BOHM'S ONTOLOGICAL INTERPRETATION AND ITS RELATIONS TO THREE FORMULATIONS OF QUANTUM MECHANICS*

Frederick M. Kronz
Dept. of Phil., The University of Texas

Abstract. The standard mathematical formulation of quantum mechanics is specified. Bohm s ontological interpretation of quantum mechanics is then shown to be incapable of providing a suitable interpretation of that formulation. It is also shown that Bohm's interpretation may well be viable for two alternative mathematical formulations of quantum mechanics, meaning that the negative result is a significant though not a devastating criticism of Bohm's interpretation. A preliminary case is made for preferring one alternative formulation over the other.

Introduction. To facilitate developing the central arguments of this essay, four layers of theorizing are distinguished: the formal, mathematical, physical, and metaphysical. The formal and mathematical are primarily syntactic categories; the physical and metaphysical are primarily semantic. The four layers are characterized here by way of examples using notions that are connected with QM (quantum mechanics). Dirac's bra-ket notation and his delta-function are formal elements. Spaces of finite or infinite sequences of complex numbers, more abstract notions such as a linear vector space over a field, and linear functionals and operators on such spaces are mathematical. The association of the square modulus of an inner product of two vectors (normalized to unity) with the probability of a measurement outcome, or the association of a self-adjoint operator with an observable physical quantity are physical. The interpretation of a state vector as corresponding to a quantum potential or of the elements of a superposition as corresponding to states of a system in different worlds are metaphysical. Hidden variables are also regarded as metaphysical elements, as are the assumptions of determinism and of the continuity of physical entities and processes.

The distinctions drawn above are pragmatic, as opposed to being hard distinctions, because the status of a notion can change over time. For example, Dirac's delta function was a purely

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formal device when he first introduced it around 1925. It became a mathematical entity about 25 years later within the context of Schwartz's theory of distributions. Similarly, it is possible that there are hidden variables such as exact particle positions or propagators of an uncontrollable nonlocality, or that there are other worlds, and that they could be measured or detected in some way, though their existence is hypothetical as matters stand right now.

For the purposes of this essay, a formulation of a physical theory is a characterization of the those elements that constitute the core notions or the essence of that theory. It may be represented as an ordered pair consisting of the essential mathematical framework and a set of physical associations for the elements of that framework. Heuristic elements (such as interpretive metaphysical notions and formal and mathematical elements having no physical associations) and contextual elements (such as a set of observations and background assumptions) are not regarded as part of a formulation of a theory.

In this essay, the standard formulation of QM is taken to be an SHS (separable Hilbert space), and the key physical associations of the elements of that space are taken to be the following: vectors with possible states of a given physical system, self-adjoint operators with measurable physical quantities, unitary operators with the dynamics of the system, and an inner product of vectors with probability amplitudes. Some explanation and justification of these identifications are provided in the next section. BI (Bohm's interpretation) is then shown not to be a viable interpretation of that formulation. Two alternative formulations of QM are then identified, one having a PHS (pre-Hilbert space) as its essential mathematical framework and the other an RHS (rigged Hilbert space). It is argued that BI is viable for both, and that the RHS formulation is substantially better than the PHS.

The Standard SHS Formulation of QM. There are two ways in which standard treatises and texts on QM present its mathematical framework. One sticks closely to the rigorous approach first established by von Neumann, and that approach is characterized in this section. The other follows a pragmatic approach made popular by Dirac, and it is characterized more fully in the next section. In the pragmatic approach, different frameworks are used; the choice of framework depends upon the nature of the problem at hand. It is curious that the standard formulation of QM is the rigorous one though most texts adopt a pragmatic approach. This uneasy tension between rigor and "what works" is characteristic of physics. This philosophical critique emphasizes the importance of rigor.

In 1927, von Neumann specified the mathematical spaces that he regarded as corresponding to the Schrödinger and Heisenberg formulations of QM and demonstrated their mathematical equivalence without using Dirac's delta function, unlike earlier demonstrations by Dirac and Jordan—see (von Neumann 1955, pp. 17-27). The lack of a rigorous mathematical definition of the delta function, a central element in Dirac's formulation, was regarded as a serious shortcoming by 1927 despite its utility and corresponding popularity. Von Neumann's proof consists in showing that the two spaces are mathematically equivalent in the sense that both are concrete realizations of an abstract SHS. It is then elementary to show that the Schrödinger and Heisenberg formulations of QM are equivalent, simply by giving physical associations to appropriate elements of the abstract SHS. These results partially explain why the abstract SHS came to be regarded as the most suitable mathematical framework for QM.[1]

Essentially, von Neumann went about proving the mathematical equivalence of matrix mechanics and wave mechanics as follows. First, he recognized the mathematical framework of matrix mechanics as an infinite-dimensional separable Hilbert space, where a Hilbert space is by definition a complete vector space with an inner product.[2] Next, he specified a tie to the physical

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[1] See (Jammer 1966, pp. 312-317). Jammer's treatise contains a superb discussion of the key developments in mathematics and physics culminating in von Neumann's equivalency proof.

[2] Informally, separability and completeness serve to restrict the size of the set of elements of the vector space. A vector space is separable if and only if there is a dense set of elements in the space that is countable, and it is complete if and only if every Cauchy sequence of vectors in the space converge on some element that is also in the space. Thus, separability places an upper cap on the size of the vector space and completeness a lower cap. A clear, concise presentation of the essentials of SHSs is given in (Jauch 1968, chapter 2) and (Jordan 1969, chapter 1). For a more detailed discussion see (von Neumann 1955, chapter 2) or (Reed and Simon 1980, chapter 2).

