Acta Philosophical Fennica, vol. 16 (1963), 83-94

# Semantical Considerations on Modal Logic

SAUL A. KRIPKE

This paper gives an exposition of some features of a semantical theory of modal logics[1]. For a certain quantified extension of S5, this theory was presented in [1], and it has been summarized in [2]. The present paper will concentrate on one aspect of the theory — the introduction of quantifiers — and it will restrict itself in the main to one method of achieving this end. The emphasis of the paper will be purely semantical, and hence it will omit the use of semantic tableaux, which is essential to a full presentation of the theory. (For these, see [1] and [11].) Proofs, also, will largely be suppressed.

We consider four modal systems. Formulae $A$, $B$, $C$, . . . are built out of atomic formulae $P$, $Q$, $R$, . . ., using the connectives $\wedge$, $\sim$, and $\Box$. The system $M$ has the following axiom schemes and rules:

A1. $\Box A \supset A$
A2. $\Box (A \supset B) \supset . \Box A \supset \Box B$
R1. $A, A \supset B \,/\, B$
R2. $A \,/\, \Box A$

If we add the following axiom scheme, we get S4:

$\Box A \supset \Box \Box A$

We get the *Brouwersche* system if we add to $M$:

$A \supset \Box \Diamond A$

S5, if we add:

$\Diamond A \supset \Box \Diamond A$

[1] The theory given here has points of contact with many authors: For lists of these, see [11] and Hintikka [6]. The authors closest to the present theory appear to be Hintikka and Kanger. The present treatment of quantification, however, is unique as far as I know, although it derives some inspiration from acquaintance with the very different methods of Prior and Hintikka.

Modal systems whose theorems are closed under the rules R1 and R2, and include all theorems of $M$, are called "normal". Although we have developed a theory which applies to such non-normal systems as Lewis's S2 and S3, we will restrict ourselves here to normal systems.

To get a semantics for modal logic, we introduce the notion of a (normal) *model structure.* A model structure (m.s.) is an ordered triple $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ where $\mathbf{K}$ is a set, $\boldsymbol{R}$ is a reflexive relation on $\mathbf{K}$, and $\mathbf{G} \varepsilon \mathbf{K}$. Intuitively, we look at matters thus: $\mathbf{K}$ is the set of all "possible worlds;" $\mathbf{G}$ is the "real world." If $\mathbf{H}_1$ and $\mathbf{H}_2$ are two worlds, $\mathbf{H}_1 \boldsymbol{R} \mathbf{H}_2$ means intuitively that $\mathbf{H}_2$ is "possible relative to" $\mathbf{H}_1$; *i.e.*, that every proposition *true* in $\mathbf{H}_2$ is *possible* in $\mathbf{H}_1$. Clearly, then, the relation $\boldsymbol{R}$ should indeed be reflexive; every world $\mathbf{H}$ is *possible* relative to itself, since every proposition *true* in $\mathbf{H}$ is, *a fortiori*, possible in $\mathbf{H}$. Reflexivity is thus an intuitively natural requirement. We may impose additional requirements, corresponding to various "reduction axioms" of modal logic: If $\boldsymbol{R}$ is transitive, we call $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ an S4-m.s.; if $\boldsymbol{R}$ is symmetric, $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is a *Brouwersche* m.s.; and if $\boldsymbol{R}$ is an equivalence relation, we call $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ an S5-m.s. A model structure without restriction is also called an $M$-model structure.

