ADDING A CLUB WITH FINITE CONDITIONS, PART II

JOHN KRUEGER

ABSTRACT. We define a forcing poset which adds a club subset of a given fat stationary set $S \subseteq \omega_2$ with finite conditions, using S-adequate sets of models as side conditions. This construction, together with the general amalgamation results concerning S-adequate sets on which it is based, is substantially shorter and simpler than our original version in [3].

The theory of adequate sets introduced in [2] provides a framework for adding generic objects on ω_2 with finite conditions using countable models as side conditions. Roughly speaking, an adequate set is a set of models A such that for all M and N in A, M and N are either equal or membership comparable below their comparison point $\beta_{M,N}$. A technique which was central to the development of adequate sets in [2], as well as to our original forcing for adding a club to a fat stationary subset of ω_2 in [3], involves taking an adequate set A and enlarging it to an adequate set which contains certain initial segments of models in A.

In this paper we prove amalgamation results for adequate sets which avoid the method of adding initial segments of models. It turns out that these new results drastically simplify the amalgamation results from [3] for strongly adequate sets. As a result we are able to develop a forcing poset for adding a club to a given fat stationary subset of ω_2 with finite conditions which is substantially shorter than our original argument in [3].

Forcing posets for adding a club to ω_2 with finite conditions were originally developed by Friedman [1] and Mitchell [5], and then later by Neeman [6]. Adequate set forcing was introduced in [2] in an attempt to simplify and generalize the methods used by the first two authors. This new framework is also flexible as it admits useful variations. For example, in a subsequent paper [4] we show that the forcing poset for adding a club presented below can be modified to preserve CH, answering a problem of Friedman [1].

1. Background

For the remainder of the paper assume that (1) $2^{\omega_1} = \omega_2$ and (2) there exists a thin stationary set $\mathcal{Y} \subseteq P_{\omega_1}(\omega_2)$, which means that $\mathcal{Y}$ is stationary and for all $\beta < \omega_2$, $|\{a \cap \beta : a \in \mathcal{Y}\}| \leq \omega_1$. Without loss of generality assume that for all $a \in \mathcal{Y}$ and $\beta < \omega_2$, $a \cap \beta \in \mathcal{Y}$. By (1) we can fix a bijection $\pi^* : \omega_2 \to H(\omega_2)$. Consider the structure $(H(\omega_2), \in, \pi^*)$. The bijection π^* induces definable Skolem functions for this structure. For any set $x \subseteq H(\omega_2)$, let $Sk(x)$ denote the closure of x under these Skolem functions.

2010 *Mathematics Subject Classification.* Primary: 03E40. Secondary: 03E05.

Key words and phrases. Adequate sets, models as side conditions, adding a club with finite conditions.

Let C^* denote the set of $\alpha < \omega_2$ such that $Sk(\alpha) \cap \omega_2 = \alpha$. Easily C^* is a club. Let $\Lambda = C^* \cap \mathrm{cof}(\omega_1)$. Let $\mathcal{X}$ be the set of a in $\mathcal{Y}$ such that $Sk(a) \cap \omega_2 = a$. The set $\mathcal{X}$ is the collection of side conditions which we use in our forcing posets. If x and y are in $\mathcal{X} \cup \Lambda$, a straightforward argument shows that $Sk(x) = \pi^*[x]$ and $Sk(x) \cap Sk(y) = Sk(x \cap y)$. It follows that if $x \in \mathcal{X}$ and $\beta \in \Lambda$, then $x \cap \beta \in \mathcal{X}$.

For $M \in \mathcal{X}$, define Λ_M as the set $\beta \in \Lambda$ such that $\Lambda \cap [\sup(M \cap \beta), \beta) = \emptyset$. Note that for $\beta < \omega_2$, $\beta \in \Lambda_M$ iff $\beta = \min(\Lambda \setminus \sup(M \cap \beta))$.