Separability was included in von Neumann's definition of a Hilbert space. It was later proven to be inessential to many of the key results concerning Hilbert spaces—see (Akhiezer and Glazman 1961, section 6). Nevertheless, the notion continued to be regarded as sufficient for doing physics for some time. In one very influential treatise on quantum field theory, it was acknowledged that there are occasions where nonseparable Hilbert spaces are used to characterize quantum phenomena, but such cases were dismissed on the grounds that they involve unrealistic assumptions such as supposing that the number of particles is actually infinite—see (Streater and Wightman 1964, section 2.6). With the rise in the importance of algebraic quantum mechanics, it is fair to say that the attitude towards infinite degrees of freedom, specifically the inequivalent representations of algebras of observables in world by regarding each column matrix (which by the definition of an SHS has a finite norm) as corresponding to a possible state of the associated physical system that determines probabilities for measurement outcomes via Born's interpretation. He then attempted to specify a set of functions that would satisfy the definition of an SHS and could be identified with wave mechanics. But, that turned out not to be possible. If the goal is mathematical equivalence, then what is needed is an equivalence class of functions.

Von Neumann began with the space of square-integrable functions—see (1955, pp. 28-33). To satisfy the completeness condition, that all Cauchy sequences of functions converge (in the mean) to some function in that space, he specified that integration must be defined in the manner of Lebesgue. To define an inner product operation, he specified that the set of Lebesgue square-integrable functions must be partitioned into equivalence classes modulo the relation of differing on a set of measure zero—see (1955, pp. 59-64) [3] This means, *according to von Neumann*, that the set of elements of the vector space corresponding to wave mechanics does not consist of a set of functions; rather, it consists of a set of equivalence classes of functions. To establish the same tie to the physical world as matrix mechanics, each equivalence class must be regarded as corresponding to a possible state of the associated physical system. It is precisely here that the SHS formulation of QM clashes with one of the key elements of BI, the single-valuedness assumption, which says that with each point of space and at each instant of time there is associated a unique value of the complex-valued wave function (Holland 1993, pp. 70-72).

It is worth noting for what follows that the mathematical structure which von Neumann linked with Schrödinger's wave mechanics, the set of equivalence classes of Lebesgue square-integrable functions, is not one that Schrödinger would have linked with it. Many discontinuous functions are square-integrable—some of which are discontinuous everywhere—and that conflicts with Schrödinger's requirement that wave functions be continuous in order to be

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nonseparable Hilbert spaces, has shifted—see (Bratteli and Robinson 1987, pp. 3-13) and (Haag 1992, pp. 105-111).

[3] The resulting space is the quotient space L^2/N, where L^2 is the space of Lebesgue square-integrable functions and $N = \{\phi \mid \langle\phi|\phi\rangle = 0\}$. The elements of L^2/N are the equivalence classes generated by N. The space L^2/N is a Hilbert space, the space L^2 (as characterized here) is not. It is worth noting that the accepted convention is to denote L^2/N simply as L^2, and to refer to the quotient space, which is denoted here as L^2/N, as "the space of Lebesgue square-integrable functions" (a potentially misleading description).

physically acceptable. That requirement is a direct consequence of the principle metaphysical structure that he associated with wave mechanics, the notion of a continuum. Schrödinger also required on similar grounds that wave-functions be single-valued.

In von Neumann's approach, a unique real number is assigned to each finite but non-zero interval of space at each instant of time; but, that is not so with respect to each point in space. This difference suggests why a different mathematical framework, specifically a PHS (which is characterized below), is best associated with the wave mechanics of Schrödinger. A case will be made to that effect by closely examining Bohm's elaboration of wave mechanics (Bohm 1951). It also indicates why BI would be viable for a PHS formulation of QM.

Two Alternative Formulations of QM. Texts that follow a pragmatic approach typically though not invariably mention that the appropriate mathematical restriction on functions is Lebesgue square-integrability. Quite often, they merely gloss over or do not even mention the delicate matter discussed above concerning the need to introduce an equivalence relation.[4] One can easily get the impression that Lebesgue square-integrable functions are the elements (i.e., may be regarded as vectors) of an SHS. But, that impression is wrong. Lebesgue square-integrable functions must be partitioned into equivalence classes of functions, and those classes are what correspond to the elements of an SHS.

The reason for this neglect is that these texts do not stay with the integrability restriction. In some contexts it is followed, but in some it is exceeded and in others it is supplemented. The three types of contexts correspond to distinct mathematical frameworks that are fundamentally different from one another. Authors wish to avoid calling attention to the fact that different spaces of functions are being used in a context dependent manner so as to give the appearance of a uniform mathematical presentation. The pedagogical advantages of this apparent uniformity are obvious. But, there are disadvantages for discussions of interpretive or philosophical issues, in which case the the potential for confusion is substantial (if not enormous). So, it is worth giving

[4] Mathematical textbooks on Hilbert spaces or, more generally, on functional analysis are careful to point this out, as are the small number of physics texts that adopt von Neumann's rigorous approach. Some examples of the latter are (Jauch 1968), (Amrein et al. 1977), (Prugovecki 1981), and (Blank et al. 1994).

an example of each of the three different contexts and specifying the corresponding space of functions that is involved.

First, a space of infinite dimensional column matrices of finite norm is often used to discuss harmonic oscillators and angular momentum observables. This space is a concrete realization of an SHS. A unitary transformation to a wave representation yields the space of equivalence classes of functions that was characterized in the previous section. The integrability requirement is sufficient.

