Mathematical Logic

The modal logic of continuous functions on the rational numbers

Philip Kremer

Abstract   Let $\mathcal{L}^{\Box\circ}$ be a propositional language with standard Boolean connectives plus two modalities: an S4-ish topological modality $\Box$ and a temporal modality $\circ$, understood as 'next'. We extend the topological semantic for S4 to a semantics for the language $\mathcal{L}^{\Box\circ}$ by interpreting $\mathcal{L}^{\Box\circ}$ in *dynamic topological systems*, i.e., ordered pairs $\langle X, f\rangle$, where X is a topological space and f is a continuous function on X. Artemov, Davoren and Nerode have axiomatized a logic S4C, and have shown that S4C is sound and complete for this semantics. S4C is also complete for continuous functions on Cantor space (Mints and Zhang, Kremer), and on the real plane (Fernández Duque); but incomplete for continuous functions on the real line (Kremer and Mints, Slavnov). Here we show that S4C is complete for continuous functions on the rational numbers.

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1 Introduction

Let $\mathcal{L}^{\Box}$ be a propositional modal language with a set $PV = \{p_1, \ldots, p_n, \ldots\}$ of propositional variables; parentheses; connectives $\neg$ and &; and a modal operator $\Box$. We assume that the Boolean connectives $\vee$, $\supset$ and $\equiv$ are defined in the standard way, and that $\Diamond$ is defined as $\neg\Box\neg$. The McKinsey-Tarski topological semantics [7,8] interprets $\mathcal{L}^{\Box}$ in topological spaces, interpreting $\Box$ as topological interior. The resulting modal logic, S4, can thus be seen as the modal logic of topological spaces.

P. Kremer (✉)
Department of Philosophy, University of Toronto, Toronto, ON, Canada
e-mail: kremer@utsc.utoronto.ca

Topology provides a minimal environment in which to study continuous functions. Thus we might want a modal logic, not only of topological spaces, but of continuous functions on topological spaces. Let a *dynamic topological system* be an ordered pair, $\langle X, f \rangle$, where X is a topological space and f is a continuous function on X. We extend the language $\mathcal{L}^\square$ to the language $\mathcal{L}^{\square\circ}$ by adding the temporal operator $\circ$: we will use f to interpret $\circ$.

Artemov et al. [1] prove the soundness and completeness of a very natural logic S4C, whose topological fragment is just S4, wrt the class of all dynamic topological systems. (See also [2,5].) Thus S4C can be seen as the modal logic of continuous functions on topological spaces. S4C is also complete for continuous functions on Cantor space [9,4]; and on the real plane, $\mathbb{R}^2$ [3]. The current paper's main theorem is that S4C is complete for continuous functions on the rational numbers, $\mathbb{Q}$. (Here we are assuming the standard topology on $\mathbb{R}$ and $\mathbb{R}^2$, and we are assuming that $\mathbb{Q}$ has the subspace topology.)

This result is in sharp contrast to what happens with the real numbers: S4C is *not* complete for continuous functions on $\mathbb{R}$. Let A be the formula $((\square \circ p \supset \circ \Diamond \square p) \vee (\circ q \supset \square \circ q))$, where $p, q \in PV$. Kremer and Mints [5] show that A is validated by $\mathbb{R}$ but that $A \notin$ S4C (See [10] for another counterexample to the completeness of S4C in $\mathbb{R}$.). Thus the modal logic of continuous functions on $\mathbb{Q}$ is strictly weaker than the modal logic of continuous functions on $\mathbb{R}$. This fact is unsurprising when we remind ourselves that there are continuum many continuous functions on $\mathbb{Q}$ that cannot be extended to continuous functions on $\mathbb{R}$. To get a continuum-sized family of simple examples, define, for each irrational number i, the function f_i as follows:

$$f_i(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \text{ and } x < i \\ 1, & \text{if } x \in \mathbb{Q} \text{ and } x > i \end{cases}$$

We now state our main theorem precisely. A *dynamic topological model* (DTM) is an ordered triple $M = \langle X, f, V \rangle$ where $\langle X, f \rangle$ is a dynamic topological system and $V : PV \to \mathcal{P}(X)$ is a valuation function assigning a subset of X to each propositional variable. The valuation V is extended to all formulas as follows:

$$V(\neg A) = X - V(A),$$
$$V(A\&B) = V(A) \cap V(B),$$
$$V(\square A) = Int(V(A)) = \text{the topological interior of } V(A), \text{ and}$$
$$V(\circ A) = f^{-1}(V(A)).$$