To complete the picture, we need the notion of *model.* Given a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, a *model* assigns to each atomic formula (propositional variable) $P$ a truth-value $\mathbf{T}$ or $\mathbf{F}$ in each world $\mathbf{H} \varepsilon \mathbf{K}$. Formally, a *model* $\varphi$ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is a binary function $\varphi(P, \mathbf{H})$, where $P$ varies over atomic formulae and $\mathbf{H}$ varies over elements of $\mathbf{K}$, whose range is the set $\{\mathbf{T}, \mathbf{F}\}$. Given a model, we can define the assignments of truth-values to non-atomic formulae by induction. Assume $\varphi(A, \mathbf{H})$ and $\varphi(B, \mathbf{H})$ have already been defined for all $\mathbf{H} \varepsilon \mathbf{K}$. Then if $\varphi(A, \mathbf{H}) = \varphi(B, \mathbf{H}) = \mathbf{T}$, define $\varphi(A \wedge B, \mathbf{H}) = \mathbf{T}$; otherwise, $\varphi(A \wedge B, \mathbf{H}) = \mathbf{F}$. $\varphi(\sim A, \mathbf{H})$ is defined to be $\mathbf{F}$ iff $\varphi(A, \mathbf{H}) = \mathbf{T}$; otherwise, $\varphi(\sim A, \mathbf{H}) = \mathbf{T}$. Finally, we define $\varphi(\Box A, \mathbf{H}) = \mathbf{T}$ iff $\varphi(A, \mathbf{H}') = \mathbf{T}$ for every $\mathbf{H}' \varepsilon \mathbf{K}$ such that $\mathbf{H} \boldsymbol{R} \mathbf{H}'$; otherwise, $\varphi(\Box A, \mathbf{H}) = \mathbf{F}$. Intuitively, this says that $A$ is necessary in $\mathbf{H}$ iff $A$ is true in all worlds $\mathbf{H}'$ possible relative to $\mathbf{H}$.

*Completeness theorem.* $\vdash A$ in $M$ (S4, S5, the *Brouwersche* system) if and only if $\varphi(A, \mathbf{G}) = \mathbf{T}$ for every model $\varphi$ on an $M$- (S4-, S5-, *Brouwersche*) model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$.

(For a proof, see [11].)

This completeness theorem equates the syntactical notion of *provability* in a modal system with a semantical notion of *validity.*

The rest of this paper concerns, with the exception of some concluding remarks, the introduction of quantifiers. To do this, we must associate with each world a domain of individuals, the individuals that exist in that world. Formally, we define a *quantificational model structure* (q.m.s.) as a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, together with a function $\psi$ which assigns to each $\mathbf{H} \varepsilon \mathbf{K}$ a set $\psi(\mathbf{H})$, called the *domain* of **H**. Intuitively $\psi(\mathbf{H})$ is the set of all individuals existing in **H**. Notice, of course, that $\psi(\mathbf{H})$ need not be the same set for different arguments **H**, just as, intuitively, in worlds other than the real one, some actually existing individuals may be absent, while new individuals, like Pegasus, may appear.

We may then add, to the symbols of modal logic, an infinite list of individual variables $x, y, z, \ldots$, and, for each nonnegative integer $n$, a list of $n$-adic predicate letters $P^n, Q^n, \ldots$, where the superscripts will sometimes be understood from the context. We count propositional variables (atomic formulae) as "0-adic" predicate letters. We then build up well-formed formulae in the usual manner, and can now prepare ourselves to define a quantificational *model*.

To define a quantificational model, we must extend the original notion, which assigned a truth-value to each atomic formula in each world. Analogously, we must suppose that in each world a given $n$-adic predicate letter determines a certain set of ordered $n$-tuples, its *extension* in that world. Consider, for example, the case of a monadic predicate letter $P(x)$. We would like to say that, in the world **H**, the predicate $P(x)$ is true of some individuals in $\psi(\mathbf{H})$ and false of others; formally, we would say that, relative to certain assignments of elements of $\psi(\mathbf{H})$ to $x$, $\varphi(P(x), \mathbf{H}) = \mathbf{T}$ and relative to others $\varphi(P(x), \mathbf{H}) = \mathbf{F}$. The set of all individuals of which $P$ is true is called the *extension* of $P$ in **H**. But there is a problem: should $\varphi(P(x), \mathbf{H})$ be given a truth-value when $x$ is assigned a value in the domain of some *other* world $\mathbf{H}'$, and not in the domain of **H**? Intuitively, suppose $P(x)$ means "$x$ is bald" — are we to assign a truth-value to the substitution instance "Sherlock Holmes is bald"? Holmes does not exist, but in other states of affairs, he would have existed. Should we assign a definite truth-value to the statement that he is bald, or not? Frege [3] and Strawson [4] would not assign the statement a truth-value; Russell [5] would [1]. For the purposes of modal logic we hold that different

