$ as the *horizontal* and *vertical* topologies, respectively.

The following table summarizes the results stated so far:

| The logic of | all | product |
|---|---|---|
| birelational frames | S4 $\oplus$ S4 | S4 $\times$ S4 |
| bitopological spaces | S4 $\oplus$ S4 | |

It is natural to expect the unfilled entry to be S4 $\times$ S4. But it isn't: [6] proves the following surprising theorem.

THEOREM 4. *$A \in$ S4 $\oplus$ S4 iff $\mathcal{X} \vDash A$ for every product space $\mathcal{X}$.*

In the unimodal case, we find not only that S4 is complete for the class of all topological spaces, but also that S4 is complete for a number of particular topological spaces, for example the rational line $\mathbb{Q}$ and the real line $\mathbb{R}$. It is natural to ask whether these results generalize to the logic S4 $\oplus$ S4 and the bitopological spaces $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{R} \times \mathbb{R}$.[4] [6] proves that the generalization does go through for $\mathbb{Q} \times \mathbb{Q}$:

THEOREM 5. *$A \in$ S4 $\oplus$ S4 iff $\mathbb{Q} \times \mathbb{Q} \vDash A$.*

[3]This terminology is at odds with the standard terminology in topology, where the *product space* $\mathcal{X}_1 \times \mathcal{X}_2$ is a topological space with a single topology defined in terms of $\tau_1$ and $\tau_2$. The current notion of a product space as a *bi*topological space is the analog of the notion of a product frame, as defined above, as a *bi*relational frame.

[4]Here we are assuming that $\mathbb{Q} \times \mathbb{Q}$ [$\mathbb{R} \times \mathbb{R}$] is equipped with horizontal and vertical topologies induced by the standard topology on $\mathbb{Q}$ [$\mathbb{R}$].

(In the exposition in [6], Theorem 4 is presented as a corollary to Theorem 5, which is proved more directly.)

[6] leaves open the question of whether S4 ⊕ S4 is complete for $\mathbb{R} \times \mathbb{R}$. The purpose of this note is to answer that question in the negative:

THEOREM 6. S4 ⊕ S4 *is not complete for* $\mathbb{R} \times \mathbb{R}$.

For Theorem 6, it suffices to find a formula $A$ such that $\mathbb{R} \times \mathbb{R} \vDash A$ and $\mathcal{M}_0 \nvDash A$ for some birelational model $\mathcal{M}_0$. Let $B$ and $C$ be the following formulas, where $p$ is a propositional variable:

$$\begin{array}{ll} B & \Box_2 p \ \& \ \Diamond_1 \neg p \ \& \ \Diamond_2 \Box_1 p \\ C & \Box_2 \neg p \ \& \ \Diamond_1 p \ \& \ \Diamond_2 \Box_1 \neg p \end{array}$$

And let $A$ be the formula $\neg \Box_1 (B \vee C)$.

Let $\mathcal{M}_0 =_{\mathrm{df}} \langle W_0, R_1, R_2, V_0 \rangle$ where

$$\begin{array}{rcl} W_0 & = & \{1, 2, 3, 4\} \\ R_1 & = & \{\langle w, w \rangle : w \in W\} \cup \{\langle 1, 2 \rangle, \langle 2, 1 \rangle\} \\ R_2 & = & \{\langle w, w \rangle : w \in W\} \cup \{\langle 1, 3 \rangle, \langle 2, 4 \rangle\} \\ V_0(p) & = & \{1, 3\} \end{array}$$

Note the following. $V_0(\Box_2 p) = \{1, 3\}$. Also, $V_0(\Box_1 p) = \{3\}$. So $V_0(\Diamond_2 \Box_1 p) = \{1, 3\}$. Also, $V_0(\Diamond_1 \neg p) = \{1, 2, 4\}$. Thus $V_0(B) = \{1\}$. Similarly, $V_0(C) = \{2\}$. So $V_0(B \vee C) = \{1, 2\}$. So $V_0(\Box_1 (B \vee C)) = \{1, 2\}$. So $V_0(A) = \{3, 4\}$. So $\mathcal{M}_0 \nvDash A$.

Our final task is to show that $\mathbb{R} \times \mathbb{R} \vDash A$. First we introduce some new terminology. An *open horizontal interval* is any subset of $\mathbb{R} \times \mathbb{R}$ of the following form, where $a, b, c \in \mathbb{R}$, where $a < b$ and where $(a, b) =_{\mathrm{df}} \{x \in \mathbb{R} : a < x < b\}$:

$$(a, b) \times \{c\}.$$

Similarly, an *open vertical interval* is any subset of $\mathbb{R} \times \mathbb{R}$ of the following form, where $a, b, c \in \mathbb{R}$ and $a < b$:

$$\{c\} \times (a, b).$$

The *unit open horizontal interval* is $I_0 =_{\mathrm{df}} (0, 1) \times \{0\}$. Note that the open horizontal intervals form a basis for the horizontal topology on $\mathbb{R} \times \mathbb{R}$, and the open vertical intervals form a basis for the vertical topology.