Second, in discussions of the observables position and momentum, the integrability requirement is relaxed. Specifically, delta functions (which are really not functions) and plane-wave functions (which are not square-integrable) are used. A mathematical space quite different from an SHS—i.e., a space of distributions—provides the foundation. This is sometimes implied by provisos that such functions really correspond to "linear functionals that operate on a set of well-behaved test functions" (Schiff 1968, p. 56), or

> …we shall always integrate the δ-function multiplied by a smooth function (called a testing function by the mathematicians) over some finite interval, and thereby in fact use the δ-function as a linear functional… (Gottfried 1974, p. 53).[5]

Notice that there is a sense in which the integrability requirement is enhanced as well as being relaxed. This is because two different mathematical spaces are being used together. The condition is enhanced to form a space of test functions and it is relaxed to form a space of distributions (generalized functions). If the two spaces are chosen in such a way that each space is the topological dual (the set of all continuous linear functionals) with respect to the other, then the pair constitutes a concrete realization of an RHS. Different sets of test functions may be chosen

[5] Other statements of this sort are given in (Messiah 1961, p. 190), (Jordan 1969, p. 59), (Merzbacher 1970, p. 82), (Cohen-Tannoudji et al. 1977, pp. 111-113), and (Sudbery 1986, p. 68). Instead of using the technical term "testing function," other modes of expression are sometimes used such as a restriction of attention to those functions that are everywhere defined, continuous, infinitely differentiable, and have a bounded domain (Cohen-Tannoudji 1977, pp. 94-95). Some texts only provide minimal guidance to the reader with respect to this point. The passage in Merzbacher's text characterizes a function that is used with a δ-function merely as "an arbitrary well-behaved function," that in Messiah's suggests that the requisite conditions are merely square integrability and continuity.

(by relaxing or enhancing requisite conditions such as those involving differentiability or rate of decrease) and correspond to different spaces of distributions as their duals.

Third, in discussions of the energy spectrum of a bounded system or of a system interacting with a potential barrier, the integrability condition is often supplemented. The additional restrictions on the "physically acceptable" wave functions typically include continuity and differentiability. Their role is usually pragmatic—they simplify the discussion. But the result of adding restrictions is an essentially different space of functions, a PHS. It is worth mentioning that there is an interpretive element served by the additional restrictions aside from the pragmatic element. That interpretive element is a remnant of an assumption about the nature of physical reality associated with Schrödinger's approach, which is closely followed in some outdated texts such as (Pauling and Wilson 1935) and (Bohm 1951). That is important from both historical and philosophical perspectives: BI was first presented in (Bohm 1952), and the backdrop for BI is (Bohm 1951), a QM text clearly in the philosophical tradition of de Broglie and Schrödinger (p. 383).

Bohm's PHS Formulation of QM. Most of (Bohm 1951) is devoted to motivating, developing, and applying wave mechanics. By contrast, only one of twenty-two chapters is devoted to matrix mechanics (pp. 361-386), and even then it is "derived from" wave mechanics and interpreted along the lines of wave mechanics. A careful reading reveals that the pre-eminent influence on Bohm is the metaphysical assumption of the fundamental character of the notion of a continuum. It is reflected in Bohm's fundamental mathematical model for elements of reality, the notion of a wave packet. That model and the physical and philosophical assumptions made in connection with it (that all actual physical systems are essentially representable as such) place Bohm firmly in the tradition of de Broglie and Schrödinger.

It is worth elaborating the points made in the previous paragraph for what follows. Bohm's text begins with two chapters that summarize the key elements of the old quantum theory. There is much discussion of the nature of light, including a discussion of Maxwell's equations and Fourier analysis. The third chapter, entitled "Wave Packets and De Broglie Waves," is transitional. It begins with a discussion of a pulse of light and its mathematical representation as a concentrated wave packet; that is, as

> …a group of waves of slightly different wavelengths, with phases and amplitudes so chosen that they interfere constructively over a small region of space, outside of which they produce an amplitude that reduces to zero rapidly as a result of destructive interference (1951, p. 60).

The chapter then informally discusses de Broglie's matter-wave hypothesis and Schrödinger's development of it. In the next two chapters, the probability interpretation of matter waves is introduced and informally discussed. In the chapter following those, Bohm contrasts wave and particle properties of matter. Wave packets are clearly regarded as fundamental. A rather telling passage occurs in Bohm's discussion of a measurement of the position of an electron: "What appears to have taken place is that when the position of the electron is observed, the wave function suffered a collapse from a broad front down to a narrow region" (1951, p. 120). Bohm's intention is to characterize some of the key elements of measurement processes.[6] What is important for the purpose at hand is that he represents the state of the electron as a wave packet having a nonzero finite width both before the measurement interaction occurs and after it is completed.[7] There are more explicit statements occurring later in the text to the effect that a physical system must be represented in this manner. This passage is worth mentioning because it shows that Bohm does not regard the effect of a position measurement as being exceptional.

Bohm presents the mathematical formulation of QM in the second part of the text, which begins at chapter 9. In sections 5-7 of that chapter, he introduces what he regards as acceptable criteria for wave functions. Square integrability is the first one mentioned, but then he indicates that a further restriction may be introduced on physical grounds; namely, that a wave function is

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[6] This passage is, for Bohm, an incomplete account of such processes. He clearly indicates that a full account must include a quantum mechanical description of the measuring device and its interaction with the measured system, and refers the reader to the penultimate chapter where such an account is given using the wave-packet model.

[7] Of course, this must be qualified, given the previous note, as being applicable only as long as the system is not interacting with some other system. Also, this qualification does not suffice to explain away the measurement problem since interactions often lead to entangled states, and those states tend to persist when the interaction ceases.

physically acceptable only if an average value exists for each observable physical quantity. This he takes to mean—in light of his discussion of products of observable operators later in the chapter—that a wave function and its nth-derivatives (for each natural number n) must vanish faster than any inverse power of x as x goes to infinity.

