A PROOF OF GAMMA

Saul A. Kripke

ABSTRACT. This paper is dedicated to the memory of Mike Dunn. His untimely death is a loss not only to logic, computer science, and philosophy, but to all of us who knew and loved him. The paper gives an argument for closure under γ in standard systems of relevance logic (first proved by Meyer and Dunn [3]). For definiteness, I chose the example of **R**. The proof also applies to **E** and to the quantified systems **RQ** and **EQ**. The argument uses semantic tableaux (with one exceptional rule not satisfying the subformula property). It avoids the previous arguments' use of cutting down inconsistent sets of formulas to consistent sets. Like all tableau arguments, it extends partial valuations to total valuations.

Keywords. Completeness, Partial valuation, Relevance logic, Rule γ, Semantic tableau

This note gives a new proof of the closure of such systems as **R**, **E**, **RQ**, and **EQ** under Ackermann's rule γ, based on the idea of a semantic tableau.* The usual proofs of γ, beginning with Meyer and Dunn [3], all "cut down" an "inconsistent valuation" to a "consistent" one. The present proof proceeds dually: no inconsistent valuation is used, but rather a partial valuation is extended to a total valuation, as is usual with tableau completeness proofs. The proof is in fact very similar in flavor to the usual completeness proofs of tableau procedures.

For convenience, we fix our attention on **R**. We assume a usual axiomatization of **R**, with *modus ponens* for the relevant conditional and adjunction as the only rules. The system obtained by adjoining γ as an additional rule is called $\mathbf{R}_\gamma$.

We assume the reader is thoroughly familiar with semantic tableaux, originally introduced by Beth [1]. However, we will informally sketch the idea of a tableau construction in the present context. Following Smullyan [4], we use signed formulae: ordered pairs $\langle A,T\rangle$ and $\langle A,F\rangle$, where A is a formula of **R**, representing in Beth's terminology that A appears on the left or the right, respectively. A tableau is then simply a set S of signed formulae. A rule extends a tableau S to one (or two) immediate descendants, S' (S' and S'') such that $S \subseteq S'$ (and $S \subseteq S''$).

The rules for conjunction and negation are usual:

$\wedge T$ Set $S' = S \cup \{\langle A,T\rangle, \langle B,T\rangle\}$, where $\langle A\wedge B,T\rangle \in S$.

$\wedge F$ Set $S' = S \cup \{\langle A,F\rangle\}$ and $S'' = S \cup \{\langle B,F\rangle\}$, where $\langle A\wedge B,F\rangle \in S$.

$\sim T$ Set $S' = S \cup \{\langle \sim A,F\rangle\}$, where $\langle \sim A,T\rangle \in S$.

$\sim F$ Set $S' = S \cup \{\langle A,T\rangle\}$, where $\langle \sim A,F\rangle \in S$.

2020 *Mathematics Subject Classification.* Primary: 03B47, Secondary: 03B20.

Bimbó, Katalin, (ed.), *Relevance Logics and other Tools for Reasoning. Essays in Honor of J. Michael Dunn*, (Tributes, vol. 46), College Publications, London, UK, 2022, pp. 261–265.

The rules for disjunction are dual to those for conjunction. As for $\rightarrow$:

$\rightarrow T$ Set $S' = S \cup \{\langle A, F\rangle\}$ and $S'' = S \cup \{\langle B, T\rangle\}$, where $\langle A \rightarrow B, T\rangle \in S$.
Mpon Set $S' = S \cup \{\langle A, F\rangle\}$ and $S'' = S \cup \{\langle A \rightarrow B, F\rangle\}$, where $\langle B, F\rangle \in S$.

Mpon is rather different from the usual tableau rules in that it does *not* decompose a formula into subformulas.

A *construction* proceeds in *stages*. The initial stage is the unit set of tableaux $\{S\}$. Each stage consists in a finite set of tableaux $\{S_1, \ldots, S_n\}$. The $n+1$th stage comes from the nth by replacing some set S_i by its immediate descendant or its two immediate descendants according to one of the rules. As usual in tableau constructions, the procedure can be diagrammed as a tree, where binary branching occurs in connection with the rules $\wedge F$, $\vee T$, $\rightarrow T$, and *Mpon*. A tableau S is *closed* if and only if for some formula A, $\langle A, F\rangle \in S$ and *either* $\langle A, T\rangle \in S$ *or* A is an axiom of $\mathbf{R}$. A stage $\{S_1, \ldots, S_n\}$ is closed if and only if each S_i is closed.