Now for our final task: suppose, for a reductio, that $\mathbb{R} \times \mathbb{R} \nvDash A$. Then there is some model $\mathcal{M} = \langle \mathbb{R} \times \mathbb{R}, \tau_1, \tau_2, V \rangle$ where $\tau_1$ and $\tau_2$ are the horizontal and vertical topologies induced on $\mathbb{R} \times \mathbb{R}$ by the standard topology on $\mathbb{R}$, where $V : PV \to \mathcal{P}(\mathbb{R} \times \mathbb{R})$, and where $\mathcal{M} \not\models A$. So there is some point $\langle a, b\rangle \in \mathbb{R} \times \mathbb{R}$ such that

$$\langle a, b\rangle \Vdash \Box_1(B \vee C).$$

So, for some horizontal interval $I$, we have $\langle a, b\rangle \in I$ and

$$I \subseteq V(B \vee C).$$

Without loss of generality, we can assume that $I = I_0$. So

$$I = I_0 \subseteq V(B) \cup V(C).$$

Let

$$\begin{array}{rcl} P & =_{\text{df}} & I_0 \cap V(B), \\ Q & =_{\text{df}} & I_0 \cap V(C), \\ P^* & =_{\text{df}} & \{x \in \mathbb{R} : \langle x, 0\rangle \in P\} = \{x \in \mathbb{R} : \exists y \in \mathbb{R}\ \langle x, y\rangle \in P\}, \text{ and} \\ Q^* & =_{\text{df}} & \{x \in \mathbb{R} : \langle x, 0\rangle \in Q\} = \{x \in \mathbb{R} : \exists y \in \mathbb{R}\ \langle x, y\rangle \in Q\}, \end{array}$$

so that $I_0 = P \cup Q$. Note that $P \subseteq V(B) \subseteq V(p)$ and $Q \subseteq V(C) \subseteq V(\neg p) = (\mathbb{R} \times \mathbb{R}) - V(p)$. So

$$\begin{array}{rcl} P & = & I_0 \cap V(p), \text{ and} \\ Q & = & I_0 \cap V(\neg p). \end{array}$$

So $P \cap Q = \emptyset$ and $I_0 = P \mathbin{\dot\cup} Q$. We will now show that $I_0 \subseteq Cl_1(P)$, where $Cl_1$ is the closure operator associated with the horizontal topology, $\tau_1$. So suppose that $\langle x, 0\rangle \in I_0$. If $\langle x, 0\rangle \in P$ then clearly $\langle x, 0\rangle \in Cl_1(P)$. On the other hand, if $\langle x, 0\rangle \notin P$, then we have $\langle x, 0\rangle \in Q = I_0 \cap V(C) \subseteq I_0 \cap V(\Diamond_1 p) = I_0 \cap Cl_1(V(p)) \subseteq Cl_1(I_0 \cap V(p))$ (since $I_0$ is horizontally open) $= Cl_1(P)$. Thus, $I_0 \subseteq Cl_1(P)$ as desired. Similarly, $I_0 \subseteq Cl_1(Q)$. We summarize: $I_0 = P \mathbin{\dot\cup} Q$ and $I_0 \subseteq Cl_1(P)$ and $I_0 \subseteq Cl_1(Q)$. Thus $(0, 1) = P^* \mathbin{\dot\cup} Q^*$, and $Cl(P^*) = Cl(Q^*) = [0, 1]$, where $Cl$ is the standard closure operator on subsets of $\mathbb{R}$ and where $[0, 1]$ is the closed unit interval. Note finally that $Int(P^*) = Int(Q^*) = \emptyset$, where $Int$ is the standard interior operator on subsets of $\mathbb{R}$.

Note that, for each $x \in (0, 1)$,

$$\begin{array}{ll} \text{if } x \in P^*, & \text{then } \langle x, 0\rangle \in V(\Box_2 p), \text{ and} \\ \text{if } x \in Q^*, & \text{then } \langle x, 0\rangle \in V(\Box_2 \neg p). \end{array}$$

Thus, for each $x \in (0, 1)$, we can choose an open vertical interval $J_x$ so that $\langle x, 0\rangle \in J_x$ and

$$\begin{array}{ll} \text{if } x \in P^*, & \text{then } J_x \subseteq V(p), \text{ and} \\ \text{if } x \in Q^*, & \text{then } J_x \subseteq V(\neg p). \end{array}$$

Note that $J_x = \{x\} \times (a, b)$ for some $a, b \in \mathbb{R}$ with $a < 0 < b$.[5]