[1] Russell, however, would conclude that "Sherlock Holmes" is therefore not a genuine name; and Frege would eliminate such empty names by an artifact.

answers to this question represent alternative *conventions*. All are tenable. The only existing discussions of this problem I have seen — those of Hintikka [6] and Prior [7] — adopt the Frege-Strawson view. This view necessarily must lead to some modification of the usual modal logic. The reason is that the semantics for modal propositional logic, which we have already given, assumed that every formula must take a truth-value in each world; and now, for a formula $A(x)$ containing a free variable $x$, the Frege-Strawson view requires that it not be given a truth-value in a world $\mathbf{H}$ when the variable $x$ is assigned an individual not in the domain of that world. We thus can no longer expect that the original laws of modal propositional logic hold for statements containing free variables, and are faced with an option: either revise modal propositional logic or restrict the rule of substitution. Prior does the former, Hintikka the latter. There are further alternatives the Frege-Strawson choice involves: Should we take $\Box A$ (in $\mathbf{H}$) to mean that $A$ is *true* in all possible worlds (relative to $\mathbf{H}$), or just *not false* in any such world? The second alternative merely demands that $A$ be either true or lack a truth-value in each world. Prior, in his system Q, in effect admits both types of necessity, one as "$L$" and the other as "$NMN$". A similar question arises for conjunction: if $A$ is false and $B$ has no truth-value, should we take $A \wedge B$ to be false or truth-valueless?

In a full statement of the semantical theory, we would explore all these variants of the Frege-Strawson view. Here we will take the other option, and assume that a statement containing free variables has a truth-value in each world for every assignment to its free variables [1]. Formally, we state the matter as follows: Let $\mathbf{U} = \bigcup_{\mathbf{H} \in \mathbf{K}} \psi(\mathbf{H})$. $\mathbf{U}^n$ is the $n$th Cartesian product of $\mathbf{U}$ with itself. We define a quantificational *model* on a q.m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ as a binary

[1] It is natural to assume that an *atomic* predicate should be *false* in a world $\mathbf{H}$ of all those individuals not existing in that world; that is, that the extension of a predicate letter must consist of actually existing individuals. We can do this by requiring semantically that $\varphi(P^n, \mathbf{H})$ be a subset of $[\psi(\mathbf{H})]^n$; the semantical treatment below would otherwise suffice without change. We would have to add to the axiom system below all closures of formulae of the form $P^n(x_1, \ldots, x_n) \wedge (y)A(y) . \supset . A(x_i)$ $(1 \leq i \leq n)$. We have chosen not to do this because the rule of substitution would no longer hold; theorems would hold for atomic formulae which would not hold when the atomic formulae are replaced by arbitrary formulae. (This answers a question of Putnam and Kalmar.)

