THE TRUTH IS SOMETIMES SIMPLE

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Note: The following version of this paper does not contain the proofs of the stated theorems. A longer version, complete with proofs, is forthcoming.

§1. Introduction. In "The truth is never simple" (1986) and its addendum (1988), Burgess conducts a breathtakingly comprehensive survey of the complexity of the set of truths which arise when you add a truth predicate to arithmetic, and interpret that predicate according to the fixed point semantics or the revision-theoretic semantics for languages expressing their own truth concepts. Burgess considers various sets that can be said to represent truth in this context, and shows that their complexity ranges from Π^1_1 or Σ^1_1 to Π^1_2 or Σ^1_2. Thus, enriching arithmetic with a truth predicate increases its complexity, which is otherwise only Δ^1_1.

In his survey, Burgess assumes that we have fixed some Gödel numbering (or some other kind of coding of sentences as numbers) effectively identifying a sentence with its Gödel number. Gödel numbering is useful because it allows the object language to mimic talk about sentences, even when our model's domain contains only numbers. And if we want a semantics for an object language expressing its own *truth* concept, it seems a minimal requirement that the object language be able to refer to or at least quantify over sentences.

Gödel numbering not only allows us to satisfy this minimal requirement; it also allows the object language to say when one sentence is the conjunction of two sentences, when one sentence is the negation of another, when one sentence is a substitution instance of another, and so on. Thus Burgess's results do not show that *truth alone* accounts for the increase in arithmetic's complexity from Δ^1_1 to Π^1_1, Σ^1_1, Π^1_2 or Σ^1_2. Rather, it is the *interplay* between truth and these rich syntactic resources that is responsible for the increase in complexity. In the current paper, we investigate, in various contexts, the complexity of *truth* rather than the complexity of *truth + rich syntactic resources*. And we discover that the truth, even in arithmetic, is sometimes simple.

Gupta (1982) makes a similar point, not about the complexity of truth, but about the ability of a languages containing its own truth predicate to express the liar's paradox and other pathologies. Tarski (1936) argues that every "semantically closed" language satisfying the laws of classical logic and the Tarski biconditionals is inconsistent, where a "semantically closed" language is one that has names for its own expressions and expresses its own truth concept (or other semantic concepts) (p. 165) . Gupta constructs a perfectly consistent semantically closed language that obeys the laws of logic and in which the Tarski biconditionals are true. (Gupta's language only has names for its sentences, but he points out that we could harmlessly add names for the other expressions.) Gupta guarantees the combination of consistency, satisfaction of the T-biconditionals, and semantic closure by restricting the syntactic resources of the language: Tarski simply assumes that richer syntactic resources are available when informally constructing the liar's paradox. And the syntactic resources assumed by Tarski are precisely the kind of resources that Gödel numbering makes available. Gupta's simple language is a special case of the languages we consider in §4, below.

§2. Basic definitions. We assume familiarity with the fixed point and revision theoretic semantics, so we skimp on the motivation for the basic definitions. Throughout, we assume that L is a first-order language with a distinguished predicate T and a quote name '*A*' for each sentence *A* of L. The T-*free fragment* of L is the fragment of L that does not contain any occurrences of T, except in the scope of quotation marks. Following Gupta and Belnap (1993) we let S = {*A*: *A* is a sentence of L}. We begin with the fixed point semantics.

A *classical model for* L is an ordered pair M = ⟨D, I⟩, where D, the *domain of discourse*, is a nonempty set; and where I is a function such that I(*c*) ∈ D, for each constant *c* of *L*; $I(f):D^n \rightarrow D$ for each n-ary function symbol *f* of *L*; and $I(R):D^n \rightarrow \{\mathbf{t}, \mathbf{f}\}$ for each n-ary relation symbol *R* of *L*. Truth in a classical model is defined as usual. A *ground model* for *L* is a classical model M = ⟨D, I⟩ for the T-free fragment of L, such that I('*A*') = *A* ∈ D for each sentence *A* of L.