Lemma 1.1. *The following statements hold:*

1. *If $\beta \in \Lambda$ and $M \in P(\beta) \cap \mathcal{X}$, then $M \in Sk(\beta)$. In particular, if $M \in \mathcal{X}$ and $\beta \in \Lambda$, then $M \cap \beta \in Sk(\beta)$.*
2. *If M and N are in $\mathcal{X}$, then $\Gamma_M \cap \Gamma_N$ has a maximum element. Let $\beta_{M,N} := \max(\Gamma_M \cap \Gamma_N)$.*
3. *$(M \cup \lim(M)) \cap (N \cup \lim(N)) \subseteq \beta_{M,N}$.*

(1) follows from the thinness of $\mathcal{Y}$. See Proposition 1.11 of [2]. (2) is proved in Lemma 2.4 of [2]. The maximum ordinal $\beta_{M,N}$ is called the *comparison point* of M and N. (3) is proved as Proposition 2.6 of [2].

Definition 1.2. *Let A be a subset of $\mathcal{X}$. We say that A is* adequate *if for all M and N in A, either $M \cap \beta_{M,N} \in Sk(N)$, $N \cap \beta_{M,N} \in Sk(M)$, or $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.*

Suppose that $\{M, N\}$ is adequate. If $M \cap \beta_{M,N} \in Sk(N)$ then we write $M < N$. If either $M \cap \beta_{M,N} \in Sk(N)$ or $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ then we write $M \le N$.

Lemma 1.3. *Let $\{M, N\}$ be adequate. Then $M < N$ iff $M \cap \omega_1 < N \cap \omega_1$, and $M \le N$ iff $M \cap \omega_1 \le N \cap \omega_1$.*

The lemma follows easily from the fact that $\omega_1 \le \beta_{M,N}$. Therefore if $\{M, N\}$ is adequate, then the relationship between M and N is determined by their intersections with ω_1. If A is an adequate set, then $M \in A$ is *minimal* in A if $M \cap \omega_1 \le N \cap \omega_1$ for all $N \in A$. If M is minimal, then for all $N \in A$, $M \le N$.

Lemma 1.4. *Suppose that $M < N$. Then $Sk(M \cap \beta_{M,N})$ is a member and a subset of $Sk(N)$. Also every limit point of $M \cap \beta_{M,N}$ and every initial segment of $M \cap \beta_{M,N}$ is in $Sk(N)$.*

This follows from the elementarity of $Sk(N)$, the fact that $Sk(M \cap \beta_{M,N}) = \pi^*[M \cap \beta_{M,N}]$, and $M \cap \beta_{M,N}$ being countable.

Definition 1.5. *Suppose that $\{M, N\}$ is adequate. Define $R_M(N)$ as the set of β satisfying either:*

1. *there is $\gamma \in M \setminus \beta_{M,N}$ such that $\beta = \min(N \setminus \gamma)$, or*
2. *$N \le M$ and $\beta = \min(N \setminus \beta_{M,N})$.*

Note that if $M < N$ then the ordinal $\min(N \setminus \beta_{M,N})$ is not required to be in $R_M(N)$. The elements of $R_M(N)$ are called *remainder points of N over M*. The set $R_M(N)$ is finite; for a proof see Proposition 2.9 of [2]. If A is adequate and $N \in A$, let $R_A(N) = \bigcup\{R_M(N) : M \in A\}$. Let $R_A = \bigcup\{R_M(K) : M, K \in A\}$.

For the purposes of adding a club to a fat stationary set, we need a stronger version of adequate. The next property was called strongly adequate in [3].

Definition 1.6. *Let S be a subset of ω_2 such that $S \cap \mathrm{cof}(\omega_1)$ is stationary and is a subset of Λ. A set $A \subseteq \mathcal{X}$ is S-*adequate *if A is adequate and for all M and N in A, $R_M(N) \subseteq S$.*

If A is S-adequate, $N \in \mathcal{X}$, and $A \subseteq Sk(N)$, then easily $A \cup \{N\}$ is S-adequate.

Below we record some technical facts, most of which follow by elementary arguments from the definitions. The reader would benefit by proving these results as a warm up before proceeding. Any difficulties in doing so can be remedied by reading Sections 1–3 of [2].