function $\varphi(P^n, \mathbf{H})$, where the first variable ranges over $n$-adic predicate letters, for arbitrary $n$, and $\mathbf{H}$ ranges over elements of $\mathbf{K}$. If $n = 0$, $\varphi(P^n, \mathbf{H}) = \mathbf{T}$ or $\mathbf{F}$; if $n \geq 1$, $\varphi(P^n, \mathbf{H})$ is a subset of $\mathbf{U}^n$. We now define, inductively, for every formula $A$ and $\mathbf{H} \,\varepsilon\, \mathbf{K}$, a truth-value $\varphi(A, \mathbf{H})$, relative to a given assignment of elements of $\mathbf{U}$ to the free variables of $A$. The case of a propositional variable is obvious. For an atomic formula $P^n(x_1, \ldots, x_n)$, where $P^n$ is an $n$-adic predicate letter and $n \geq 1$, given an assignment of elements $a_1, \ldots, a_n$ of $\mathbf{U}$ to $x_1, \ldots, x_n$, we define $\varphi(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{T}$ if the $n$-tuple $(a_1, \ldots, a_n)$ is a member of $\varphi(P^n, \mathbf{H})$; otherwise, $\varphi(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{F}$, relative to the given assignment. Given these assignments for atomic formulae, we can build up the assignments for complex formulae by induction. The induction steps for the propositional connectives $\wedge$, $\sim$, $\Box$, have already been given. Assume we have a formula $A(x, y_1, \ldots, y_n)$, where $x$ and the $y_i$ are the only free variables present, and that a truth-value $\varphi(A(x, y_1, \ldots, y_n), \mathbf{H})$ has been defined for each assignment to the free variables of $A(x, y_1, \ldots, y_n)$. Then we define $\varphi((x)A(x, y_1, \ldots, y_n), \mathbf{H}) = \mathbf{T}$ relative to an assignment of $b_1, \ldots, b_n$ to $y_1, \ldots, y_n$ (where the $b_i$ are elements of $\mathbf{U}$), if $\varphi((A(x, y_1, \ldots, y_n), \mathbf{H}) = \mathbf{T}$ for every assignment of $a, b_1, \ldots, b_n$ to $x, y_1, \ldots, y_n$, respectively, where $a \,\varepsilon\, \psi(\mathbf{H})$; otherwise, $\varphi((x)A(x, y_1, \ldots, y_n), \mathbf{H}) = \mathbf{F}$ relative to the given assignment. Notice that the restriction $a \,\varepsilon\, \psi(\mathbf{H})$ means that, in $\mathbf{H}$, we quantify only over the objects actually existing in $\mathbf{H}$.

To illustrate the semantics, we give counterexamples to two familiar proposals for laws of modal quantification theory — the "Barcan formula" $(x)\Box A(x) \supset \Box(x)A(x)$ and its converse $\Box(x)A(x) \supset (x)\Box A(x)$. For each we consider a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, where $\mathbf{K} = \{\mathbf{G}, \mathbf{H}\}$, $\mathbf{G} \neq \mathbf{H}$, and $\boldsymbol{R}$ is simply the Cartesian product $\mathbf{K}^2$. Clearly $\boldsymbol{R}$ is reflexive, transitive, and symmetric, so our considerations apply even to S5.

For the Barcan formula, we extend $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ to a quantificational model structure by defining $\psi(\mathbf{G}) = \{a\}$, $\psi(\mathbf{H}) = \{a, b\}$, where $a$ and $b$ are distinct. We then define, for a monadic predicate letter $P$, a model $\varphi$ in which $\varphi(P, \mathbf{G}) = \{a\}$, $\varphi(P, \mathbf{H}) = \{a\}$. Then clearly $\Box P(x)$ is true in $\mathbf{G}$ when $x$ is assigned $a$; and since $a$ is the only object in the domain of $\mathbf{G}$, so is $(x)\Box P(x)$. But, $(x)P(x)$ is clearly false in $\mathbf{H}$ (for $\varphi(P(x), \mathbf{H}) = \mathbf{F}$ when $x$ is assigned $b$), and hence $\Box(x)P(x)$ is false in $\mathbf{G}$. So we have a counterexample to the Barcan formula. Notice that this counterexample is quite independent of whether $b$ is assigned a truth-value in **G** or not, so also it applies to the systems of Hintikka and Prior. Such counterexamples can be disallowed, and the Barcan formula reinstated, only if we require a model structure to satisfy the condition that $\psi(\mathbf{H}') \subseteq \psi(\mathbf{H})$ whenever $\mathbf{H}\,\boldsymbol{R}\,\mathbf{H}'$ $(\mathbf{H}, \mathbf{H}' \varepsilon \mathbf{K})$.