Given a ground model M = ⟨D, I⟩ for L, an *hypothesis* is a function $h:D \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ and a *classical* hypothesis, a function $h:D \rightarrow \{\mathbf{t}, \mathbf{f}\}$. Let $h^+ = \{d \in D: h(d) = \mathbf{t}\}$ and $h^- = \{d \in D: h(d)$

$= \mathbf{f}\}$. Let M + h be the model $M' = \langle D, I' \rangle$ for all of L, where I′ and I agree on the T-free fragment of L and where $I'(T) = h$. In a *three-valued* model M + h, we assign to each sentence a truth-value by using, for example, the *strong Kleene scheme*, SK. According to SK,

$\neg\mathbf{t} = \mathbf{f}$ and $\neg\mathbf{f} = \mathbf{t}$ and $\neg\mathbf{n} = \mathbf{n}$;

$(\mathbf{t} \,\&\, \mathbf{t}) = \mathbf{t}$, and $(\mathbf{f} \,\&\, \mathbf{x}) = \mathbf{f}$ for any $\mathbf{x} \in \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, and $(\mathbf{t} \,\&\, \mathbf{n}) = (\mathbf{n} \,\&\, \mathbf{t}) = (\mathbf{n} \,\&\, \mathbf{n}) = \mathbf{n}$;

$(\mathbf{t} \vee \mathbf{x}) = \mathbf{t}$ for any $\mathbf{x} \in \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, and $(\mathbf{f} \vee \mathbf{f}) = \mathbf{f}$, and $(\mathbf{f} \vee \mathbf{n}) = (\mathbf{n} \vee \mathbf{f}) = (\mathbf{n} \vee \mathbf{n}) = \mathbf{n}$;

and the quantifiers are treated analogous to & and ∨. SK delivers, for each sentence *A* of L a value $Val_{M + h, SK}(A) \in \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$. The value $Val_{M + h, WK}(A)$ using the *weak Kleene scheme*, WK, is defined similarly except that $(\mathbf{f} \,\&\, \mathbf{n}) = (\mathbf{n} \,\&\, \mathbf{f}) = \mathbf{n} = (\mathbf{n} \vee \mathbf{t}) = (\mathbf{t} \vee \mathbf{n})$ and analagously for the quantifiers. The value $Val_{M + h, SV}(A)$ using the *supervaluation scheme*, SV, is defined as follows. First say that $h \leq h'$ iff if $h(d) = \mathbf{t}$ [$\mathbf{f}$] then $h'(d) = \mathbf{t}$ [$\mathbf{f}$] for every $d \in D$. When M + h is classical, we denote the truth value of a sentence *A* by $Val_{M + h, CL}(A)$: CL is the *classical scheme*. Finally,

$$Val_{M + h, SV}(A) =_{df} \begin{cases} \mathbf{t}\ [\mathbf{f}], \text{ if } Val_{M + h', CL}(A) = \mathbf{t}\ [\mathbf{f}] \text{ for every classical } h' \geq h. \\ \mathbf{n}, \text{ otherwise.} \end{cases}$$

For X = CL, WK, SK, or SV, define the *jump operator* X_M on the set of hypotheses as follows, restricting this definition to classical hypotheses for X = CL:

$$X_M(h)(A) = Val_{M + h, X}(A), \text{ if } A \in S$$
$$X_M(h)(d) = \mathbf{f} \text{ if } d \in D - S.$$

The models M + h in which T plausibly expresses *truth* are those in which h is a *fixed point* of X_M, i.e. when $X_M(h) = h$. Kripke (1975) proves that WK_M, [SK_M, SV_M] has a fixed point, for every ground model M. In fact, Kripke's results are stronger. Say that a function X on hypotheses is *monotone* iff, for all hypotheses h and h′, if $h \leq h'$ then $X(h) \leq X(h')$. WK_M, SK_M, and SV_M are monotone, for every ground model M. Each monotone function X not only has a fixed point, but a *least* fixed point, lfp(X). Say that h and h′ are *compatible* iff $h \leq h''$ and $h' \leq h''$ for some hypothesis h″, and that h is *intrinsic* iff h is compatible with every fixed point. Each monotone function X not only has a *least* fixed point, but a *greatest* intrinsic fixed point, gifp(X), which is not in general identical to lfp(X).

Given a ground model M and an evaluation scheme X, a sentence *A* is *everywhere* (*somewhere*) (*nowhere*) true [false] iff h(*A*) = **t** [**f**] for every (some) (no) fixed point h, and is *intrinsically* true [false] iff h(*A*) = **t** [**f**] at some intrinsic fixed point h. With Burgess (1986) we note that *A* is everywhere true [false] iff $A \in$ lfp$(X_M)^+$ [lfp$(X_M)^-$], and is intrinsically true [false] iff $A \in$ gifp$(X_M)^+$ [gifp$(X_M)^-$]. Define sfp$(X_M)^+$ [sfp$(X_M)^-$] to be the set of sentences that are somewhere true [false].