In sections 13-22 of chapter 10, Bohm discusses plane-wave functions and δ-functions. Concerning plane-wave functions, he says that they are used "with the understanding that they really refer to packets which are very broad in position space and very narrow in momentum space," and similar remarks are made about δ-functions, except that the roles of position and momentum are switched (1951, pp. 211-214). These remarks are important because Bohm clearly does not intend to weaken the square-integrability requirement for physical states, unlike the texts that are mentioned above in connection with Dirac's approach. That there is such a difference with respect to this issue should come as no surprise since Schwartz did not provide the first rigorous definition of δ-functions until 1951—the texts discussed above were published a sufficient interval after Schwartz's formulation.

The considerations above clearly indicate the nature of the mathematical space corresponding to Bohm's set of physically acceptable wave functions. It is the space of continuous, complex-valued functions having continuous first derivatives which together with all their derivatives are of rapid decrease. This space satisfies the conditions of a vector space. The standard inner product, the integral of a product of a pair of functions, is well defined on this space since its elements belong to the space of square-integrable functions. It is not complete because sequences of functions in the space can easily be defined that converge (with respect to the standard norm) on a discontinuous function that is not in the space. Finally, the space is separable since the set of simple functions (which are characterized in any text on Lebesgue integration) is countable and it is dense in Bohm's set of functions. That is to say, this set of functions together with the standard inner product on the associated Hilbert space constitutes a PHS.

Bohm's "Hidden Variables" Interpretation of QM. Bohm's initial presentation of BI in (Bohm 1952) is referred to there as a "hidden variables" interpretation of QM. Bohm regards it as providing an account of elements of reality

> …which in principle determine the precise behavior of an individual system, but which are in practice averaged over in measurements of the type that we can now carry out (1952, p. 166).

To ensure a precise determination (i.e., a deterministic account) of reality, certain assumptions about the nature of wave functions are necessary.[8] What is interesting is that those assumptions are grounded in an assumption about the nature of physical reality:

> Since ψ is now being regarded as a mathematical representation of an objectively real force field, it follows that (like the electromagnetic field) it should be everywhere finite, continuous, and single valued (1952, p. 173).

As in (Bohm 1951), a wave function ψ is regarded as corresponding to something real. In the passage above it is regarded as corresponding to a force field, which is quite different from the way in which it is regarded in that earlier work. What is crucial is that because it is regarded as corresponding to something real, it is assumed to have certain associated mathematical features that are very similar (though not equivalent) to those characterized in (Bohm 1951).[9]

More to the point, the mathematical features specified by Bohm in the passage above are really necessary in order for the wave function to perform its double duty in (Bohm 1952), serving both as a representation of a quantum potential that precisely guides a material point-particle and as a probability amplitude. By contrast, it can serve the single role of a probability amplitude (meaning that it can specify a unique real number—see the discussion of SHSs above), which is the only role specified in (Bohm 1951), if it is merely a representative element of an

[8] It is important to mention in connection with this metaphysical assumption that Bohm later developed a stochastic—meaning essentially indeterministic—hidden variables interpretation of QM. See sections (Cushing 1994, chapter 9) for a characterization of this stochastic interpretation and others that are similar to it. The stochastic Bohmian interpretation is not considered in this essay.

equivalence class of square-integrable functions. Only the additional duty of serving to represent a quantum potential really requires the use of stronger constraints, particularly that of being represented by a continuous function (as opposed to an equivalence class of functions). It is worth noting that those constraints were already in place in (Bohm 1951), which suggests that they played a substantial role in facilitating the conceptual transition from the interpretation of the wave function there to the one given in (Bohm 1952).

It is worth dwelling on one of the points mentioned above. In the SHS formulation of QM, an equivalence class of square-integrable functions that is associated with a system at a given time can serve the second role characterized above in connection with BI only if it is assumed that there is one function in the class that corresponds to the real state of the system. Otherwise, there can be no precise determination of the system since it is possible to assign a continuum of different values to the quantum potential at any spatial point at any given time, depending on the element of the equivalence class that is chosen to represent the potential. But this means from a physical point of view that functions belonging to an equivalence class are distinguishable in principle. That is to say, in BI differences on a set of measure zero correspond to real physical differences that could in principle show up in measurements that cannot now be carried out, but may be possible in the future. This indicates that the partitioning into equivalence classes is not appropriate for BI, meaning that the associated mathematical space cannot be an SHS.

A "Dressed" PHS is Still a PHS. As noted earlier, the main difference between SHS and PHS formulations of QM is the completeness property: An SHS has it, a PHS does not. That difference is crucial because completeness plays an essential role in proving the existence of key geometric properties of infinite-dimensional SHSs (von Neumann 1955, pp. 46-59), which have to do with the existence and completeness of orthonormal sets of vectors. This suggests that a good analogy with formulating QM in PHS rather than in SHS is formulating analytic geometry using the rational numbers rather then the real numbers. So, although BI is a viable interpretation

[9] These criteria for a physically acceptable wave-function are precisely the ones listed in (Pauling and Wilson 1935, pp. 58-59)—a widely used and very influential text in its day that is cited several times in (Bohm 1951), and with which it shares a common approach.

of the PHS formulation of QM, its being such does not constitute a compelling argument for sustaining serious interest in developing BI.

One way to attempt to make a more compelling argument for developing BI might be to keep the SHS framework and distinguish a suitable subspace—the term “subspace” is being used in a broad sense here meaning a closed linear space, which may or may not be a Hilbert space, that is contained in a larger space—as corresponding to the state space of wave functions, such as the space of functions specified by Bohm or a space of test functions. The physical theory is presumably formulated in the chosen subspace, meaning that only vectors in that subspace are tied to the physical world. The others remain as purely mathematical elements that serve as a useful analytical tool for the analysis of the physical theory (Berndl 1996, pp. 79-80).