We define four validity relations, where X is a topological space, f is a continuous function on X, $V : PV \to \mathcal{P}(X)$, A is a formula, and $M = \langle X, f, V \rangle$:

$$
\begin{array}{rl}
M \models A & \text{iff } V(A) = X. \\
\langle X, f \rangle \models A & \text{iff } \langle X, f, V \rangle \models A \text{ for every } V : PV \to \mathcal{P}(X). \\
X \models A & \text{iff } \langle X, f \rangle \models A \text{ for every continuous function } f. \\
\models A & \text{iff } M \models A \text{ for every dynamic topological model } M.
\end{array}
$$

The logic S4C is the logic determined by the axioms of S4 for $\square$, plus

$$(\circ(A \,\&\, B) \equiv (\circ A \,\&\, \circ B)),$$
$$(\circ \neg A \equiv \neg \circ A), \text{ and}$$
$$(\circ \square A \supset \square \circ A);$$

and the rules of Modus Ponens, and necessitation for $\square$ and $\circ$. Artemov et al. [1] prove the following soundness and completeness theorem:

Theorem 1.1 $\models A$ *iff* $A \in$ S4C.

Our main theorem is as follows:

Theorem 1.2 $\mathbb{Q} \models A$ *iff* $A \in$ S4C.

The ($\Leftarrow$) direction of the biconditional in Theorem 1.2, i.e., soundness, follows from the ($\Leftarrow$) direction of the biconditional in Theorem 1.1. It remains for us to prove the ($\Rightarrow$) direction of the biconiditon in Theorem 1.2, i.e., completeness.

First, a couple of remarks about *linear orderings* and *countable dense linear orderings without endpoints*. Let $\mathbb{D} = \langle D, \ll \rangle$ be a linear ordering. Given $a, b \in D$ with $a \ll b$, we define the open rays (a, ∞) and (∞, b) and the open interval (a, b) as follows:

$$(a, \infty) =_{\mathrm{df}} \{d \in D : a \ll d\}$$
$$(\infty, b) =_{\mathrm{df}} \{d \in D : d \ll b\}$$
$$(a, b) =_{\mathrm{df}} \{d \in D : a \ll d \ll b\}$$

The open intervals together with the open rays form a basis for the *order topology* on D. The subspace topology on $\mathbb{Q}$ is, in fact, identical to the order topology on $\mathbb{Q}$.[1] If $\mathbb{D} = \langle D, \ll \rangle$ has no endpoints, then the open intervals (without the open rays) form a basis for the order topology on D.

A classic result of Cantor's states that any two countable dense linear orderings without endpoints are order-isomorphic. From this it follows that, if $\mathbb{D} = \langle D, \ll \rangle$ is a countable dense linear ordering without endpoints, then there is a homeomorphism, i.e., a continuous open bijection, $h : D \to \mathbb{Q}$, where continuity and openness are defined via the order topologies. It is also easy to prove

Lemma 1.3 *If X and Y are topological spaces and $h : X \to Y$ is a homeomorphism, then for any formula A of $\mathcal{L}$, we have $X \models A$ iff $Y \models A$.*

Given Lemma 1.3, in order to prove Theorem 1.2 it suffices to prove

Lemma 1.4 *If $A \notin$ S4C then there is a countable dense linear ordering $\mathbb{D} = \langle D, \ll \rangle$ without endpoints such that $D \not\models A$ (where we have imposed the order topology on D).*

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[1] This does not go without saying. For, when D is linearly ordered by $\ll$ and $D' \subseteq D$ is linearly ordered by the induced ordering from D (i.e., by $\ll' = \ll \cap D' \times D'$), it is not always the case that the subspace topology on D' is identical to the order topology on D'. For a counterexample, see the PlanetMath article, *order topology* (http://planetmath.org/encyclopedia/OrderTopology.html, accessed 16/04/07).

2 Dynamic Kripke models

Our proof of Lemma 1.4 relies on an alternate, Kripke-style, semantics for S4C. A *dynamic Kripke frame* is an ordered triple $K = \langle W, R, f \rangle$ where W is a non-empty set (of worlds), R is a reflexive and transitive relation on W, and $f : W \to W$ is monotonic wrt R, i.e., $wRw' \Rightarrow fwRfw'$. We define the interior of $S \subseteq W$:

$$Int(S) =_{df} \{w \in W : \forall w' \in W, \text{ if } wRw' \text{ then } w' \in S\}.$$

A *dynamic Kripke model* is a quartuple $M = \langle W, R, f, V \rangle$, where $\langle W, R, f \rangle$ is a dynamic Kripke frame and V is a valuation function assigning a subset of W to each propositional variable. The valuation V is extended to all formulas as follows:

$$V(\neg A) = X - V(A),$$
$$V(A\&B) = V(A) \cap V(B),$$
$$V(\Box A) = Int(V(A)), \text{ and}$$
$$V(\circ A) = f^{-1}(V(A)).$$