Lemma 1.7. *Let K, L, and M be in $\mathcal{X}$.*

1. *If $M \subseteq L$ then $\Lambda_M \subseteq \Lambda_L$. Hence $\beta_{K,M} \leq \beta_{K,L}$.*
2. *If $L \subseteq \beta$ and $\beta \in \Lambda$, then $\Lambda_L \subseteq \beta + 1$. Therefore $\beta_{K,L} \leq \beta$.*
3. *If $\beta < \beta_{K,L}$ and $\beta \in \Lambda$, then $K \cap [\beta, \beta_{K,L}) \neq \emptyset$.*
4. *Suppose that $K \cap \beta_{K,M} \subseteq L$. Then $\beta_{K,M} \leq \beta_{L,M}$.*

Proof. (4) By definition $\beta_{K,M} \in \Lambda_M$. By our assumptions, $\sup(K \cap \beta_{K,M}) \leq \sup(L \cap \beta_{K,M})$. Since $\beta_{K,M} \in \Lambda_K$, $\beta_{K,M} = \min(\Lambda \setminus \sup(K \cap \beta_{K,M}))$. So clearly $\beta_{K,M} = \min(\Lambda \setminus \sup(L \cap \beta_{K,M}))$. Hence $\beta_{K,M} \in \Lambda_L$. Since $\beta_{L,M}$ is maximal in $\Lambda_L \cap \Lambda_M$, $\beta_{K,M} \leq \beta_{L,M}$. □

Lemma 1.8. *Let M and N be in $\mathcal{X}$ and assume that $\{M, N\}$ is adequate.*

1. *If there is $\zeta \in M \setminus N$ with $\zeta < \beta_{M,N}$, then $N < M$.*
2. *If $M \leq N$ then $M \cap \beta_{M,N} = M \cap N$.*
3. *If $\beta < \beta_{M,N}$ and $\beta \in \Lambda$, then $(M \cap N) \setminus \beta \neq \emptyset$.*

2. Amalgamation of S-adequate sets

The basic method for preserving cardinals when forcing with side conditions is the amalgamation of conditions over elementary substructures. In this section we prove general results for amalgamating S-adequate sets over countable structures and structures of size ω_1. This material is a simplification of the analogous results from [3].

Lemma 2.1. *Let A be an adequate set. Let $\zeta \in R_A$ and $K \in A$ with $K \setminus \zeta \neq \emptyset$. Then $\min(K \setminus \zeta) \in R_A$.*

Proof. The proof splits into a large number of cases. Fix M and L in A such that $\zeta \in R_M(L)$. Let $\sigma := \min(K \setminus \zeta)$ and we will show that $\sigma \in R_A$. If $\zeta = \sigma$, then we are done since $\zeta \in R_A$. So assume that $\zeta < \sigma$. Then $\zeta \notin K$. If $\beta_{K,L} \leq \zeta$, then since $\zeta \in L$, $\sigma \in R_L(K)$ and we are done. So assume that $\zeta < \beta_{K,L}$. As $\zeta \in L \setminus K$, it follows that $K < L$.

If $\beta_{K,L} \leq \sigma$, then $\sigma = \min(K \setminus \beta_{K,L})$, so $\sigma \in R_L(K)$ and we are done. So assume that $\sigma < \beta_{K,L}$. Since $K \cap \beta_{K,L} \subseteq L$, it follows that $\sigma \in L$ and $K \cap \sigma \subseteq L$.

Case 1: $L \leq M$. Then $K < M$. Since $\zeta \in R_M(L)$, $\beta_{L,M} \leq \zeta$. So $\sigma \in L \setminus \beta_{L,M}$, and hence $\sigma \notin M$. Therefore $\sigma \in K \setminus M$. Since $K < M$, this implies that $\beta_{K,M} \leq \sigma$. Hence $K \cap \beta_{K,M} \subseteq L$. By Lemma 1.7(4), $\beta_{K,M} \leq \beta_{L,M}$.

Subcase 1.1: $\zeta = \min(L \setminus \gamma)$ for some $\gamma \in M \setminus \beta_{L,M}$. Since $\beta_{K,M} \leq \beta_{L,M}$, $\gamma \in M \setminus \beta_{K,M}$ and easily $\sigma = \min(K \setminus \gamma)$. So $\sigma \in R_M(K)$.

Subcase 1.2: $\zeta = \min(L \setminus \beta_{L,M})$. Let $\sigma' := \min(K \setminus \beta_{K,M})$, which is in $R_M(K)$. We claim that $\sigma' = \sigma$. Since $K \cap \zeta \subseteq L$, $K \cap [\beta_{L,M}, \zeta) = \emptyset$. So if $\sigma' < \sigma$, then $\sigma' < \beta_{L,M}$. But then $\sigma' \in L \cap \beta_{L,M}$ and hence $\sigma' \in M$ since $L \leq M$. Hence $\sigma' \in (K \cap M) \setminus \beta_{K,M}$, which is impossible.