Now we move on to the revision-theoretic semantics. Fix a ground model M = ⟨D, I⟩. Given any function X on hypotheses an X-*sequence*, or a *revision sequence for* X, is an ordinal-length sequence ***S*** of hypotheses such that $\boldsymbol{S}_{\alpha+1} = X(\boldsymbol{S}_\alpha)$ for every ordinal α; and such that for every limit ordinal λ, every truth value **x** and every $d \in D$, $\boldsymbol{S}_\lambda(d) = \mathbf{x}$ if there is a $\beta < \lambda$ such that $\boldsymbol{S}_\alpha(d) = \mathbf{x}$ for every ordinal α between β and λ. This second clause is the *limit* rule for X-sequences. Note that if ***S*** is an X-sequence then X is defined on $\boldsymbol{S}_\alpha$ for every ordinal α; so, if ***S*** is a CL_M-sequence then $\boldsymbol{S}_\alpha$ is classical for every ordinal α. A sentence *A* is *stably* **x** in ***S*** iff there is a β such that $\boldsymbol{S}_\alpha(d) = \mathbf{x}$ for every ordinal $\alpha > \beta$. If *A* is not stably anything in ***S*** then *A* is *unstable* in ***S***. ***S*** *culminates in* h iff there is a β such that $\boldsymbol{S}_\alpha = h$ for every $\alpha \geq \beta$. Note that if X = WK, SK or SV, then there is a unique X_M-sequence ***S*** such that $\boldsymbol{S}_0(d) = \mathbf{n}$ for every $d \in D$. Furthermore, that X_M-sequence culminates in lfp(X_M). For revision theory, we are primarily interested in the case where $X = CL_M$. Following Burgess, we distinguish one particular sequence, the *negative sequence* or the N-*sequence* for M, which is the CL_M-sequence ***S*** for which (1) $\boldsymbol{S}_0(d) = \mathbf{f}$ for every $d \in D$, and (2) for every limit ordinal λ and every $d \in D$, $\boldsymbol{S}_\lambda(d) = \mathbf{t}$ if *and only if* there is a $\beta < \lambda$ such that $\boldsymbol{S}_\alpha(d) = \mathbf{t}$ for every ordinal α between β and λ. In the N-sequence, every sentence is assessed as false to begin with, and a sentence is assessed as false at the limit, unless if it forced to be assessed as true.

Following Burgess, we define, for ground models M, the sets $\Box T_M$, $\Box F_M$, $\Box U_M$, $\Diamond T_M$, $\Diamond F_M$, $\Diamond U_M$ where the appearance of $\Box$ (of $\Diamond$) and of T [of F] [of U] indicates the set of sentences that are stably true [stably false] [unstable] in every (some) CL_M-sequence. And we define $\lim_M N$ as the set of sentences stably true in the N-sequence for M.

§3. Burgess's results. For this section, let L be the language of arithmetic enriched with a predicate T and a quote name '*A*' for each sentence *A* of L. Fix some Gödel numbering establishing a bijection between sentences of L and natural numbers, so that Gn(*A*) is the Gödel number of *A*. Let $N = \langle \omega, I \rangle$ be the standard model of arithmetic, understood as a ground model for L when we identify a sentence with its Gödel number, so that I('*A*') = Gn(*A*).

Remark. Burgess works with a language L that has a function symbol for every primitive recursive function, but this does not affect the basic results. Also, given that I('*A*') = Gn(*A*), the quote names are redundant since the sentence *A* is named by $\mathrm{Gn}(A)^{\mathrm{th}}$ numeral. We include the inessential quote names to make the language like the languages used in §2, but their presence or absence has no effect on complexity.

Remark. Another approach would be to let $N = \langle S, I \rangle$ where I is defined so that N is isomorphic to the standard model of arithmetic where the isomorphism maps each sentence *A* to Gn(*A*).

Burgess proves the following:

	The set	is
(1)	$\mathrm{lfp}(SK_N)^+$	complete Π^1_1.
(2)	$\mathrm{sfp}(SK_N)^+$	complete Σ^1_1.
(3)	$\mathrm{sfp}(SK_N)^+ \cap \mathrm{sfp}(SK_N)^-$	complete Σ^1_1.
(4)	$\omega - (\mathrm{sfp}(SK_N)^+ \cup \mathrm{sfp}(SK_N)^-)$	complete Π^1_1.
(5)	$\mathrm{sfp}(SK_N)^+ - \mathrm{sfp}(SK_N)^-$	complete difference of two Σ^1_1.
(6)	$\mathrm{gifp}(SK_N)^+$	complete Σ^1_1-in-a-Π^1_1-parameter.
(7)	$\mathrm{lfp}(SV_N)^+$	complete Π^1_1.