A formula A is *validated by* $M = \langle W, R, f, V \rangle$ iff $V(A) = W$. Notation: $M \models A$.
    Artemov et al. [1] prove soundness and completeness wrt dynamic Kripke models:

Theorem 2.1 $A \in$ S4C *iff A is validated by every dynamic Kripke model.*

They also prove that S4C has the finite model property (for an alternate proof, see [6]):

Theorem 2.2 $A \in$ S4C *iff A is validated by every finite dynamic Kripke model.*

Suppose that $K = \langle W, R, f \rangle$ is a dynamic Kripke frame, that $\langle X, g \rangle$ is a dynamic topological system and that $h : X \to W$ is onto. We say that h is a *simulation of K in* $\langle X, g \rangle$ iff

1. for each $w \in W$, the set $O_w = \{x \in X : wRh(x)\}$ is open in X;
2. for each open set $O \subseteq X$, the image of O under h is closed under the accessibility relation R; and
3. for each $x \in X$, we have $f(h(x)) = h(g(x))$.

Our *simulation lemma* is as follows:

Lemma 2.3 *Suppose that $K = \langle W, R, f \rangle$ is a dynamic Kripke frame, that $\langle X, g \rangle$ is a dynamic topological system and that $h : X \to W$ is a simulation of K in $\langle X, g \rangle$. Also suppose that $V : PV \to \mathcal{P}(W)$, that $V' : PV \to \mathcal{P}(X)$, and that $V'(p) = \{x \in X : h(x) \in V(p)\}$ for each $p \in PV$. Then $V'(A) = \{x \in X : h(x) \in V(A)\}$ for each formula A.*

Proof Given the stated assumptions, we prove that $V'(A) = \{x \in X : h(x) \in V(A)\}$ for each formula A, by induction on A. If $A \in PV$, then the result is given by our

assumption that $V'(p) = \{x \in X : h(x) \in V(p)\}$ for each $p \in PV$. Suppose that $A = \neg B$. Then $V'(A) = V'(\neg B) = X - V'(B) = X - \{x \in X : h(x) \in V(B)\} = \{x \in X : h(x) \notin V(B)\} = \{x \in X : h(x) \in V(\neg B)\} = \{x \in X : h(x) \in V(A)\}$. The argument is similar if $A = B \mathbin{\&} C$. The only tricky cases are the induction steps for $\circ$ and $\square$.

Suppose that $A = \circ B$. Then $V'(A) = V'(\circ B) = g^{-1}(V'(B)) = \{x \in X : g(x) \in V'(B)\} = \{x \in X : h(g(x)) \in V(B)\} = \{x \in X : f(h(x)) \in V(B)\} = \{x \in X : h(x) \in V(\circ B)\} = \{x \in X : h(x) \in V(A)\}$.

Suppose that $A = \square B$. For every $x \in X$, we will show that $x \in V'(A)$ iff $h(x) \in V(A)$. ($\Rightarrow$) Suppose that $x \in V'(A) = V'(\square B) = Int(V'(B))$. To see that $h(x) \in V(A) = V(\square B)$, choose $w \in W$ with $h(x)Rw$. Note that $w \in V(B)$, since the image of the open set $Int(V'(B))$ under h is closed under R, by the definition of *simulation*. Thus $h(x) \in V(\square B)$, as desired. ($\Leftarrow$) Suppose that $h(x) \in V(A) = V(\square B)$. We want to show that $x \in Int(V'(B))$. By the definition of *simulation*, the set $O_{h(x)} = \{y \in X : h(x)Rh(y)\}$ is open in X. Also notice that $x \in O_{h(x)}$. So it suffices to show that $O_{h(x)} \subseteq V'(B)$. Suppose that $y \in O_{h(x)}$. Then $h(x)Rh(y)$. So $h(y) \in V(B)$, since $h(x) \in V(\square B)$. So $y \in V'(B)$ by the Inductive Hypothesis. $\square$

Corollary 2.4 *Suppose that $K = \langle W, R, f \rangle$ is a dynamic Kripke frame, that $\langle X, g \rangle$ is a dynamic topological system and that $h : X \to W$ is a simulation of K in $\langle X, g \rangle$. Then, for each formula A, if $K \not\models A$ then $\langle X, g \rangle \not\models A$.*

Given Corollary 2.4 and Theorem 2.2, in order to prove Lemma 1.4 it suffices to prove