Case 2: $M < L$. Then there is $\gamma \in M \setminus \beta_{L,M}$ such that $\zeta = \min(L \setminus \gamma)$. Since $K \cap \sigma \subseteq L$, $\sigma = \min(K \setminus \gamma)$. So if $\beta_{K,M} \leq \gamma$, then $\sigma \in R_M(K)$. Assume that $\beta_{K,M} > \gamma$. Then since $\gamma \in M \setminus K$, $K < M$. If $\beta_{K,M} \leq \sigma$, then $\sigma = \min(K \setminus \beta_{K,M})$ and hence $\sigma \in R_M(K)$. Assume that $\beta_{K,M} > \sigma$. Then $\sigma \in M$. But then $\sigma \in (L \cap M) \setminus \beta_{L,M}$, which is impossible. □

For the rest of the section assume that S is a subset of ω_2 such that $S \cap \mathrm{cof}(\omega_1)$ is stationary and is a subset of Λ.

The next result describes the amalgamation of adequate sets over countable models, and replaces the material of 2.2-2.11 of [3].

Proposition 2.2. *Let A be adequate and let $N \in A$. Let B be adequate and assume that $A \cap Sk(N) \subseteq B \subseteq Sk(N)$. Suppose that:*

1. *for all $M < N$ in A, there is $M' \in B$ such that $M \cap \beta_{M,N} = M' \cap \beta_{M,N}$;*
2. *there is N' in B such that $R_A(N) \subseteq R_B(N')$;*
3. *for all $M < N$ in A, $M' < N'$ and $\beta_{M,N} = \beta_{M',N'}$.*

Then $A \cup B$ is adequate and $R_{A \cup B} = R_A \cup R_B$. Therefore if A and B are S-adequate, then so is $A \cup B$.

Proof. Let $M \in A$ and $L \in B$. We will prove that either $M \leq L$ or $L < M$, and $R_L(M)$ and $R_M(L)$ are subsets of $R_A \cup R_B$.

First suppose that $N \leq M$. Since $L \in Sk(N)$, $\beta_{L,M} \leq \beta_{M,N}$, and so $L \cap \beta_{L,M} \in Sk(N) \cap Sk(\beta_{M,N}) = Sk(N \cap \beta_{M,N}) \subseteq Sk(M)$. This proves that $L < M$. Let $\zeta \in R_L(M)$. Then there exists $\gamma \in L \setminus \beta_{L,M}$ such that $\zeta = \min(M \setminus \gamma)$. Since $\gamma \in N \setminus M$ and $N \leq M$, $\beta_{M,N} \leq \gamma$. So ζ is in $R_N(M)$.

Now consider $\zeta \in R_M(L)$. Then ζ is equal to either (a) $\min(L \setminus \beta_{L,M})$, or (b) $\min(L \setminus \gamma)$ for some $\gamma \in M \setminus \beta_{L,M}$. Since $\zeta \in N \setminus M$ and $N \leq M$, $\beta_{M,N} \leq \zeta$. So $\beta_{L,M} \leq \beta_{M,N} \leq \zeta$. Let $\xi := \min(N \setminus \beta_{M,N})$. Since $N \leq M$, $\xi \in R_M(N)$. So by property (2), $\xi \in R_B$. (a) If $\zeta = \min(L \setminus \beta_{L,M})$, then clearly $\zeta = \min(L \setminus \xi)$. Since ξ is in R_B, $\zeta \in R_B$ by Lemma 2.1. (b) If $\gamma \leq \beta_{M,N}$, then again $\zeta = \min(L \setminus \xi)$ so $\zeta \in R_B$. Otherwise $\gamma > \beta_{M,N}$. Then $\gamma \in M \setminus \beta_{M,N}$. Let $\tau := \min(N \setminus \gamma)$. Then $\tau \in R_M(N)$ and hence $\tau \in R_B$. Clearly $\zeta = \min(L \setminus \tau)$, so $\zeta \in R_B$ by Lemma 2.1.