(8)	$\mathrm{sfp}(\mathrm{SV_N})^+$	complete Σ^1_2.
(9)	$\mathrm{gifp}(\mathrm{SK_N})^+$	Δ^1_2-in-a-Π^1_2-parameter.
(10)	$\Box\mathrm{T_N}$	complete Π^1_2.
(11)	$\Diamond\mathrm{U_N}$	complete Σ^1_2.
(12)	$\Diamond\mathrm{T_N}$	at least Σ^1_1 and at most Σ^1_2.
(13)	$\Box\mathrm{U_N}$	at least Π^1_1 and at most Π^1_2.
(14)	$\mathrm{lim_N N}$	Δ^1_2.

Furthermore, every Π^1_1 or Σ^1_1 set is reducible to $\mathrm{lim_N N}$, and every set Σ^1_1-in-a-Π^1_2-parameter or Π^1_1-in-a-Π^1_2-parameter is reducible to $\mathrm{gifp}(\mathrm{SK_N})^+$.

Remark. Burgess attributes (1)-(5), (7) and the upper bound in (6) to Kripke – (1) and (7) are stated without proof by Kripke (1975).

§4. The truth is sometimes decidable. For this section, let L be a first-order language with a finite list, c_1, ..., c_n, of constants other than quote names; with no function symbols; and with only two relation symbols: the identity sign, =, and the truth predicate, T. We also assume that L has a quote name '*A*' for each sentence *A* of L.

Remark. If the list of constants in nonempty, then there is a ground model expressing the liar's paradox: consider any ground model $\mathrm{M} = \langle \mathrm{D}, \mathrm{I} \rangle$ such that $\mathrm{I}(c_1) = \neg \mathrm{T}c_1$.

Theorem 4.1. If M is any ground model for L, then the following sets are all decidable: $\mathrm{lfp}(\mathrm{WK_M})^+$, $\mathrm{lfp}(\mathrm{SK_M})^+$, $\mathrm{lfp}(\mathrm{SV_M})^+$, $\mathrm{gifp}(\mathrm{WK_M})^+$, $\mathrm{gifp}(\mathrm{SK_M})^+$, $\mathrm{gifp}(\mathrm{SV_M})^+$, $\mathrm{sfp}(\mathrm{WK_M})^+$, $\mathrm{sfp}(\mathrm{SK_M})^+$, $\mathrm{sfp}(\mathrm{SV_M})^+$, $\Box\mathrm{T_M}$, $\Diamond\mathrm{T_M}$, $\Box\mathrm{U_M}$, $\Diamond\mathrm{U_M}$, $\mathrm{lim_M N}$.

Consider the special case where L has no constants other than quote names. Let M be any ground model for L. Then, in addition to the above theorem, we have the following.

Theorem 4.2. (i) $\Box\mathrm{U_M} = \Diamond\mathrm{U_M} = \mathrm{sfp}(\mathrm{WK_M})^+ \cap \mathrm{sfp}(\mathrm{WK_M})^- = \mathrm{sfp}(\mathrm{SK_M})^+ \cap \mathrm{sfp}(\mathrm{SK_M})^- = \varnothing$.

(ii) $\mathrm{sfp}(\mathrm{WK_M})^+ \cup \mathrm{sfp}(\mathrm{WK_M})^- = \mathrm{sfp}(\mathrm{SK_M})^+ \cup \mathrm{sfp}(\mathrm{SK_M})^- = \mathrm{S}$.

Theorem 4.3. $\mathrm{sfp}(\mathrm{WK_M})^+ = \mathrm{sfp}(\mathrm{SK_M})^+ = \mathrm{sfp}(\mathrm{SV_M})^+ = \mathrm{lfp}(\mathrm{SV_M})^+ = \mathrm{gifp}(\mathrm{WK_M})^+ = \mathrm{gifp}(\mathrm{SK_M})^+ = \mathrm{gifp}(\mathrm{SV_M})^+ = \Box\mathrm{T_M} = \Diamond\mathrm{T_M}$.

Remark. When L has no constants other than quote names, CL_M has exactly one fixed point, say h_0, and h_0 = gifp(WK_M) = gifp(SK_M) = lfp(SV_M) = gifp(SV_M). Furthermore every CL_M-revision sequence converges to h_0. Let $\mathfrak{L}$ be the classical interpreted language $\langle L, M + h_0\rangle$, where M is the ground model whose domain is the set of sentences of L. $\mathfrak{L}$ is a simple version of Gupta's (1982) classical, semantically closed, language expressing its own truth concept and verifying all the T-biconditionals.