One thing that should be clear is that the subspace is the essential mathematical framework, not the SHS. The formulation is a PHS formulation which is mathematically dressed. It does not correspond to the standard formulation of QM. There are ineliminable, fundamental syntactic and semantic differences: a partitioning of the equivalence classes of functions into two distinct sets (syntactic), with only one of the two sets being assigned physical counterparts (semantic). So, the claim that BI does not provide a suitable interpretation of the standard mathematical formulation of QM still holds. The force of that claim may be cushioned somewhat by the pragmatic move, but the extent to which that is so is unclear.

One problem with the move is that the partitioning characterized above would not in itself be sufficient for BI to function properly. Equivalence classes that have a continuous function as one of its members have a continuum of members that differ from that function on a set of measure zero. So, not only must there be a partitioning of the Hilbert space, there must also be a selection of a particular function (the continuous one) from the reduced set of equivalence classes.It seems appropriate to say that this move from the reduced set of equivalence classes to elements of the class having specific mathematical features is a move out of the SHS framework.

Another problem, leaving aside the first for the sake of argument, is that suitable constraints would have to be imposed on Hamiltonians to ensure that a function in the subspace that corresponds to the state of a system is not transformed out of that subspace by the corresponding unitary transformation. Reasonable constraints on Hamiltonians must be specified, and the subspace must then be shown to be closed under transformations corresponding to Hamiltonians of that type.

A third problem that besets any PHS formulation of QM is that this formulation creates havoc for matrix mechanics. The crucial question is whether there is a simple, straightforward way to partition normalized column matrices with regards to their physical significance or lack thereof. If so, that must be shown. If not, then there are at least two possible responses. One is to give up matrix mechanics. But that seems to be too extreme given the SHS backdrop in which the PHS formulation is now to be played out. Another is to argue along the following lines: this situation is just an "inversion" of that which arose when von Neumann's strategy for "proving" equivalency was adopted. Von Neumann needed to add to Schrödinger's initial set of physically acceptable wave functions and then partition that enhanced set to prove mathematical equivalency. The inversion keeps Schrödinger's set of functions and "proves" equivalence within the SHS framework by subtracting from Heisenberg's original set of normalized column matrices.

But, there is a crucial difference between these two strategies. The havoc created for wave mechanicsby von Neumann's approach is interpretive (that is, from the standpoint of de Broglie and Schrödinger) but not mathematical, whereas that of the inverted strategy for matrix mechanics is mathematical and not interpretive (the latter being so perhaps only because it really did not have a substantial interpretive framework to begin with, by contrast with wave mechanics).

Böhm's RHS formulation of QM and BI. The disadvantages of the PHS approach considered above suggest that it is appropriate and perhaps necessary to search for an alternative mathematical framework for QM with respect to which BI is better suited. As it turns out, there is a more natural and powerful way of formulating QM, the RHS formulation,[10] for which BI

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[10] A superb introductory article on RHS is (Nagel 1989). For other elementary mathematical presentations of the essentials, see (Böhm and Gadella 1989, chapter 1) and (Bogolubov et al. 1975, chapter 1). A detailed formal presentation is given in (Gel'fand and Vilenkin 1964, chapter 1). Three quantum mechanics texts that use the RHS construction are (Böhm and Gadella 1989), which contains references to the primary literature on the RHS formulation of QM, (Bolgolubov et al. 1975), and (Dubin and Hennings 1990). For a closely related approach, see (Van Eijndhoven and de Graaf 1986).

may be viable. Moreover, it has both mathematical and pragmatic advantages. One advantage is that the partitioning of the SHS introduced above for BI, which is done in a completely ad hoc manner, is an essential part of the structure of an RHS. In addition, there are other advantages of the RHS approach to QM over the SHS approach that are independent of considerations involving BI. Before characterizing some of these advantages, it is appropriate to say a bit more about the historical context.

The RHS construction emerged from Schwartz's efforts to provide a rigorous foundation for the delta function. A rigorous definition of the delta-function first became possible with his development of distribution theory during 1945-1949. He succeeded in formulating its definition during 1950-1951 (Schwartz 1966). These developments inspired Gel'fand and collaborators during 1955-1959 to develop a new mathematical structure, the notion of a rigged Hilbert space (Gel'fand and Vilenkin 1964), which later made it possible to associate a mathematical structure with Dirac's formulation of QM introduced independently by Arno Böhm and J.E. Roberts—see (Böhm 1966) and (Roberts 1966).[11] Now, consider some of the advantages of this approach.

In an RHS, an inner product operation is defined using two spaces: one a nuclear Fr chet space,[12] and the other the topological dual of (the set of all continuous linear functionals on) the first. One particularly nice nuclear Fr chet space is the test-function space S consisting of all infinitely differentiable functions which vanish faster than any inverse power of x.[13] It's dual space S′ is the space of tempered distributions, which contains a complete set of eigenvalues and generalized eigenvectors for position and momentum. This illustrates one important advantage of

[11]Since the introduction of Schwartz's theory, the majority of standard physics texts use delta functions and plane-wave functions and justify doing so on mathematical grounds by appeal to Schwartz's theory of distributions. A small number of serious texts diverge from this pattern and actually develop a rigged Hilbert space formulation of QM—such as those mentioned in the previous note—or at least suggest that this is the appropriate foundation, as in (Sudbery 1986).