Assume now that $M < N$. Since $L \subseteq N$, $\beta_{L,M} \leq \beta_{M,N}$. As $M \cap \beta_{M,N} = M' \cap \beta_{M,N}$, $M \cap \beta_{L,M} \subseteq M'$. By Lemma 1.7(4), $\beta_{L,M} \leq \beta_{L,M'}$. We claim that either $\beta_{L,M} = \beta_{L,M'}$ or $\beta_{L,M'} > \beta_{M,N}$. Suppose that $\beta_{L,M} < \beta_{L,M'}$. Since $\{L, M'\}$ is adequate, by Lemma 1.8(3) fix $\theta \in (L \cap M') \setminus \beta_{L,M}$. Then $\theta < \beta_{L,M'}$. If $\theta < \beta_{M,N}$, then $\theta \in M' \cap \beta_{M,N} = M \cap \beta_{M,N}$. So $\theta \in (L \cap M) \setminus \beta_{L,M}$, which is impossible. Hence $\beta_{M,N} \leq \theta < \beta_{L,M'}$.

Since B is adequate, either $M' \leq L$ or $L < M'$. Suppose that $M' \leq L$. Then $M' \cap \beta_{L,M'}$ is either equal to $L \cap \beta_{L,M'}$ or is in $Sk(L)$. Since $\beta_{L,M} \leq \beta_{L,M'}$, $M' \cap \beta_{L,M}$ is either equal to $L \cap \beta_{L,M}$ or is in $Sk(L)$. But as $\beta_{L,M} \leq \beta_{M,N}$, $M' \cap \beta_{L,M} = M \cap \beta_{L,M}$. So $M \cap \beta_{L,M}$ is either equal to $L \cap \beta_{L,M}$ or is in $Sk(L)$. Therefore $M \leq L$. Note that L and M compare the same way as do L and M'.

Consider $\zeta \in R_L(M)$. Then ζ is equal to either (a) $\min(M \setminus \beta_{L,M})$ or (b) $\min(M \setminus \gamma)$ for some $\gamma \in L \setminus \beta_{L,M}$. First assume that $\zeta < \beta_{M,N}$. Then $\zeta \in M \cap \beta_{M,N} = M' \cap \beta_{M,N}$. Since $M' \leq L$ and $\zeta \in M' \setminus L$, we must be in the case that $\beta_{L,M} = \beta_{L,M'}$. In case (a), clearly $\zeta = \min(M' \setminus \beta_{L,M'})$, and in case (b), $\gamma \in L \setminus \beta_{L,M'}$ and $\zeta = \min(M' \setminus \gamma)$. In either case, $\zeta \in R_L(M')$. Now assume that $\beta_{M,N} \leq \zeta$. If (a) holds or if (b) holds and $\gamma < \beta_{M,N}$, then clearly

$\zeta = \min(M \setminus \beta_{M,N})$ and hence $\zeta \in R_N(M)$. Otherwise (b) holds and $\beta_{M,N} \leq \gamma$. Since $\gamma \in N$, $\zeta \in R_N(M)$.

Now let $\zeta \in R_M(L)$. Then either (a) $L \cap \beta_{L,M} = M \cap \beta_{L,M}$ and $\zeta = \min(L \setminus \beta_{L,M})$ or (b) $\zeta = \min(L \setminus \gamma)$ for some $\gamma \in M \setminus \beta_{L,M}$. Assume (a). Then $L \cap \beta_{L,M'} = M' \cap \beta_{L,M'}$. If $\beta_{L,M} = \beta_{L,M'}$, then $\zeta = \min(L \setminus \beta_{L,M'})$ and hence is in $R_{M'}(L)$. Otherwise $\beta_{M,N} < \beta_{L,M'}$. Note that ζ cannot be below $\beta_{M,N}$, because otherwise it would be in $L \cap \beta_{M,N} = M' \cap \beta_{M,N} = M \cap \beta_{M,N}$, and hence in $(L \cap M) \setminus \beta_{L,M}$, which is impossible. Therefore $\zeta = \min(L \setminus \beta_{M,N})$. By Lemma 1.8(3), $L \cap M'$ meets the interval $[\beta_{M,N}, \beta_{L,M'})$. Hence $\zeta < \beta_{L,M'}$. Then $\zeta = \min(M' \setminus \beta_{M,N})$. But $\beta_{M,N} = \beta_{M',N'}$ and $M' < N'$. So $\zeta \in R_{N'}(M')$.