§5. Arithmetic plus truth is sometimes as simple as arithmetic. Suppose that we want to study the effects of adding the truth predicate to arithmetic, without assuming the rich syntactic resources provided by Gödel numbering. One approach (§5.1) is to consider a language that talks about both numbers and sentences, but that keeps the two distinct: so we focus on a single model whose domain D is $\omega \cup S$. On the first approach, adding truth to the language of arithmetic introduces no complexity: theresulting sets of sentences plausibly representing truth are all reducible to arithmetic. Another approach (§5.2) is to consider models that, like the Burgess model in §2, above, essentially identify each sentence with some number, but not through Gödel numbering: we consider models M = $\langle D, I\rangle$ which are isomorphic to the standard model of arithmetic, and for which $S \subseteq D$.

§5.1. Numbers and sentences. For this subsection, let L be a first order language with identity with a constant o (zero), a binary relation symbol s (successor), two trinary relation symbols + (addition) and × (multiplication), a unary relation symbol Num (number), a truth predicate T, and quote names for the sentences of L. Our reasons for treating s, + and × as relations rather than functions will become apparent. Let M = $\langle D, I\rangle$ be the following classical ground model for which,

D	=	$\omega \cup S$
I(Num)(d) = **t**	iff	$d \in \omega$
I(s)(d, d′) = **t**	iff	$d \in \omega$ and $d' \in \omega$ and $d' = d + 1$
I(+)(d, d′, d″) = **t**	iff	$d \in \omega$ and $d' \in \omega$ and $d'' \in \omega$ and $d'' = d + d'$

$I(\times)(d, d', d'') = \mathbf{t}$ iff $d \in \omega$ and $d' \in \omega$ and $d'' \in \omega$ and $d'' = d \times d'$

$I('A') = A$, for each $A \in S$.

Theorem 5.1.1. $lfp(WK_M)^+$ and $lfp(SK_M)^+$ are reducible to true arithmetic.

Theorem 5.1.2. (i) $\Box U_M = \Diamond U_M = sfp(WK_M)^+ \cap sfp(WK_M)^- = sfp(SK_M)^+ \cap sfp(SK_M)^- = \varnothing$.

(ii) $sfp(WK_M)^+ \cup sfp(WK_M)^- = sfp(SK_M)^+ \cup sfp(SK_M)^- = S$.

Theorem 5.1.3. $sfp(WK_M)^+ = sfp(SK_M)^+ = sfp(SV_M)^+ = lfp(SV_M)^+ = gifp(WK_M)^+ = gifp(SK_M)^+ = gifp(SV_M)^+ = \Box T_M = \Diamond T_M = lim_M N$ is reducible to arithmetic.

§5.2. Sentences as numbers. For this subsection, let L be the first order language of arithmetic enlarged with a truth predicate T and quote names for the sentences of L. The symbols s, + and × can be treated as function symbols or relation symbols. A ground model M = ⟨D, I⟩ is *standard* iff M restricted the the language of arithmetic (i.e. the fragment of L without T and without quote names) is isomorphic to the standard model of arithmetic.

Theorem 5.2.1. There is a standard model M and a classical hypothesis h, such that h is a fixed point of CL_M and such that $h^+ = \{A: h(A) = \mathbf{t}\} = \{A: Val_{M + h, CL}(A) = \mathbf{t}\}$ is reducible to true arithmetic.

Remark. Thus there is a classical interpreted language $\mathfrak{L} = \langle L, M + h\rangle$ whose T- and quote-name-free fragment is isomorphic to the standard model of arithmetic, in which T plausibly means truth, and whose set of truths is reducible to true arithmetic.

Question. What is the least complexity of $lfp(WK_M)^+$ as M ranges over standard ground models? How about $lfp(SK_M)^+$, $lfp(SV_M)^+$, $gifp(WK_M)^+$, $gifp(SK_M)^+$, $gifp(SV_M)^+$, $sfp(WK_M)^+$, $sfp(SK_M)^+$, $sfp(SV_M)^+$, $\Box T_M$, $\Diamond T_M$, $\Box U_M$ and $\Diamond U_M$?