[12] A Fr chet space is a complete metric space. Hilbert spaces and, more generally, Banach spaces are examples of Fr chet spaces. The type of Fr chet space that is used to define an RHS is a countably-Hilbert space, which is not a Banach space, meaning that it does not have a norm that induces a metric with respect to which the space is complete. It has a set of ordered norms, and all of these norms are used together to define such a metric. Countably-Hilbert spaces are characterized in more detail below. It is most convenient to introduce the notion of a nuclear space in that context.

[13] One reason that it is "particularly nice" is that the operators corresponding to position and momentum are treated on an equal footing in the sense that the Fourier transformation from one to the other is an isomorphism (i.e., a bi-continuous, one-one mapping of the space onto itself). See (Nagel 1989) for an elaboration of this point.

the RHS approach over the SHS approach: in an SHS, the operators corresponding to position and momentum have no eigenvalues or eigenvectors, and these operators are only defined on a dense domain of the vectors of the SHS, meaning that there is a constant worry about domains of definition of products of these operators.

It is important to note that there are many other test function spaces that can be used aside from S.[14] The question as to which concrete realization of an RHS is most suitable for characterizing a physical system will not be addressed here, except to say this freedom of choice (which depends on the initial and boundary conditions of the system, and perhaps others of a semantic or metaphysical nature as with issues involving BI) should be regarded as another very substantial advantage of the RHS formulation of QM over the SHS formulation.[15] It is analogous to that claimed for the algebraic QM over QM in SHS: in algebraic QM, there are unitarily inequivalent representations which serves to explain the existence of superselection, whereas all representations of an SHS are unitarily equivalent.[16]

The advantages mentioned above are mathematical and theoretical.[17] So, it is also worth noting that these advantages have led to new and important physical models, and promise to lead to others. Until very recently, substantial applications of the RHS formulation of QM were restricted to addressing the problems of decaying states and resonances in scattering theory. That is substantial, since it is possible to give rigorous accounts of these phenomena, which has been

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[14] For a brief characterization of the most common test-function spaces, see (Constantinescu 1980, pp. 24-30). For a more extensive classification of types of test-function spaces and a concise characterization of their properties and the relations between them, see (Horvath 1966, pp. 439-442).

[15] For some examples of choices of test-function spaces for the purpose of describing resonances and virtual states in scattering experiments, see (Böhm and Gadella 1989, pp. 53-96). In each of these cases, the test-function space chosen consists of elements that belong both to one of the standard types of test function spaces and to one of the types of Hardy-function spaces.

[16] The allusion here is to a system having an infinite number of degrees of freedom and its associated C*-algebra of observables, such as a quantum field or a many-body system in the thermodynamic limit. For such systems there are unitarily inequivalent representations of the associated algebra in a Hilbert space via a GNS construction (devised by the mathematicians Gel'fand, Naimark, and Segal). The resulting models have proven to be very useful in characterizing many important effects in quantum statistical mechanics and quantum field theory including crystallization, ferromagnetism, superfluidity, structural phase transition, Bose-Einstein condensation, and superconductivity. For more details, see (Emch 1972, pp. 247-253), (Streater and Wightman 1978, pp. 191-198), or (Bratteli and Robinson 1987, pp. 3-15). For a contrasting point of view concerning inequivalent representations, see (Wald 1994, pp. 73-85).

[17] It is worth noting that this theoretical role also extends to the context of the axiomitization of quantum field theory, as in (Bogolubov et al. 1975).

intractable within the standard formulation—see (Böhm and Gadella 1989) for references. Within the last five years, new and important discoveries have been made using this mathematical structure in a broad range of physical domains including chaos theory and statistical mechanics in both their classical and quantum formulations, and in connection with philosophically oriented issues in the physics literature including the measurement problem in quantum mechanics and irreversibility at the macroscopic and microscopic levels.[18]

In light of the advantages of the RHS approach characterized above, it is certainly worth considering how this formulation of QM meshes with BI. To do so, it is necessary to provide more details concerning the construction of an RHS. There are several different construction procedures. Three are considered here: one begins with an SHS in the manner of Gel'fand (1964, pp. 103-119), another begins with a topological semi-inner product space in the manner of Roberts (1966, pp. 108-110), and the third begins with a PHS in the manner of Böhm (1978, pp. 14-34).[19] For BI, Böhm's approach seems to be the best (most natural and easily understandable) of the three. Each is presented below. Technical elements are kept to a minimum, but cannot be completely avoided.

In Gel'fand's approach, an RHS is a triplet of spaces $\langle \Phi, \mathrm{H}, \Phi' \rangle$ known as a "Gel'fand triplet." The crucial elements are Φ and Φ'. The space H is an SHS with the usual norm $\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$, where ψ is any arbitrary element of H. H serves as a starting place for generating Φ and Φ'. As noted earlier, the space Φ is a nuclear countably-Hilbert space and Φ' is its topological dual.[20] To facilitate explaining what this means, denote H by H_0 and $\|\psi\|$ by $\|\psi\|_0$. To say Φ is a countably-Hilbert space means that there is a sequence of non-decreasing norms

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[18] For example, RHSs are used in connection with chaos theory in (Antoniou and Tasaki 1993) and (Suchanecki et al. 1996), in connection with statistical mechanics, macroscopic irreversibility, and the measurement problem in (Petrosky and Prigogine 1997), and in connection with microscopic irreversibility in (Böhm et al. 1997). See these essays for additional sources. These characterizations of irreversibility raise a very interesting and important challenge for BI. It may be that the irreversibility here is essentially so, meaning that a deterministic hidden variables account such as BI must fail.

[19] In his early work (1966, pp. 276-283), Böhm adopts the approach of Gel'fand. A simplified presentation of Böhm's own approach is given in (Böhm and Gadella 1989, pp. 4-30).