Given $y > 0$, we define the sets $P_y$, $Q_y$, and $R_y \subseteq (0, 1)$ as follows:

$$\begin{array}{lll} P_y & =_{\text{df}} & \{x \in P^* : \langle x, y\rangle \in J_x \text{ and } \langle x, -y\rangle \in J_x\} \\ Q_y & =_{\text{df}} & \{x \in Q^* : \langle x, y\rangle \in J_x \text{ and } \langle x, -y\rangle \in J_x\} \\ R_y & =_{\text{df}} & \{x \in (0, 1) : \langle x, y\rangle \in J_x \text{ and } \langle x, -y\rangle \in J_x\} = P_y \mathbin{\dot\cup} Q_y. \end{array}$$

Here are some useful facts about $P_y$, $Q_y$ and $R_y$. First,

$$\begin{array}{lll} P_y & = & \{x \in P^* : \big(\{x\} \times [-y, y]\big) \subseteq J_x\}, \\ Q_y & = & \{x \in Q^* : \big(\{x\} \times [-y, y]\big) \subseteq J_x\}, \text{ and} \\ R_y & = & \{x \in (0, 1) : \big(\{x\} \times [-y, y]\big) \subseteq J_x\}. \end{array}$$

Second, if $y > y' > 0$, then $P_y \subseteq P_{y'} \subseteq P^*$ and $Q_y \subseteq Q_{y'} \subseteq Q^*$ and $R_y \subseteq R_{y'} \subseteq (0, 1)$. And third,

$$\begin{array}{rll} P^* & = & \bigcup_{n\geq 1} P_{\frac{1}{n}}, \\ Q^* & = & \bigcup_{n\geq 1} Q_{\frac{1}{n}}, \text{ and} \\ (0, 1) & = & \bigcup_{n\geq 1} R_{\frac{1}{n}}. \end{array}$$

LEMMA 7. *$Cl(P_y) \cap (0, 1) \subseteq P^*$, for each $y > 0$.*

PROOF. Suppose not. Then for some $y > 0$ and some $x \in (0, 1)$ we have $x \in Cl(P_y)$ and $x \notin P^*$. So $x \in Cl(P_y)$ and $x \in Q^*$. We will now show the following:

$$\forall z \in (-y, y),\ \langle x, z\rangle \Vdash \Diamond_1 p. \qquad (\dagger)$$

So choose any $z \in (-y, y)$. We consider two cases: (1) $z = 0$ and (2) $z \neq 0$.

In case (1), since $x \in Q^*$, we have the following: $\langle x, z\rangle = \langle x, 0\rangle \in Q \subseteq V(C) = V(\Box_2\neg p \ \&\ \Diamond_1 p \ \&\ \Box_2\Diamond_1\neg p) \subseteq V(\Diamond_1 p)$. So $\langle x, z\rangle \Vdash \Diamond_1 p$ as desired.

In case (2), consider any open horizontal interval $K$ such that $\langle x, z\rangle \in K$. We want to show that $K \cap V(p)$ is nonempty. Let $K^* =_{\text{df}} \{w \in (0, 1) : \langle w, z\rangle \in K\}$. Note that $x \in K^*$ and that $K^*$ is an open interval in the real line. So, since $x \in Cl(P_y)$, there is some $v \in K^* \cap P_y$. Also, since $0 < |z| < y$, we have $P_y \subseteq P_{|z|}$. So $v \in K^*$ and $v \in P_{|z|}$. Thus, $\langle v, z\rangle \in K$ and $\langle v, z\rangle \in J_v$.

[5] We do not need the axiom of choice to choose the $J_x$'s. For each $x \in P^*$, we can define

$$J_x =_{\text{df}} (\{x\} \times (-1, 1)) \ \cap\ \bigcup\{J \text{ an open vertical interval} : \langle x, 0\rangle \in J \subseteq V(p)\}.$$

And for each $x \in Q^*$, we can define $J_x$ similarly in terms of $V(\neg p)$.

Since $J_v \subseteq V(p)$, we have $\langle v, z\rangle \in K \cap V(p)$. So $K \cap V(p)$ is nonempty, as desired, and (†) is shown.