§6. Logics of truth. Fix some language L with a distinguished predicate T and quote names for all the sentences of L, and fix some valuation scheme X. The following *logics of truth* suggest themselves:

Every fixed point logic, efp(X) $\bigcap_{M \text{ is a ground model}} \bigcap_{h \text{ is a fixed point of } X} h^+$

Least fixed point logic, lfp(X) $\bigcap_{M \text{ is a ground model}} lfp(X_M)^+$

Intrinsic fixed point logic, ifp(X)	$\cap_{\text{M is a ground model}} \cup_{\text{h is an intrinsic fixed point of X}} h^+$
Greatest intrinsic fixed point logic, gifp(X)	$\cap_{\text{M is a ground model}}\ \text{gifp}(X_M)^+$
Some fixed point logic, sfp(X)	$\cap_{\text{M is a ground model}}\ \text{sfp}(X_M)^+$
Universal revision logic, $\Box T$	$\cap_{\text{M is a ground model}}\ \Box T_M$
Existential revision logic, $\Diamond T$	$\cap_{\text{M is a ground model}}\ \Diamond T_M$

It is immediately clear that efp(X) = lfp(X) and that ifp(X) = gifp(X).

Theorem 6.1. lfp(WK), lfp(SK) and lfp(SV) are recursively enumerable.

Remark. The complexity of the remaining sets is open.

Remark. $\Box T$ is the set $\mathbf{V}_L^*$ of P. Kremer (1993). We caution *against* inferring from Burgess's result that $\Box T_N$ is complete Π_2^1 any conclusion that $\Box T$ is complete Π_2^1. Consider the analogue in the fixed point semantics: $\text{lfp}(SK_N)^+$ is complete Π_1^1, but lfp(SK) is recursively enumerable. We can see just how much work Gödel numbering is doing in such a context. Welch 2001 claims to have solved the complexity of $\Box T$ (see p. 351, Remark 5), but he assumes Gödel numbering. The problem remains open.

Remark. We also caution against inferring from Burgess's result that $\Box T_N$ is complete Π_2^1 any conclusion that Gupta and Belnap's logic S* is complete Π_2^1 (as in Welch 2001, p. 348). S* is a general revision-theoretic logic of circular concepts, and $\Box T$ is the result of applying that logic to truth. Again, if we defined a fixed-point analogue of S* using the least fixed point and the Strong Kleene Scheme the resulting logic of circular definitions would be recursively enumerable despite the fact that $\text{lfp}(SK_N)^+$ is complete Π_1^1. As a matter of fact, S* *is* complete Π_2^1 (Kremer (1993) and Antonelli (1993)), but that does not mean that Burgess's results could have been appealed to for this result.

REFERENCES

A. Antonelli 1993, "The complexity of revision", *Notre Dame Journal of Formal Logic* 34.

N.D. Belnap 1982, "Gupta's rule of revision theory of truth", *Journal of Philosophical Logic* 11, 103-116.

J. Burgess 1986, "The truth is never simple", *Journal of Symbolic Logic* 51, 663-681.

J. Burgess 1988, "Addendum to 'The truth is never simple'", *Journal of Symbolic Logic* 53, 390-392.

A. Gupta 1982, "Truth and paradox", *Journal of Philosophical Logic* 11, 1-60.

A. Gupta and N. Belnap 1993, *The Revision Theory of Truth,* MIT Press.

H. Herzberger 1982, "Notes on naive semantics", *Journal of Philosophical Logic* 11, 61-102.

M. Kremer 1988, "Kripke and the logic of truth", *Journal of Philosophical Logic* 17, 225-278.

P. Kremer 1993, "The Gupta-Belnap systems S^* and $S^\#$ are not axiomatisable", *Notre Dame Journal of Formal Logic* 34, 583-596.

P. Kremer 2002, "Comparing fixed point and revision theories of truth", manuscript.

S. Kripke 1975, "Outline of a theory of truth", *Journal of Philosophy* 72, 690-716.

R.L. Martin and P.W. Woodruff 1975, "On representing 'true-in-L' in L", *Philosophia* 5, 217-221.

A. Tarski 1936, "Der Wahrheitsbegriff in den formalisierten Sprachen", *Studia Philosophica* 1, 261-405. (Reprint dated 1935.) Translated by J.H. Woodger as "The concept of truth in formalized languages", in A. Tarski 1956, *Logic, Semantics, Metamathematics* (J.H. Woodger, trans.), OUP, 152-278. Page references are to this translation.

P.D. Welch 2001, "On Gupta-Belnap revision theories of truth, Kripkean fixed points, and the next stable set", *Bulletin of Symbolic Logic* 7, 345-360.