We can stipulate a fixed priority ordering for applying rules if we wish. The point is to make the stages of a construction determinate, given the initial stage. We assume that the ordering is such that every applicable rule is eventually applied. The construction for A is the construction whose initial stage is $\{\{\langle A, F\rangle\}\}$. A construction is *closed* if some one of its stages is closed.

A *valuation* is a map ν whose domain is the set of formulae of $\mathbf{R}$, and whose range is $\{T, F\}$. A valuation is *admissible* if and only if:

(i) It respects the usual conditions for truth functions.
(ii) If $\nu(A \rightarrow B) = \nu(A) = T$, then $\nu(B) = T$ (equivalently, given (i): if $\nu(A \rightarrow B) = T$, $\nu(A \supset B) = T$).
(iii) If A is an axiom of $\mathbf{R}$, $\nu(A) = T$.

A formula A is *valid* if and only if for every admissible valuation ν, $\nu(A) = T$.

Theorem 1. *If A is a theorem of $\mathbf{R}_\gamma$, A is a theorem of $\mathbf{R}$.*

Proof. The theorem follows from Lemmas 2–4. The crucial step is Lemma 3. $\triangleleft$

Lemma 2. *If A is a theorem of $\mathbf{R}_\gamma$, A is valid.*

Proof. The axioms of $\mathbf{R}_\gamma$ are valid, and the rules preserve validity. $\triangleleft$

Lemma 3. *If A is valid, the construction for A is closed.*

Proof. We prove the contrapositive. Suppose the construction for A is not closed. Then by the usual argument from König's Lemma, there is an infinite set S of signed formulae such that:[1]

(i) S is closed under the rules (e.g., for $\wedge F$, if $\langle B \wedge C, F\rangle \in S$, either $\langle B, F\rangle \in S$ or $\langle C, F\rangle \in S$; for $\wedge T$, if $\langle B \wedge C, T\rangle \in S$, $\langle B, T\rangle \in S$ and $\langle C, T\rangle \in S$; for *Mpon*, if $\langle B, F\rangle \in S$, then for any C, either $\langle C \rightarrow B, F\rangle \in S$ or $\langle C, F\rangle \in S$; etc.).
(ii) $\langle A, F\rangle \in S$.
(iii) For no formula B are both $\langle B, T\rangle$ and $\langle B, F\rangle \in S$.
(iv) If B is an axiom of $\mathbf{R}$, $\langle B, F\rangle \notin S$.

Define a valuation $\nu(B)$ by induction on the complexity of B. If B is atomic, set $\nu(B) = T$ (F) if and only if $\langle B, T\rangle \in S$ $(\langle B, T\rangle \notin S)$. For truth-functional formulas, define ν so as to respect the truth-functions. $\nu(B \to C) = T$ if and only if $\nu(B) = F$ or $\nu(C) = T$, *and* $\langle B \to C, F\rangle \notin S$; otherwise, $\nu(B \to C) = F$.

We needed to show that A is not valid. This will follow if $\nu(A) = F$ and ν is an admissible valuation. That $\nu(A) = F$ follows from Sublemma 3.1, given that $\langle A, F\rangle \in S$. ν obviously satisfies conditions (i) and (ii) for admissibility. Condition (iii) is Sublemma 3.2.

Sublemma 3.1. For any formula B, if $\langle B, T\rangle \in S$, then $\nu(B) = T$; if $\langle B, F\rangle \in S$, then $\nu(B) = F$.

Proof. This is the usual lemma for the completeness of a tableau procedure. It is proved by induction on the number of connectives in A. If A is atomic and $\langle A, T\rangle \in S$, the result follows by the definition of ν. If $\langle A, F\rangle \in S$, then $\langle A, T\rangle \notin S$. So by the definition of ν, $\nu(A) = F$. Suppose B is $C \wedge D$, and the lemma holds for C and D. Then if $\langle B, T\rangle \in S$, then by the closure of S under the rules, $\langle C, T\rangle \in S$ and $\langle D, T\rangle \in S$, so $\nu(C) = \nu(D) = T$. So $\nu(B) = \nu(C \wedge D) = T$. If $\langle B, F\rangle \in S$, then either $\langle C, F\rangle \in S$ or $\langle D, F\rangle \in S$, so by inductive hypothesis $\nu(C) = F$ or $\nu(D) = F$, so $\nu(C \wedge D) = F$. Similarly, for the other truth functional formulas. If B is $C \to D$ and $\langle B, T\rangle \in S$, then either $\langle C, F\rangle \in S$ or $\langle D, T\rangle \in S$. So, by inductive hypothesis, either $\nu(C) = F$ or $\nu(D) = T$. Also, since $\langle C \to D, T\rangle \in S$, $\langle C \to D, F\rangle \notin S$, so by definition of ν, $\nu(B) = \nu(C \to D) = T$. If $\langle C \to D, F\rangle \in S$, then by definition of ν, $\nu(C \to D) = F$.