Lemma 2.5 *For every finite dynamic Kripke frame $K = \langle W, R, f \rangle$, there is a countable dense linear ordering $\mathbb{D} = \langle D, \ll \rangle$ without endpoints, a continuous function $g : D \to D$ (where we have imposed the order topology on D), and a simulation h of K in $\langle D, g \rangle$.*

3 Proof of Lemma 2.5

Suppose that $K = \langle W, R, f \rangle$ is a finite dynamic Kripke frame. Our task is to define a countable dense linear ordering $\mathbb{D} = \langle D, \ll \rangle$ without endpoints, a continuous function g on D, and a simulation h of K in $\langle D, g \rangle$.[2] First, some preliminaries. A *marked world* is an ordered pair $\langle z, w \rangle \in \{-1, 0, 1\} \times W$. If **d** is an infinite sequence of marked worlds, we write

$$\mathbf{d} = \langle z_0^{\mathbf{d}}, w_0^{\mathbf{d}} \rangle, \langle z_1^{\mathbf{d}}, w_1^{\mathbf{d}} \rangle, \langle z_2^{\mathbf{d}}, w_2^{\mathbf{d}} \rangle, \ldots$$

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[2] The construction we will present is motivated by a construction in [11]. Let a *trinary* number be any rational number of the form $\pm m/3^n$, where $m, n \in \mathbb{N}$. Van Benthem et al. [11] label the trinary numbers in the open interval $(-\frac{3}{2}, \frac{3}{2})$ with nodes from the infinite binary tree, T_2. These nodes can, in turn, be labeled with the worlds of any finite Kripke frame $\langle W, R \rangle$: this induces a labeling of the trinary numbers by the worlds. Rather than labeling a preexisting dense linear ordering with worlds, we proceed by constructing our dense linear ordering directly out of the worlds in the Kripke frame.

We say that $\mathbf{d}$ is *R-increasing* iff $w_n^{\mathbf{d}} R w_{n+1}^{\mathbf{d}}$, for every $n \geq 0$. We say that $k \geq 0$ is a *stabilization point for* $\mathbf{d}$ iff $z_m^{\mathbf{d}} = z_k^{\mathbf{d}} = 0$ and $w_m^{\mathbf{d}} = w_k^{\mathbf{d}}$, for every $m \geq k$. We say that $\mathbf{d}$ is *stable* iff $\mathbf{d}$ has a stabilization point. Note that stable R-increasing sequences of marked worlds exist, since R is reflexive. Our set D will be the set of stable R-increasing infinite sequences of marked worlds. D is clearly countably infinite.

Given any distinct $\mathbf{d}, \mathbf{d}' \in D$, it will be useful to define $diff(\mathbf{d}, \mathbf{d}') =_{df}$ the smallest number n such that $\langle z_n^{\mathbf{d}}, w_n^{\mathbf{d}} \rangle \neq \langle z_n^{\mathbf{d}'}, w_n^{\mathbf{d}'} \rangle$. Thus $diff(\mathbf{d}, \mathbf{d}')$ is the first point at which $\mathbf{d}$ and $\mathbf{d}'$ differ.

Also, suppose that $\mathbf{c} \in D$, and that $k \geq 0$ is a stabilization point for $\mathbf{c}$. Thus,

$$\mathbf{c} = \langle z_0^{\mathbf{c}}, w_0^{\mathbf{c}} \rangle, \langle z_1^{\mathbf{c}}, w_1^{\mathbf{c}} \rangle, \ldots, \langle z_{k-1}^{\mathbf{c}}, w_{k-1}^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \ldots$$

(If $k = 0$, then the finite sequence $\langle z_0^{\mathbf{c}}, w_0^{\mathbf{c}} \rangle, \langle z_1^{\mathbf{c}}, w_1^{\mathbf{c}} \rangle, \ldots, \langle z_{k-1}^{\mathbf{c}}, w_{k-1}^{\mathbf{c}} \rangle$ is understood to be the empty sequence.) It will be useful to define two variants of $\mathbf{c}$ as follows:

$$\mathbf{c}^{k-} =_{df} \langle z_0^{\mathbf{c}}, w_0^{\mathbf{c}} \rangle, \langle z_1^{\mathbf{c}}, w_1^{\mathbf{c}} \rangle, \ldots, \langle z_{k-1}^{\mathbf{c}}, w_{k-1}^{\mathbf{c}} \rangle, \langle -1, w_k^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \ldots$$
$$\mathbf{c}^{k+} =_{df} \langle z_0^{\mathbf{c}}, w_0^{\mathbf{c}} \rangle, \langle z_1^{\mathbf{c}}, w_1^{\mathbf{c}} \rangle, \ldots, \langle z_{k-1}^{\mathbf{c}}, w_{k-1}^{\mathbf{c}} \rangle, \langle 1, w_k^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \langle 0, w_k^{\mathbf{c}} \rangle, \ldots$$

In other words, $\mathbf{c}^{k-}$ and $\mathbf{c}^{k+}$ are exactly like $\mathbf{c}$ except that $z_k^{\mathbf{c}^{k-}} = -1$ and $z_k^{\mathbf{c}^{k+}} = 1$, while $z_k^{\mathbf{c}} = 0$. Note that $\mathbf{c}^{k-}, \mathbf{c}^{k+} \in D$.