[20] See (Gel'fand and Vilenkin 1964, pp. 106-110) or (Böhm 1966, pp. 273-283) for more details. In the earlier work (Gel'fand and Shilov 1968, pp. 53-59), which was translated later, Φ is specified as being a perfect countably-Banach space. The later specification is a refinement of the earlier one since all nuclear spaces are perfect (but not

$$\|\psi\|_0 \le \|\psi\|_1 \le \|\psi\|_2 \cdots$$

which corresponds a sequence of nested Hilbert spaces

$$H_0 \supset H_1 \supset H_2 \cdots$$

for which Φ is by definition the projective limit of this second sequence, meaning that

$$\Phi \equiv \bigcap_{k=0}^{\infty} H_k \,.$$

A countably-Hilbert space is nuclear if and only if the sequence of norms is generated by a self-adjoint operator A in H whose inverse is a compact Hilbert-Schmidt operator, meaning that the trace of its square is finite. If A is such an operator, then it may be used to define a sequence of k-norms $\|h\|_k \equiv \|A^k h\|$ for k=0,1,2… of increasing strength, where h is any arbitrary vector in H_0. The term "nuclear" comes into play here because the square of a compact Hilbert-Schmidt operator, which is used in defining the sequence of norms,[21] is a nuclear operator. The initial space H_0 is identified with H in the Gel'fand triplet, and the set of elements of H with a finite k-norm together with its set of limit elements corresponds to a Hilbert space H_k. The elements of the resulting countable set of Hilbert spaces are suitably nested.

The construction of Φ′ goes in a similar manner with the k-norms replaced by (—k)-norms. The inequality and subset relations are then reversed for increasing k, and Φ′ corresponds to the union of the elements of the corresponding sequence of Hilbert spaces. It is easy to see that H_k and H_{-k} are the dual spaces with respect toone another (for each natural number k). It turns out that Φ is also the topological dual of Φ′, and this means that it is possible to define a generalized inner product operation from Φ and Φ′ into C.[22]

vice versa), and all Hilbert spaces are Banach spaces (but not vice versa). These refinements were introduced in an attempt to develop a canonical construction procedure for rigged Hilbert spaces. It seems (in light of what follows) that this procedure is not regarded as being the definitve one. Another refinement on perfect spaces that is used sometimes is that of being a Montel space—see (Horvath 1966, pp. 231-243) or (Treves 1967, pp. 356-358).

[21] In effect, this is so since $\|h\|_k = \sqrt{\langle h | A^{2k} h \rangle}$.

[22] There are other types of pairs of space on which a generalized inner product may be defined. For example, a generalized inner product may be defined on a pair of Banach spaces, one a set of (equivalence classes of) Lebesgue p-integrable functions, where p is a positive real number greater than one, and the other a set of q-integrable

BI may seem to be no better off in an RHS than it is in an SHS, but it is better off. Suppose, for example, that H corresponds to the space L^2/N (see note 3 above). The elements of the spaces Φ and Φ′ corresponding to this H will then be well defined only up to a set of measure zero. But this difficulty may be avoided in an RHS. Gel'fand shows that a concrete realization of the space Φ can be induced in which the elements correspond to single-valued functions (1964, pp. 110-113). He first proves that for any φ in Φ there is a linear functional f_x in Φ′ such that

$$\phi(x) = f_x(\phi)$$

is true for almost every x (i.e., except for a set of measure zero) on the real line. The concrete realization of Φ, which is really the crucial space for the physics (and especially for BI), is then obtained by associating each φ in Φ with the function (in the equivalence class corresponding to φ) for which the equation above is true for every x, and it is. No such functionals exist in the dual of H (which is just H). So, BI is in fact much better off in the RHS than in the SHS.

Gel'fand's approach seems contrived rather than natural, which may explain why some regard the term "rigged" as suggesting that there is something questionable in the mathematics. So, it is worth emphasizing that the notion of an RHS is as elegant if not more so than an SHS, and that it is certainly more powerful and adaptable. Perhaps either Robert's or Böhm's more natural looking and equally rigorous approaches will serve to supplant this unfortunate connotation with a positive one such as the sense of "rigged" in the phrase "a rigged ship," which suggests the ship is fully equipped.

In the second approach, that of Roberts, the generating space is a topological semi-inner product space. The difference between an inner product and a semi-inner product involves this condition: $\langle\phi|\phi\rangle = 0$ if and only if $\phi = 0$. An inner product must satisfy this condition, a semi-inner product need not. An example of a semi-inner product space is L^2, the space of Lebesgue square-integrable functions (in the literal rather than the conventional sense). This space together with a self-adjoint operator of the appropriate sort (one whose inverse is Hilbert-Schmidt and

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functions, if p and q satisfy this relation: (1/p)+(1/q)=1. See any text on functional analysis, such as (Reed and Simon 1980), for details.

compact) can be used to induce a sequence of semi-normed spaces the projective limit of which is a space of test-functions. The space **Φ** is the completion of this projective limit.