From (†) it follows that $\langle x, 0\rangle \Vdash \Box_2\Diamond_1 p$. So $\langle x, 0\rangle \nVdash \Diamond_2\Box_1\neg p$. On the other hand, $x \in Q^*$. So

$$\langle x, 0\rangle \in Q \subseteq V(C) = V(\Box_2\neg p \,\&\, \Diamond_1 p \,\&\, \Diamond_2\Box_1\neg p) \subseteq V(\Diamond_2\Box_1\neg p).$$

So $\langle x, 0\rangle \Vdash \Diamond_2\Box_1\neg p$. A contradiction. ■

Given that $Cl(P_y) \cap (0, 1) \subseteq P^*$ (Lemma 7), and given that $P_y \subseteq (0, 1)$, we conclude that

$$Int(Cl(P_y)) = Int(Cl(P_y) \cap (0, 1)) \subseteq Int(P^*) = \emptyset.$$

So $P_y$ is nowhere dense, for each $y > 0$. A completely parallel argument shows that $Q_y$ is nowhere dense, for each $y > 0$. So $R_y = P_y \cup Q_y$ is nowhere dense, for each $y > 0$. Recall that

$$(0, 1) \quad = \quad \bigcup_{n\geq 1} R_{\frac{1}{n}}.$$

Thus, the open unit interval is a countable union of nowhere dense sets, i.e. it is meagre. But this contradicts the Baire Category Theorem. This ends our proof that $\mathbb{R} \times \mathbb{R} \vDash A$.

To summarize: We have shown that the following formula, though not a theorem of S4 $\oplus$ S4, is validated by $\mathbb{R} \times \mathbb{R}$:

$$\neg\Box_1((\Box_2 p \,\&\, \Diamond_1\neg p \,\&\, \Diamond_2\Box_1 p) \vee (\Box_2\neg p \,\&\, \Diamond_1 p \,\&\, \Diamond_2\Box_1\neg p)).$$

So S4 $\oplus$ S4 is not complete for $\mathbb{R} \times \mathbb{R}$. A slight reworking of the above argument shows that this formula is also validated by $\mathbb{R} \times \mathbb{Q}$: thus S4 $\oplus$ S4 is not complete for $\mathbb{R} \times \mathbb{Q}$. If we define a formula $A'$ by switching the subscripted 1's and 2's in the formula $A$, then we get a formula that, though not a theorem of S4 $\oplus$ S4, is validated by $\mathbb{Q} \times \mathbb{R}$: thus S4 $\oplus$ S4 is not complete for $\mathbb{Q} \times \mathbb{R}$.

For any bitopological space $\mathcal{X}$ and any class $\mathfrak{X}$ of bitopological spaces, define the logics $\mathsf{Log}(\mathcal{X}) =_{\text{df}} \{A : \mathcal{X} \vDash A\}$ and $\mathsf{Log}(\mathfrak{X}) =_{\text{df}} \{A : \mathcal{X} \vDash A$, for every $\mathcal{X} \in \mathfrak{X}\}$. For any (uni)topological space $\mathcal{X}$ and any class $\mathfrak{Y}$ of (uni)topological spaces, define the class of bitopological spaces $\mathcal{X} \times \mathfrak{Y} =_{\text{df}} \{\mathcal{X} \times \mathcal{Y} : \mathcal{Y} \in \mathfrak{Y}\}$; and let $\mathfrak{T}$ be the class of all (uni)topological spaces and $\mathfrak{A}$ be the class of all Alexandroff spaces.[6] There remains the question of

[6]A (uni)topological space $\mathcal{X} = \langle X, \tau\rangle$ is *Alexandroff* iff $\tau$ is closed under arbitrary intersections. There is a well-known duality between Alexandroff spaces and unirelational Kripke frames: For each unirelational Kripke frame $\mathcal{U} = \langle W, R\rangle$, define the toplogical space $\mathcal{X}_{\mathcal{U}} =_{\text{df}} \langle W, \tau\rangle$, where $O \in \tau$ iff $(\forall x, y \in W)(x \in O \Rightarrow y \in O)$. Note that a topological space $\mathcal{X}$ is Alexandroff iff $\mathcal{X} = \mathcal{X}_{\mathcal{U}}$ for some unirelational Kripke frame $\mathcal{U}$.

the properties (axiomatizability, etc.) of $\mathsf{Log}(\mathbb{R} \times \mathbb{R})$, $\mathsf{Log}(\mathbb{R} \times \mathbb{Q})$ and other related logics, such as $\mathsf{Log}(\mathbb{R} \times \mathfrak{T})$ and $\mathsf{Log}(\mathbb{R} \times \mathfrak{A})$. Finally, let $\mathfrak{Triv}$ be the class of trivial (uni)topological spaces, i.e. spaces with only two open sets. Valentin Shehtman has suggested, in personal correspondence, a possibly easier but still open question: what are the properties (axiomatizability, etc.) of $\mathsf{Log}(\mathbb{R} \times \mathfrak{Triv})$?[7]

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[7] Let $com_{\supset}$ be the left-to-right direction of the axioms scheme *com*: $\Box_1\Box_2 A \supset \Box_2\Box_1 A$. In [3], we show that $\mathsf{Log}(\mathbb{Q} \times \mathfrak{Triv})$ can be axiomatized by adding to S4 $\oplus$ S5 the axioms schemes $com_{\supset}$ and *chr*, and that $\mathsf{Log}(\mathbb{R} \times \mathfrak{Triv})$ cannot be so axiomatized.