3.1 A linear ordering $\ll$ on D

Choose any linear ordering $\prec$ of W. The ordering $\prec$ need not have anything in particular to do with the accessibility relation R. We order the marked worlds lexicographically as follows: $\langle z, w \rangle \lhd \langle z', w' \rangle$ iff either $z < z'$ or both $z = z'$ and $w \prec w'$. Now we order D lexicographically as follows: $\mathbf{d} \ll \mathbf{d}'$ iff $\mathbf{d} \neq \mathbf{d}'$ and $\langle z_n^{\mathbf{d}}, w_n^{\mathbf{d}} \rangle \lhd \langle z_n^{\mathbf{d}'}, w_n^{\mathbf{d}'} \rangle$, where $n = diff(\mathbf{d}, \mathbf{d}')$. Since $\prec$ and $<$ are linear orderings, so are $\lhd$ and $\ll$.

Note that $\mathbf{c}^{k-} \ll \mathbf{c} \ll \mathbf{c}^{k+}$, for any $\mathbf{c} \in D$. So $\ll$ has no endpoints. To see that $\ll$ is dense, choose any $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \ll \mathbf{b}$. Let $n = diff(\mathbf{a}, \mathbf{b})$ and choose any stabilization point k for $\mathbf{a}$, such that $k > n$. Clearly $\mathbf{a} \ll \mathbf{a}^{k+}$. So, for the denseness of $\ll$, it suffices to show that $\mathbf{a}^{k+} \ll \mathbf{b}$. Note that $diff(\mathbf{a}, \mathbf{a}^{k+}) = k > n = diff(\mathbf{a}, \mathbf{b})$. So $diff(\mathbf{a}^{k+}, \mathbf{b}) = n$. Now, $\langle z_n^{\mathbf{a}^{k+}}, w_n^{\mathbf{a}^{k+}} \rangle = \langle z_n^{\mathbf{a}}, w_n^{\mathbf{a}} \rangle$, since $n < k$. And $\langle z_n^{\mathbf{a}}, w_n^{\mathbf{a}} \rangle \lhd \langle z_n^{\mathbf{b}}, w_n^{\mathbf{b}} \rangle$, since $\mathbf{a} \ll \mathbf{b}$ and $n = diff(\mathbf{a}, \mathbf{b})$. So $\langle z_n^{\mathbf{a}^{k+}}, w_n^{\mathbf{a}^{k+}} \rangle \lhd \langle z_n^{\mathbf{b}}, w_n^{\mathbf{b}} \rangle$. So $\mathbf{a}^{k+} \ll \mathbf{b}$, since $n = diff(\mathbf{a}^{k+}, \mathbf{b})$.

So $\mathbb{D} = \langle D, \ll \rangle$ is a countable dense linear ordering without endpoints.

3.2 A continuous function g on D

Given any $\mathbf{d} \in D$, let

$$g(\mathbf{d}) =_{df} \langle z_0^{\mathbf{d}}, f(w_0^{\mathbf{d}}) \rangle, \langle z_1^{\mathbf{d}}, f(w_1^{\mathbf{d}}) \rangle, \langle z_2^{\mathbf{d}}, f(w_2^{\mathbf{d}}) \rangle, \ldots$$

That is, $z_i^{g(\mathbf{d})} = z_i^{\mathbf{d}}$ and $w_i^{g(\mathbf{d})} = f(w_i^{\mathbf{d}})$, for each $i \geq 0$. Since $\mathbf{d} \in D$ and f is monotonic, $g(\mathbf{d}) \in D$, i.e., $g(\mathbf{d})$ is a stable R-increasing infinite sequence of marked worlds. Our task now is to prove that g is a continuous function on D. As noted in Sect. 1, the intervals form a basis for the order topology on D. So it suffices to show that the inverse image of any interval is open.