For the converse of the Barcan formula, set $\psi(\mathbf{G}) = \{a, b\}$, $\psi(\mathbf{H}) = \{a\}$, where again $a \neq b$. Define $\varphi(P, \mathbf{G}) = \{a, b\}$, $\varphi(P, \mathbf{H}) = \{a\}$, where $P$ is a given monadic predicate letter. Then clearly $(x)P(x)$ holds in both **G** and **H**, so that $\varphi(\Box(x)P(x), \mathbf{G}) = \mathbf{T}$. But $\varphi(P(x), \mathbf{H}) = \mathbf{F}$ when $x$ is assigned $b$, so that, when $x$ is assigned $\varphi(\Box P(x), \mathbf{G}) = \mathbf{F}$. Hence $\varphi((x)\Box P(x), \mathbf{G}) = \mathbf{F}$, and we have the desired counterexample to the converse of the Barcan formula. This counterexample, however, depends on asserting that, in **H**, $P(x)$ is actually *false* when $x$ is assigned $b$; it might thus disappear if, for this assignment, $P(x)$ were declared to lack truth-value in **H**. In this case, we will still have a counterexample if we require a necessary statement to be *true* in all possible worlds (Prior's "$L$"), but not if we merely require that it never be false (Prior's "$NMN$"). On our present convention, we can eliminate the counterexample only by requiring, for each q.m.s., that $\psi(\mathbf{H}) \subseteq \psi(\mathbf{H}')$ whenever $\mathbf{H}\,\boldsymbol{R}\,\mathbf{H}'$.

These counterexamples lead to a peculiar difficulty: We have given countermodels, in quantified S5, to both the Barcan formula and its converse. Yet Prior appears to have shown in [8] that the Barcan formula is derivable in quantified S5; and the converse seems derivable even in quantified $M$ by the following argument:

(A) $(x)A(x) \supset A(y)$ (by quantification theory)

(B) $\Box((x)A(x) \supset A(y))$ (by necessitation)

(C) $\Box((x)A(x) \supset A(y)) \supset \Box(x)A(x) \supset \Box A(y)$ (Axiom A2)

(D) $\Box(x)A(x) \supset \Box A(y)$ (from (B) and (C))

(E) $(y)(\Box(x)A(x) \supset \Box A(y))$ (generalizing on (D))

(F) $\Box(x)A(x) \supset (y)\Box A(y)$ (by quantification theory, and (E))

We seem to have derived the conclusion using principles that should all be valid in the model-theory. Actually, the flaw lies in the application of necessitation to (A). In a formula like (A), we give the free variables the generality interpretation:[1] When (A) is asserted as a theorem, it abbreviates assertion of its ordinary universal closure

(A′) $(y)\ ((x)A(x) \supset A(y))$

Now if we applied necessitation to (A′), we would get

(B′) $\Box\ (y)\ ((x)A(x) \supset A(y))$

On the other hand, (B) itself is interpreted as asserting

(B″) $(y)\ \Box\ ((x)A(x) \supset A(y))$

To infer (B″) from (B′), we would need a law of the form $\Box\ (y)C(y) \supset (y)\Box C(y)$, which is just the converse Barcan formula that we are trying to prove. In fact, it is readily checked that (B″) fails in the countermodel given above for the converse Barcan formula, if we replace $A(x)$ by $P(x)$.

We can avoid this sort of difficulty if, following Quine [15], we formulate quantification theory so that only *closed* formulae are asserted. Assertion of formulae containing free variables is at best a convenience; assertion of $A(x)$ with free $x$ can always be replaced by assertion of $(x)A(x)$.

If $A$ is a formula containing free variables, we define a *closure* of $A$ to be any formula without free variables obtained by prefixing universal quantifiers and necessity signs, in any order, to $A$. We then define the axioms of quantified $M$ to be the closures of the following schemata:

(0) Truth-functional tautologies

(1) $\Box A \supset A$

(2) $\Box (A \supset B) \,.\supset.\, \Box A \supset \Box B$

(3) $A \supset (x)A$, where $x$ is not free in $A$.

(4) $(x)(A \supset B) \,.\supset.\, (x)A \supset (x)B$

(5) $(y)\ ((x)A(x) \supset A(y))$

---

[1] It is not asserted that the generality interpretation of theorems with free variables is the only possible one. One might wish a formula $A$ to be provable iff, for each model $\varphi$, $\varphi(A, \mathbf{G}) = \mathbf{T}$ for every assignment to the free variables of $A$. But then $(x)A(x) \supset A(y)$ will not be a theorem; in fact, in the countermodel above to the Barcan formula, $\varphi\ ((x)P(x) \supset P(y), \mathbf{G}) = \mathbf{F}$ if $y$ is assigned $b$. Thus quantification theory would have to be revised along the lines of [9] or [10]. This procedure has much to recommend it, but we have not adopted it since we wished to show that the difficulty can be solved without revising quantification theory or modal propositional logic.