Sublemma 3.2. If B is an axiom of **R**, $\nu(B) = T$.

Proof. This goes case by case. We give two sample cases. The reader can verify the others.

Suppose B is $(C \to (C \to D)) \to (C \to D)$. To show $\nu(B) = T$, suppose for *reductio* that $\nu(B) = F$. Then by definition of ν, since B is an implicational formula, either $\langle B, F\rangle \in S$, or $\nu(C \to (C \to D)) = T$ and $\nu(C \to D) = F$. Since B is an axiom, $\langle B, F\rangle \in S$ is impossible, so $\nu(C \to (C \to D)) = T$ and $\nu(C \to D) = F$. Since $\nu(C \to D) = F$, either $\langle C \to D, F\rangle \in S$, or $\nu(C) = T$ and $\nu(D) = F$. Suppose $\langle C \to D, F\rangle \in S$. Then by closure of S under *Mpon*, either $\langle (C \to (C \to D)) \to (C \to D), F\rangle \in S$ or $\langle C \to (C \to D), F\rangle \in S$. But the first alternative is impossible, as already observed, and the second is impossible, since $\nu(C \to (C \to D)) = T$. So $\nu(C) = T$ and $\nu(D) = F$. But then, by definition of ν, $\nu(C \to (C \to D)) = F$. This is a contradiction.

Suppose B is $(C \to (D \to E)) \to (D \to (C \to E))$. Suppose $\nu(B) = F$. $\langle B, F\rangle \in S$ is impossible, so $\nu(C \to (D \to E)) = T$ and $\nu(D \to (C \to E)) = F$. So either $\langle D \to (C \to E), F\rangle \in S$ or $\nu(D) = T$ and $\nu(C \to E) = F$. In the former case, by *Mpon*, either $\langle (C \to (D \to E)) \to (D \to (C \to E)), F\rangle \in S$, or $\langle C \to (D \to E), F\rangle \in S$. Both are already ruled out, since the first is an axiom and the second contradicts $\nu(C \to (D \to E)) = T$. So $\langle D \to (C \to E), F\rangle \notin S$, so $\nu(D) = T$ and $\nu(C \to E) = F$. Since $\nu(C \to E) = F$, either $\langle C \to E, F\rangle \in S$, or $\nu(C) = T$ and $\nu(E) = F$. If $\langle C \to E, F\rangle \in S$, by *Mpon*, either $\langle D \to (C \to E), F\rangle \in S$ or $\langle D, F\rangle \in S$. But $\langle D \to (C \to E), F\rangle \in S$ has already been ruled out, and $\langle D, F\rangle \in S$ is impossible, since then, by Sublemma 2.1, $\nu(D) = F$, which has already been ruled out. So, $\langle C \to E, F\rangle \notin S$, hence $\nu(C) = T$ and $\nu(E) = F$. Also we already have $\nu(D) = T$. Hence by definition of ν, $\nu(C \to (D \to E)) = F$. This is a contradiction. ◁

Lemma 4. *If the construction for A is closed, A is a theorem of* **R**.