Now assume (b). First suppose that $\beta_{L,M} = \beta_{L,M'}$. Then $\gamma \in M \setminus \beta_{L,M'}$. If $\gamma < \beta_{M,N}$, then $\gamma \in M'$. So $\zeta = \min(L \setminus \gamma)$ and $\gamma \in M' \setminus \beta_{L,M'}$, and hence ζ is in $R_{M'}(L)$. Otherwise $\gamma \in M \setminus \beta_{M,N}$. Let $\xi := \min(N \setminus \gamma)$. Then $\xi \in R_M(N)$ and hence $\xi \in R_B$. Clearly $\zeta = \min(L \setminus \xi)$, so $\zeta \in R_B$ by Lemma 2.1. Now assume that $\beta_{M,N} < \beta_{L,M'}$. We claim that $\gamma \geq \beta_{M,N}$. Otherwise $\gamma \in M \cap \beta_{M,N} = M' \cap \beta_{M,N}$. So $\gamma \in M' \cap \beta_{L,M'}$, and since $M' \leq L$, this implies that $\gamma \in L$. But then $\gamma \in (L \cap M) \setminus \beta_{L,M}$ which is impossible. Since $\gamma \in M \setminus \beta_{M,N}$, $\xi := \min(N \setminus \gamma)$ is in $R_M(N)$ and hence in R_B. Clearly $\zeta = \min(L \setminus \xi)$, so $\zeta \in R_B$ by Lemma 2.1.

In the final comparison, suppose that $L < M'$. Then $L \cap \beta_{L,M'} \in Sk(M')$. Since $\beta_{L,M} \leq \beta_{L,M'}, \beta_{M,N}$, $L \cap \beta_{L,M} \in Sk(M') \cap Sk(\beta_{M,N}) = Sk(M' \cap \beta_{M,N}) = Sk(M \cap \beta_{M,N}) \subseteq Sk(M)$. So $L < M$. Let $\zeta \in R_L(M)$ be given. Then $\zeta = \min(M \setminus \gamma)$ for some $\gamma \in L \setminus \beta_{L,M}$. If $\gamma \geq \beta_{M,N}$, then $\zeta \in R_N(M)$. Suppose that $\gamma < \beta_{M,N}$. If $\beta_{L,M'} > \beta_{M,N}$ then since $L < M'$, $\gamma \in M'$. But then $\gamma \in M' \cap \beta_{M,N} = M \cap \beta_{M,N}$. So $\gamma \in (L \cap M) \setminus \beta_{L,M}$ which is impossible. Therefore $\beta_{L,M} = \beta_{L,M'}$. If $\zeta < \beta_{M,N}$, then clearly $\zeta = \min(M' \setminus \gamma)$, and ζ is in $R_L(M')$. Otherwise $\gamma < \beta_{M,N} \leq \zeta$. Then clearly $\zeta = \min(M \setminus \beta_{M,N})$. Since $M < N$, $\zeta \in R_N(M)$.

Consider $\zeta \in R_M(L)$. Then ζ is equal to either (a) $\min(L \setminus \beta_{L,M})$ or (b) $\min(L \setminus \gamma)$ for some $\gamma \in M \setminus \beta_{L,M}$. First assume that $\beta_{L,M} = \beta_{L,M'}$. Then (a) implies that $\zeta \in R_{M'}(L)$. Suppose (b). If $\gamma < \beta_{M,N}$, then $\gamma \in M \cap \beta_{M,N} = M' \cap \beta_{M,N}$, so $\gamma \in M' \setminus \beta_{L,M'}$. Therefore $\zeta \in R_{M'}(L)$. Assume that $\beta_{M,N} \leq \gamma$. Let $\xi := \min(N \setminus \gamma)$. Since $\gamma \in M$, $\xi \in R_M(N)$ and hence $\xi \in R_B$. Clearly $\zeta = \min(L \setminus \xi)$, so $\zeta \in R_B$ by Lemma 2.1.