The rule of inference is detachment for material implication. Necessitation can be obtained as a derived rule.

To obtain quantified extensions of S4, S5, the *Brouwersche* system, simply add to the axiom schemata all closures of the appropriate reduction axiom.

The systems we have obtained have the following properties: They are a straightforward extension of the modal propositional logics, without the modifications of Prior's Q; the rule of substitution holds without restriction, unlike Hintikka's presentation; and nevertheless neither the Barcan formula nor its converse is derivable. Further, all the laws of quantification theory — modified to admit the empty domain — hold. The semantical completeness theorem we gave for modal propositional logic can be extended to the new systems.

We can introduce *existence as a predicate* in the present system if we like. Semantically, existence is a monadic predicate $E(x)$ satisfying, for each model $\varphi$ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, the identity $\varphi(E, \mathbf{H}) = \psi(\mathbf{H})$ for every $\mathbf{H} \varepsilon \mathbf{K}$. Axiomatically, we can introduce it through the postulation of closures of formulae of the form: $(x)A(x) \wedge E(y). \supset . A(y)$, and $(x)E(x)$. The predicate $P$ used above in the counter-example to the converse Barcan formula can now be recognized as simply existence. This fact shows how existence differs from the tautological predicate $A(x) \vee \sim A(x)$ even though $\Box(x)E(x)$ is provable. For although $(x)\Box(A(x) \vee \sim A(x))$ is valid, $(x)\Box E(x)$ is not; although it is necessary that every thing exists, it does not follow that everything has the property of necessary existence.

We can introduce identity semantically in the model theory by defining $x = y$ to be true in a world $\mathbf{H}$ when $x$ and $y$ are assigned the same value and otherwise false; existence could then be defined in terms of identity, by stipulating that $E(x)$ means $(\exists y)(x = y)$. For reasons not given here, a broader theory of identity could be obtained if we complicated the notion of quantificational model structure.

We conclude with some brief and sketchy remarks on the "provability" interpretations of modal logics, which we give in each case for propositional calculus only. The reader will have obtained the main point of this paper if he omits this section. Provability interpretations are based on a desire to adjoin a necessity operator to a formal system, say Peano arithmetic, in such a way that, for any formula $A$ of the system, $\Box A$ will be interpreted as true iff $A$ is provable in the system. It has been argued that such "provability" interpretations of a modal operator are dispensable in favor of a provability *predicate*, attaching to the Gödel number of $A$; but Professor Montague's contribution to the present volume casts at least some doubt on this viewpoint.

Let us consider the formal system **PA** of Peano arithmetic, as formalized in Kleene [12]. We adjoin to the formation rules operators $\wedge$, $\sim$, and $\Box$ (the conjunction and negation adjoined are to be distinct from those of the original system), operating on closed formulae only. In the model theory we gave above, we took atomic formulae to be propositional variables, or predicate letters followed by parenthesized individual variables; here we take them to be simply the closed well-formed formulae of **PA** (*not* just the atomic formulae of **PA**). We define a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, where **K** is the set of all distinct (non-isomorphic) countable models of **PA**, **G** is the standard model in the natural numbers, and $\boldsymbol{R}$ is the Cartesian product $\mathbf{K}^2$. We define a model $\varphi$ by requiring that, for any atomic formula $P$ and $\mathbf{H} \varepsilon \mathbf{K}$, $\varphi(P, \mathbf{H}) = \mathbf{T}$ ($\mathbf{F}$) iff $P$ is true (false) in the model **H**. (Remember, $P$ is a wff of **PA**, and **H** is a countable model of **PA**.) We then build up the evaluation for compound formulae as before.[1] To say that $A$ is true is to say it is true in the real world **G**; and, for any atomic $P$, $\varphi(\Box P, \mathbf{G}) = \mathbf{T}$ iff $P$ is provable in **PA**. (Notice that $\varphi(P, \mathbf{G}) = \mathbf{T}$ iff $P$ is true in the intuitive sense.) Since $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an S5-m.s., all the laws of S5 will be valid on this interpretation; and we can show that *only* the laws of S5 are generally valid. (For example, if $P$ is Gödel's undecidable formula, $\varphi(\Box P \vee \Box \sim P, \mathbf{G}) = \mathbf{F}$, which is a counterexample to the "law" $\Box A \vee \Box \sim A$.)