Proof. Any of the usual methods of proving that a tableau procedure is contained in a corresponding axiomatic system will do. For example, let a tableau $S = \{\langle B_1, T\rangle, \ldots, \langle B_m, T\rangle, \langle C_1, F\rangle, \ldots, \langle C_n, F\rangle\}$ (m or n may $= 0$). Then define the *characteristic formula* of S as $\neg B_1 \vee \cdots \vee \neg B_m \vee C_1 \vee \cdots \vee C_n$. Note that if S is closed, its characteristic formula is provable in **R**, since either it has two disjuncts of the forms B and $\neg B$, or some disjunct is an axiom of **R**. If a stage of a construction is $\{S_1, \ldots, S_q\}$ with characteristic formulae $D_1, \ldots, D_q$, let the characteristic formula of the stage be $D_1 \wedge \cdots \wedge D_q$. Then the characteristic formula of a closed stage is provable in **R**. By inspection of the various tableau rules, if C is the characteristic formula of a non-initial stage of a construction and C' is the characteristic formula of the preceding stage, $C \to C'$ is provable in **R**. Hence by transitivity of $\to$, if C is the characteristic formula of any stage of a construction and D is the characteristic formula of the initial stage, $C \to D$ is provable in **R**. Note that the characteristic formula of the initial stage of the construction for A is A itself. So, if the construction for A is closed, and C is the characteristic formula of the closed stage, $C \to A$ and C are both theorems of **R**. Thus, A is. $\triangleleft$

The proof above, except for its treatment of $\to$ and the axioms of **R**, is very close to the usual completeness proofs of tableau procedures. It is shown that the theorems of **R**, of $\mathbf{R}_\gamma$, and the valid formulae are all coextensive, though the semantical notion of validity used is of little independent interest. In Lemma 3, the partial function defined by $v(A) = T$ if $\langle A, T\rangle \in S$, and $v(A) = F$ if $\langle A, F\rangle \in S$, and undefined otherwise, is shown to extend to an admissible valuation defined on all formulae.

Although for definiteness the theorem was stated for **R**, the proof applies equally well, for example, to **E**. If quantifiers are added, as in **RQ** or **EQ**, the proof extends readily. Here we define an admissible valuation over a nonempty domain D, and the quantifiers are evaluated in the usual way. For the tableaux, quantifier rules of the usual kind are added.

Acknowledgments. My thanks to Oliver Marshall, Yale Weiss, and especially to Romina Padró for their help in producing this version. I would like to thank Katalin Bimbó for some technical help. This paper has been completed with support from the Saul A. Kripke Center at the City University of New York, Graduate Center.

Notes

* Unfortunately, the original handwritten manuscript of this paper was undated. Mike Dunn, however, recalled that I verbally reported this result to him in the summer of 1978 at Oxford. In an email to the Saul Kripke Center, dated February 6th, 2017, Dunn said: "Saul's communication to me was verbal without much detail. We were both visitors at Oxford in the spring of 1978 and [I] know that I at least stayed through mid-summer. I think that early summer/late spring at Oxford might be when/where he told me about his proof, but my memory is not clear on this. Anyway, I wrote up his proof, probably within a couple of months after he told me about it, and sent him a copy. I attach what I sent him. It is dated July 23, 1978, and I think was sent to him shortly after then." (See also Dunn and Meyer [2, §5].)

I originally thought that I was influenced by the result in Dunn and Meyer [2] connecting the proofs of γ to a method of proving Gentzen's cut elimination, but in the email mentioned above, Dunn said: "I sent him [me] a copy of 'Gentzen's cut ...' Feb. 12, 1980. It wasn't published until 1989, because of delay in publication of the Norman–Sylvan volume." So unless I had seen another copy or heard them give a talk, it is unlikely that I was influenced by their paper.

Other proofs of γ in relevance logic had long been around since the original paper by Meyer and Dunn [3], some of them making it much simpler. However, the present proof was novel in that it was based on the usual completeness proofs of tableau (cut-free) methods, and it is not based on cutting down an inconsistent set of statements to a consistent set.

[1] *S* is not a tableau of the construction but is the union of an increasing sequence of non-closed tableaux, each of which is an immediate descendant of its predecessor in the sequence.

REFERENCES

[1] Beth, E. W. (1955). Semantic entailment and formal derivability, *Mededelingen der koninklijke, Nederlandse Akademie van Wetenschappen, Afdeling Letterkunde* (Nieuwe Reeks) **18**(13): 309–342.

[2] Dunn, J. M. and Meyer, R. K. (1989). Gentzen's cut and Ackermann's gamma, *in* J. Norman and R. Sylvan (eds.), *Directions in Relevant Logic*, Kluwer, Dordrecht, pp. 229–240.

[3] Meyer, R. K. and Dunn, J. M. (1969). E, R and γ, *Journal of Symbolic Logic* **34**(3): 460–474.

[4] Smullyan, R. M. (1968). *First-order Logic*, Springer-Verlag, New York, NY.

SAUL KRIPKE CENTER, THE GRADUATE CENTER, CITY UNIVERSITY OF NEW YORK, NEW YORK, U.S.A.