Choose any interval $(\mathbf{a}, \mathbf{b})$, and suppose that $\mathbf{c} \in g^{-1}((\mathbf{a}, \mathbf{b}))$. It will suffice to find a stabilization point k for $\mathbf{c}$ so that $\mathbf{c} \in (\mathbf{c}^{k-}, \mathbf{c}^{k+}) \subseteq g^{-1}((\mathbf{a}, \mathbf{b}))$. Note that $g(\mathbf{c}) \in (\mathbf{a}, \mathbf{b})$. Let $n_\mathbf{a} = diff(g(\mathbf{c}), \mathbf{a})$ and $n_\mathbf{b} = diff(g(\mathbf{c}), \mathbf{b})$. Choose any stabilization point k for $\mathbf{c}$, such that $k > max(n_\mathbf{a}, n_\mathbf{b})$. Since $\mathbf{c} \in (\mathbf{c}^{k-}, \mathbf{c}^{k+})$, it suffices to show that $(\mathbf{c}^{k-}, \mathbf{c}^{k+}) \subseteq g^{-1}((\mathbf{a}, \mathbf{b}))$.

So choose $\mathbf{d} \in (\mathbf{c}^{k-}, \mathbf{c}^{k+})$. We want to show that $g(\mathbf{d}) \in (\mathbf{a}, \mathbf{b})$. Since $g(\mathbf{c}) \in (\mathbf{a}, \mathbf{b})$, we only consider the case when $g(\mathbf{d}) \neq g(\mathbf{c})$. By the definition of $\mathbf{c}^{k-}$ and $\mathbf{c}^{k+}$, the first k entries of $\mathbf{c}^{k-}$ and of $\mathbf{c}^{k+}$ are the same as the first k entries of $\mathbf{c}$, namely,

$$\langle z_0^{\mathbf{c}}, w_0^{\mathbf{c}} \rangle, \langle z_1^{\mathbf{c}}, w_1^{\mathbf{c}} \rangle, \ldots, \langle z_{k-1}^{\mathbf{c}}, w_{k-1}^{\mathbf{c}} \rangle.$$

These are also the first k entries of $\mathbf{d}$, since $\mathbf{c}^{k-} \ll \mathbf{d} \ll \mathbf{c}^{k+}$. So the first k entries of $g(\mathbf{d})$ are the same as the first k entries of $g(\mathbf{c})$, namely,

$$\langle z_0^{\mathbf{c}}, f(w_0^{\mathbf{c}}) \rangle, \langle z_1^{\mathbf{c}}, f(w_1^{\mathbf{c}}) \rangle, \ldots, \langle z_{k-1}^{\mathbf{c}}, f(w_{k-1}^{\mathbf{c}}) \rangle.$$

So $diff(g(\mathbf{c}), g(\mathbf{d})) \geq k$. Thus,

$$diff(g(\mathbf{c}), g(\mathbf{d})) \geq k > n_\mathbf{a} = diff(g(\mathbf{c}), \mathbf{a}); \text{ and}$$
$$diff(g(\mathbf{c}), g(\mathbf{d})) \geq k > n_\mathbf{b} = diff(g(\mathbf{c}), \mathbf{b}).$$

Thus,

$$diff(g(\mathbf{d}), \mathbf{a}) = n_\mathbf{a} \text{ and}$$
$$diff(g(\mathbf{d}), \mathbf{b}) = n_\mathbf{b}.$$

Also notice the following:

$\langle z_{n_\mathbf{a}}^{\mathbf{a}}, w_{n_\mathbf{a}}^{\mathbf{a}} \rangle \lhd \langle z_{n_\mathbf{a}}^{g(\mathbf{c})}, w_{n_\mathbf{a}}^{g(\mathbf{c})} \rangle$, since $\mathbf{a} \ll g(\mathbf{c})$ and $n_\mathbf{a} = diff(g(\mathbf{c}), \mathbf{a})$; and
$\langle z_{n_\mathbf{a}}^{g(\mathbf{c})}, w_{n_\mathbf{a}}^{g(\mathbf{c})} \rangle = \langle z_{n_\mathbf{a}}^{g(\mathbf{d})}, w_{n_\mathbf{a}}^{g(\mathbf{d})} \rangle$, since $n_\mathbf{a} < diff(g(\mathbf{c}), g(\mathbf{d}))$.
So, $\langle z_{n_\mathbf{a}}^{\mathbf{a}}, w_{n_\mathbf{a}}^{\mathbf{a}} \rangle \lhd \langle z_{n_\mathbf{a}}^{g(\mathbf{d})}, w_{n_\mathbf{a}}^{g(\mathbf{d})} \rangle$.
So, $\mathbf{a} \ll g(\mathbf{d})$, since $n_\mathbf{a} = diff(g(\mathbf{d}), \mathbf{a})$.