Another provability interpretation is the following: Again we take the atomic formulae to be the closed wffs of **PA**, and then build up new formulae using the adjoined connectives $\wedge$, $\sim$, and $\Box$.

[1] It may be protested that **PA** already contain symbols for conjunction and negation, say "&" and "¬"; so why do we adjoin new symbols "$\wedge$" and "$\sim$"? The answer is that if $P$ and $Q$ are atomic formulae, then $P$ & $Q$ is *also* atomic in the present sense, since it is well-formed in **PA**; but $P \wedge Q$ is not. In order to be able to apply the previous theory, in which the conjunction of atomic formulae is not atomic, we need "$\wedge$". Nevertheless, for any $\mathbf{H} \varepsilon \mathbf{K}$ and atomic $P$ and $Q$, $\varphi(P\&Q, \mathbf{H}) = \varphi(P \wedge Q, \mathbf{H})$, so that confusion of "&" with "$\wedge$" causes no harm in practice. Similar remarks apply to negation, and to the provability interpretation of S4 in the next paragraph.

Let **K** be the set of all ordered pairs (**E**, $\alpha$), where **E** is a consistent extension of **PA**, and $\alpha$ is a (countable) model of the system **E**. Let **G** = (**PA**, $\alpha_0$), where $\alpha_0$ is the standard model of **PA**. We say (**E**, $\alpha$) $\boldsymbol{R}$ (**E**′, $\alpha'$), where (**E**, $\alpha$) and (**E**′, $\alpha'$) are in **K**, iff **E**′ is an extension of **E**. For atomic $P$, define $\varphi(P, (\mathbf{E}, \alpha)) = \mathbf{T}$ (**F**) iff $P$ is true (false) in $\alpha$. Then we can show, for atomic $P$, that $\varphi(\Box P, (\mathbf{E}, \alpha)) = \mathbf{T}$ iff $P$ is provable in **E**; in particular, $\varphi(\Box P, \mathbf{G}) = \mathbf{T}$ iff $P$ is provable in **PA**. Since (**G**, **K**, $\boldsymbol{R}$) is an S4-m.s., all the laws of S4 hold. But not all the laws of S5 hold; if $P$ is Gödel's undecidable formula, $\varphi((\sim \Box P \supset \Box \sim \Box P), \mathbf{G}) = \mathbf{F}$. But some laws are valid which are not provable in S4; in particular, we can prove for any $A$, $\varphi(\Box \sim \Box (\Diamond A \wedge \Diamond \sim A), \mathbf{G}) = \mathbf{T}$, which yields the theorems of McKinsey's S4.1 (cf. [13]). By suitable modifications this difficulty could be removed; but we do not go into the matter here.

Similar interpretations of $M$ and the *Brouwersche* system could be stated; but, in the present writer's opinion, they have less interest than those given above. We mention one more class of provability interpretations, the "reflexive" extensions of **PA**. Let **E** be a formal system containing **PA**, and whose well-formed formulae are formed out of the closed formulae of **PA** by use of the connectives &, $\neg$, and $\Box$. (I say " &" and "$\neg$" to indicate that I am using the same conjunction and negation as in **PA** itself, not introducing new ones. See footnote 1, p. 91.) Then **E** is called a reflexive extension of **PA** iff: (1) It is an inessential extension of **PA**; (2) $\Box A$ is provable in **E** iff $A$ is; (3) there is a valuation $\alpha$, mapping the closed formulae of **E** into the set $\{\mathbf{T}, \mathbf{F}\}$, such that conjunction and negation obey the usual truth tables, all the true closed formulae of **PA** get the value **T**, $\alpha(\Box A) = \mathbf{T}$ iff $A$ is provable in **E**, and all the theorems of **E** get the value **T**. It can be shown that there are reflexive extensions of **PA** containing the axioms of S4 or even S4.1, but none containing S5.