In Böhm's approach, the generating space is a PHS. He focuses on sequence spaces rather than function spaces in presenting the RHS framework in the works cited above. His paragdigm case involves the matrix representation of the quantum harmonic oscillator. The corresponding PHS is the space of infinite sequences of complex numbers having non-zero elements in the first k places only (meaning that there is a zero at place k+1 and beyond), for some natural number k (or other). This PHS is then completed (1) with respect to the standard Hilbert space topology, and (2) with respect to the topology induced by a the sequence of norms generated using successive integral powers of the operator (N+I)—N is the number operator and I is the identity operator, which entails that (N+I) is a self-adjoint operator in the Hilbert space of the harmonic oscillator whose inverse is a compact Hilbert-Schmidt operator. The PHS completion (1) above corresponds to a Hilbert space, and (2) above to a nuclear Fr chet space. [23]

Böhm's paradigm of an RHS is about as simple as it gets, and is definitely worth studying. But it is not revealing with respect to the issues involving BI, the main concern of this essay, without further elaboration. The reason is that sequence spaces do not require passage to a quotient space to define a norm or a countable sequence of norms, unlike the function spaces considered above. Nevertheless, it becomes reasonably clear that Böhm's approach will work for function spaces after noting that the space of continuous Lebesgue square-integrable functions together with the associated Hilbert space norm is a PHS.

Aside from issues concerning the mathematical construction of the RHS framework, there are issues concerning the physical interpretation of that framework. The literature that addresses the question of how to interpret physically the RHS formulation of QM is scant and, as the reader may have anticipated, controversial. Nevertheless, the issue has been addressed, and there is at

[23] A sequence of elements in a countably normed topological vector space P is a Cauchy sequence if and only if the sequence is Cauchy with respect to each norm in the countable sequence of norms. Such a space is easily completed by using its sequence of norms to define a suitable metric—there is a standard construction of this metric from a countable sequence of norms as indicated in (Gel'fand and Shilov 1968, pp. 21-25)—and then taking the union of P and the set of all of its limit elements (a limit element being the element to which a Cauchy sequence converges). A Fr chet space is a complete, separable metric space. An example of such a space is the space of test functions S characterized above.

least one proposal for doing so that appears compatible with BI. Part of the problem is that in the RHS formulation there are two mathematical spaces involved in calculating average values of observables—in the special case mentioned above, it is the space S and its dual S′—instead of just a single space as in the SHS formulation, and those spaces are very different in character. One view is that the elements of S correspond to truly realizable states of an actual systems and that those of S′ correspond to potential experiments (Antoine 1969, p. 2276ff). A dissenting point of view is expressed in (Melsheimer 1972, pp. 902ff). A discussion of the nature of this controversy is beyond the expressed scope of this essay. It is appropriate to briefly consider the promise that Antoine’s physical interpretation of the RHS formulation of QM has to offer with respect to the viability of BI in that formulation.

If the elements of S correspond to the realizable states of actual systems, then those elements can serve the double duty required of them by BI. Clearly, they are suitable elements for defining the type of quantum potential that Bohm specifies in his (1952) since S is a subspace of the space of functions that Bohm characterizes as being suitable. Indeed, if one were to take Schrödinger’s continuum model to its logical conclusion, then S would be one obvious choice in light of the criteria for suitable wave functions that are listed by Bohm in his (1951), especially those from chapter 9 (which are characterized above). The other duty, yielding probabilities for measurement outcomes, is served in conjunction with elements from the space S′.

In light of the considerations above, it is fair to say that the inverted strategy considered in the previous section is “rigged” in a bad sense of the term that means “makeshift or contrived.” By contrast, in the RHS strategy, the term “rigged” is used in a much more positive manner that means “fully equipped.” It is fully equipped to provide a rigorous foundation for an extremely pragmatic but unrigorous formal framework of Dirac. [24] Thus, the RHS approach is a bolder and better strategy for BI.[25]

Concluding Remarks. The main arguments of this essay may now be summarized as follows. A key assumption of BI, the single-valuedness of the wave-function, is incompatible with a central feature of SHSs, a topological feature referred to as "completeness." Thus, BI cannot serve as a viable interpretation of the standard formulation of QM. Since any modification of BI sufficient for viability is likely to distort it beyond recognition, it appears that BI is a failure. But, that is just not so. BI may be bolstered by showing its viability with respect to a PHS formulation of QM. That is not a particularly compelling formulation of QM, but that situation can be improved somewhat. Moreover, the viability of a PHS formulation provides reasonable grounds for supposing that BI may also serve as a viable interpretation of an RHS formulation of QM. I want to suggest in closing why I believe that to be a line of research worth pursuing.

Despite the significant advantages of the RHS formulation of QM that were indicated in the previous section, there has been virtually no attention in the philosophy literature to this approach with regards to the philosophical foundations of physics. This essay is a first step in that direction. The considerations above indicate that the most suitable mathematical framework for developing a Bohmian interpretation of QM is the RHS formulation. It is hoped that this essay will serve to draw more attention to this beautiful and promising mathematical structure and its uses in physics and its foundations.

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[24] It might be argued that the partitioning strategy is not as makeshift relative to the usual SHS formulation of QM, since a similar move must be made in order to deal with unbounded operators, which is a necessity since some of the most important measurable physical quantities correspond to operators of that sort—see (Jordan 1969, pp. 29-33) or (Reed and Simon 1980, pp. 249-312). But, it seems that this argument could just as well be used as an additional argument for considering an RHS formulation of QM which does not require such a partitioning for unbounded observables, and yields in a much simpler account of such observables than that given in an SHS.

[25] Jim Cushing, a major proponent of Bohmian mechanics, has made the following comments in a private communication that are worth noting: "I guess that my overall reaction to your case is (much like your own, it seems to me) that one interested in a Bohmian approach to quantum mechanics would, before too much reflection, opt for PHS and then, if he or she has technical scruples, perhaps for RHS. I suppose my 'guiding' principle would be to do whatever one must with/to the formalism to make it compatible with an objective (i.e., observer-independent) ontology. I do believe it is valuable to make clear, as you have done, the intricate relations between the specific formalism one adopts (even if only informally) and the ontological commitments one hopes to press for." This is a fair assessment, and one with which I agree.

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