Similarly,

$\langle z_{n_\mathbf{b}}^{g(\mathbf{c})}, w_{n_\mathbf{b}}^{g(\mathbf{c})} \rangle \lhd \langle z_{n_\mathbf{b}}^{\mathbf{b}}, w_{n_\mathbf{b}}^{\mathbf{b}} \rangle$, since $g(\mathbf{c}) \ll \mathbf{b}$ and $n_\mathbf{b} = diff(g(\mathbf{c}), \mathbf{b})$; and
$\langle z_{n_\mathbf{b}}^{g(\mathbf{c})}, w_{n_\mathbf{b}}^{g(\mathbf{c})} \rangle = \langle z_{n_\mathbf{b}}^{g(\mathbf{d})}, w_{n_\mathbf{b}}^{g(\mathbf{d})} \rangle$, since $n_\mathbf{b} < diff(g(\mathbf{c}), g(\mathbf{d}))$.

So, $\langle z_{n_\mathbf{b}}^{g(\mathbf{d})}, w_{n_\mathbf{b}}^{g(\mathbf{d})} \rangle \lhd \langle z_{n_\mathbf{b}}^{\mathbf{b}}, w_{n_\mathbf{b}}^{\mathbf{b}} \rangle$.

So $g(\mathbf{d}) \ll \mathbf{b}$, since $n_\mathbf{b} = \mathit{diff}(g(\mathbf{d}), \mathbf{b})$.

So $\mathbf{a} \ll g(\mathbf{d}) \ll \mathbf{b}$. So $g(\mathbf{d}) \in (\mathbf{a}, \mathbf{b})$, as desired.

3.3 A simulation h of K in $\langle D, g \rangle$

For each $\mathbf{d} \in D$, define $h(\mathbf{d}) = w_k^\mathbf{d}$, where k is the smallest (or any other) stabilization point for $\mathbf{d}$. To see that h maps D *onto* W, choose any $w \in W$, and consider the following stable R-increasing infinite sequence of marked worlds:

$$\mathbf{d} = \langle 0, w \rangle, \langle 0, w \rangle, \langle 0, w \rangle, \langle 0, w \rangle, \dots$$

Clearly, $h(\mathbf{d}) = w$. Our final task is to prove that h is a simulation of K in $\langle D, g \rangle$. So we want to show,

1. for each $w \in W$, the set $O_w = \{\mathbf{d} \in D : wRh(\mathbf{d})\}$ is open in D;
2. for each open set $O \subseteq D$, the image of O under h is closed under the accessibility relation R; and
3. for each $\mathbf{d} \in D$, we have $f(h(\mathbf{d})) = h(g(\mathbf{d}))$.

Re 1. Choose $w \in W$, and suppose that $\mathbf{c} \in O_w = \{\mathbf{d} \in D : wRh(\mathbf{d})\}$. It will suffice to find a stabilization point k for $\mathbf{c}$ so that $\mathbf{c} \in (\mathbf{c}^{k-}, \mathbf{c}^{k+}) \subseteq O_w$. Choose any stabilization point l for $\mathbf{c}$, and let $k = l + 1$. Note that $h(\mathbf{c}) = w_{k-1}^\mathbf{c}$. Since $\mathbf{c} \in (\mathbf{c}^{k-}, \mathbf{c}^{k+})$, it suffices to show that $(\mathbf{c}^{k-}, \mathbf{c}^{k+}) \subseteq O_w$.

So suppose that $\mathbf{d} \in (\mathbf{c}^{k-}, \mathbf{c}^{k+})$. The first k entries of $\mathbf{c}^{k-}$ and $\mathbf{c}^{k+}$ are the same as the first k entries of $\mathbf{c}$, namely, $\langle z_0^\mathbf{c}, w_0^\mathbf{c} \rangle, \langle z_1^\mathbf{c}, w_1^\mathbf{c} \rangle, \dots, \langle z_{k-1}^\mathbf{c}, w_{k-1}^\mathbf{c} \rangle$.

These are also the first k entries of $\mathbf{d}$, since $\mathbf{c}^{k-} \ll \mathbf{d} \ll \mathbf{c}^{k+}$. So $w_{k-1}^\mathbf{d} = w_{k-1}^\mathbf{c} = h(\mathbf{c})$. Also, since $\mathbf{d}$ is R-increasing, $w_{k-1}^\mathbf{d} R w_m^\mathbf{d}$ for every $m \geq k$. So $w_{k-1}^\mathbf{d} Rh(\mathbf{d})$. So $h(\mathbf{c})Rh(\mathbf{d})$. Also, $wRh(\mathbf{c})$, since $\mathbf{c} \in O_w$. So $wRh(\mathbf{d})$, since R is transitive. So $\mathbf{d} \in O_w$, as desired.