Finally, we remark that, using the usual mapping of intuitionistic logic into S4, we can get a model theory for the intuitionistic predicate calculus. We will not give this model theory here, but instead will mention, for propositional calculus only, a particular useful interpretation of intuitionistic logic that results from the model theory. Let **E** be any consistent extension of **PA**. We say a formula $P$ of **PA** is *verified* in **E** iff it is provable in **E**. We take the closed wffs $P$ of **PA** as atomic, and build formulae out of them using the intuitionistic connectives $\wedge$, $\vee$, $\neg$, and $\supset$. We then stipulate inductively: $A \wedge B$ is verified in **E** iff $A$ and $B$ are; $A \vee B$ is verified in **E** iff $A$ or $B$ is; $\neg A$ is verified in **E** iff there is no consistent extension of **E** verifying $A$; $A \supset B$ is verified in **E** iff every consistent extension **E**′ of **E** verifying $A$ also verifies $B$.

Then every instance of a law of intuitionistic logic is verified in **PA**; but, e.g., $A \vee \neg A$ is not, if $A$ is the Gödel undecidable formula. In future work, we will extend this interpretation further, and show that using it we can find an interpretation for Kreisel's system FC of absolutely free choice sequences (cf. [14]). It is clear, incidentally, that **PA** can be replaced in the provability interpretations of S4 and S5 by any truth functional system (i.e., by any system whose models determine each closed formula as true or false); while the interpretation of intuitionism applies to any formal system whatsoever.

Harvard University.

## References

[1] Saul A. Kripke. *A completeness theorem in modal logic.* **The journal of symbolic logic,** vol. 24 (1959), pp. 1—15.

[2] Saul A. Kripke. *Semantical analysis of modal logic* (abstract). **The journal of symbolic logic,** vol. 24 (1959), pp. 323—324.

[3] Gottlob Frege. *Über Sinn und Bedeutung.* **Zeitschrift für Philosophie und philosophische Kritik,** vol. 100 (1892), pp. 25—50. English translations in P. Geach and M. Black, **Translations from the philosophical writings of Gottlob Frege,** Basil Blackwell, Oxford 1952, and in H. Feigl and W. Sellars (ed.), **Readings in philosophical analysis,** Appleton-Century-Crofts, Inc., New York 1949.

[4] P. F. Strawson. *On referring.* **Mind,** n. s., vol. 59 (1950), pp. 320—344.

[5] Bertrand Russell. *On denoting.* **Mind,** n. s., vol. 14 (1905), pp. 479—493.

[6] Jaakko Hintikka. *Modality and quantification.* **Theoria** (Lund), vol. 27 (1961), pp. 119—128.

[7] A. N. Prior. **Time and modality.** Clarendon Press, Oxford 1957, viii + 148 pp.

[8] A. N. Prior. *Modality and quantification in S5.* **The journal of symbolic logic,** vol. 21 (1956), pp. 60—62.

[9] Jaakko Hintikka. *Existential presuppositions and existential commitments.* **The journal of philosophy,** vol. 56 (1959), pp. 125—137.

[10] Hugues Leblanc and Theodore Hailperin. *Nondesignating singular terms.* **Philosophical review,** vol. 68 (1959), pp. 239—243.

[11] SAUL A. KRIPKE. *Semantical analysis of modal logic I.* **Zeitschrift für mathematische Logik und Grundlagen der Mathematik,** vol. 9 (1963), pp. 67—96.

[12] STEPHEN C. KLEENE. **Introduction to metamathematics.** D. Van Nostrand, New York 1952, x + 550 pp.

[13] J. C. C. MCKINSEY. *On the syntactical construction of systems of modal logic.* **The journal of symbolic logic,** vol. 10 (1945), pp. 83—94.

[14] G. KREISEL. *A remark on free choice sequences and the topological completeness proofs.* **The journal of symbolic logic,** vol. 23 (1958), pp. 369—388.

[15] W. VAN O. QUINE. **Mathematical logic.** Harvard University Press, Cambridge, Mass., 1940; second ed., revised, 1951, xii + 346 pp.