Re 2. It suffices to show this in the special case when O is an interval, say $(\mathbf{a}, \mathbf{b})$. So we want to show that the image of $(\mathbf{a}, \mathbf{b})$ is closed under the accessibility relation R. So suppose that $w = h(\mathbf{c})$ for some $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$, and that wRw'. We want to find a $\mathbf{d} \in (\mathbf{a}, \mathbf{b})$ such that $w' = h(\mathbf{d})$. Let $n_\mathbf{a} = \mathit{diff}(\mathbf{a}, \mathbf{c})$ and $n_\mathbf{b} = \mathit{diff}(\mathbf{b}, \mathbf{c})$. Choose any stabilization point k for $\mathbf{a}$, such that $k > \mathit{max}(n_\mathbf{a}, n_\mathbf{b})$. Thus,

$$\mathbf{c} = \langle z_0^\mathbf{c}, w_0^\mathbf{c} \rangle, \langle z_1^\mathbf{c}, w_1^\mathbf{c} \rangle, \dots, \langle z_{k-1}^\mathbf{c}, w_{k-1}^\mathbf{c} \rangle, \langle 0, w \rangle, \langle 0, w \rangle, \langle 0, w \rangle, \dots$$

Define $\mathbf{d}$ as follows:

$$\mathbf{d} =_{\mathrm{df}} \langle z_0^\mathbf{c}, w_0^\mathbf{c} \rangle, \langle z_1^\mathbf{c}, w_1^\mathbf{c} \rangle, \dots, \langle z_{k-1}^\mathbf{c}, w_{k-1}^\mathbf{c} \rangle, \langle 1, w' \rangle, \langle 0, w' \rangle, \langle 0, w' \rangle, \dots$$

Note that $\mathbf{d} \in D$ and $h(\mathbf{d}) = w'$. It remains for us to show that $\mathbf{d} \in (\mathbf{a}, \mathbf{b})$, i.e., that $\mathbf{a} \ll \mathbf{d} \ll \mathbf{b}$. Note that

$$diff(\mathbf{d}, \mathbf{c}) = k > n_{\mathbf{a}} = diff(\mathbf{a}, \mathbf{c}); \text{ and}$$
$$diff(\mathbf{d}, \mathbf{c}) = k > n_{\mathbf{b}} = diff(\mathbf{b}, \mathbf{c}).$$

Thus,

$$diff(\mathbf{a}, \mathbf{d}) = n_{\mathbf{a}} \quad \text{and} \quad diff(\mathbf{b}, \mathbf{d}) = n_{\mathbf{b}}.$$

Thus, by reasoning similar to the reasoning at the end of Sect. 3.2,

$$\langle z^{\mathbf{a}}_{n_{\mathbf{a}}}, w^{\mathbf{a}}_{n_{\mathbf{a}}} \rangle \lhd \langle z^{\mathbf{c}}_{n_{\mathbf{a}}}, w^{\mathbf{c}}_{n_{\mathbf{a}}} \rangle = \langle z^{\mathbf{d}}_{n_{\mathbf{a}}}, w^{\mathbf{d}}_{n_{\mathbf{a}}} \rangle; \text{ and}$$
$$\langle z^{\mathbf{d}}_{n_{\mathbf{b}}}, w^{\mathbf{d}}_{n_{\mathbf{b}}} \rangle = \langle z^{\mathbf{c}}_{n_{\mathbf{b}}}, w^{\mathbf{c}}_{n_{\mathbf{b}}} \rangle \lhd \langle z^{\mathbf{b}}_{n_{\mathbf{b}}}, w^{\mathbf{b}}_{n_{\mathbf{b}}} \rangle.$$

So $\mathbf{a} \ll \mathbf{d} \ll \mathbf{b}$, as desired.

Re 3. Choose $\mathbf{d} \in D$. Choose any stabilization point k for $\mathbf{d}$. Note that k is also a stabilization point for $g(\mathbf{d})$. Thus, $f(h(\mathbf{d})) = f(w^{\mathbf{d}}_k) = w^{g(\mathbf{d})}_k = h(g(\mathbf{d}))$.

As promised, we have defined a countable dense linear ordering $\mathbb{D} = \langle D, \ll \rangle$ without endpoints (Sect. 3.1), a continuous function g on D (Sect. 3.2), and a simulation h of K in $\langle D, g \rangle$ (Sect. 3.3). This completes our proof of Lemma 2.5, and therefore of our main result, Theorem 1.2.

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