Now assume that $\beta_{M,N} < \beta_{L,M'}$. We claim that $\zeta \geq \beta_{M,N}$. Otherwise since $L < M'$, $\zeta \in M' \cap \beta_{M,N} = M \cap \beta_{M,N}$. But then $\zeta \in (L \cap M) \setminus \beta_{L,M}$, which is impossible. In case (a) or in case (b) when $\gamma \leq \beta_{M,N}$, $\zeta = \min(L \setminus \beta_{M,N})$. By Lemma 1.7(3), $\zeta < \beta_{L,M'}$. Recall that $\beta_{M,N} = \beta_{M',N'}$ and $M' < N'$. Let $\tau := \min(M' \setminus \beta_{M,N})$, which is in $R_{N'}(M')$. Then $\zeta = \min(L \setminus \tau)$, so $\zeta \in R_B$ by Lemma 2.1. Suppose case (b) and $\gamma > \beta_{M,N}$. Then $\gamma \in M \setminus \beta_{M,N}$. Let $\xi := \min(N \setminus \gamma)$, which is in $R_M(N)$ and hence in R_B. Then clearly $\zeta = \min(L \setminus \xi)$, so $\zeta \in R_B$ by Lemma 2.1. □

The next proposition decribes the amalgamation of adequate sets over models of size ω_1 and replaces 2.12–2.15 of [3].

Proposition 2.3. *Let A be adequate and $\beta^* \in \Lambda$. Let B be adequate and assume that $A \cap Sk(\beta^*) \subseteq B \subseteq Sk(\beta^*)$. Suppose that there is $\beta < \beta^*$ in Λ such that for all $M \in A$, there is M' in B with $M \cap \beta^* = M' \cap \beta$. Let $r_A = \{\min(M \setminus \beta^*) : M \in A\}$ and $r_B = \{\min(K \setminus \beta) : K \in B\}$. Then $A \cup B$ is adequate and $R_{A \cup B} \subseteq R_A \cup R_B \cup r_A \cup r_B$. In particular, if A and B are S-adequate, $r_A \subseteq S$, and $r_B \subseteq S$, then $A \cup B$ is S-adequate.*

Proof. Let $M \in A$ and $L \in B$. Note that $M \cap \beta^* \subseteq \beta$ and $M \cap \beta = M' \cap \beta$. Since $L \subseteq \beta^*$, $\beta_{L,M} \le \beta^*$ by Lemma 1.7(2). We claim that $\beta_{L,M} \le \beta$. Otherwise $\beta < \beta_{L,M}$, which implies that $M \cap [\beta, \beta_{L,M}) \neq \emptyset$ by Lemma 1.7(3). But then $M \cap [\beta, \beta^*) \neq \emptyset$, which is false. Since $\beta_{L,M} \le \beta$, $M \cap \beta_{L,M} \subseteq M'$. So $\beta_{L,M} \le \beta_{L,M'}$ by Lemma 1.7(4).

We claim that either $\beta_{L,M} = \beta_{L,M'}$ or $\beta_{L,M'} > \beta$. Assume that $\beta_{L,M'} > \beta_{L,M}$. Since $\{L, M'\}$ is adequate, by Lemma 1.8(3) we can fix $\theta \in (L \cap M') \setminus \beta_{L,M}$. Then $\theta \in \beta_{L,M'}$. If $\theta < \beta$, then $\theta \in M' \cap \beta = M \cap \beta$, so $\theta \in (L \cap M) \setminus \beta_{L,M}$, which is impossible. Hence $\beta \le \theta < \beta_{L,M'}$.

Since L and M' are in B, either $M' \le L$ or $L < M'$. Assume that $M' \le L$. Then $M' \cap \beta_{L,M'}$ is either equal to $L \cap \beta_{L,M'}$ or is a member of $Sk(L)$. Since $\beta_{L,M} \le \beta_{L,M'}$, $M' \cap \beta_{L,M}$ is either equal to $L \cap \beta_{L,M}$ or is a member of $Sk(L)$. But as $\beta_{L,M} \le \beta$, $M' \cap \beta_{L,M} = M \cap \beta_{L,M}$. So $M \le L$. Also note that L and M compare the same way as do L and M'.

Let $\zeta \in R_M(L)$. Then either (a) $\zeta = \min(L \setminus \beta_{L,M})$ and $M \cap \beta_{L,M} = L \cap \beta_{L,M}$, or (b) $\zeta = \min(L \setminus \gamma)$ for some $\gamma \in M \setminus \beta_{L,M}$. Assume (a). Then $M' \cap \beta_{L,M'} = L \cap \beta_{L,M'}$. If $\beta_{L,M} = \beta_{L,M'}$ then clearly $\zeta \in R_{M'}(L)$. Otherwise $\beta_{L,M'} > \beta$. Then $L \cap \beta = M' \cap \beta = M \cap \beta$. Since $\zeta \notin M$, $\zeta \ge \beta$. So $\zeta = \min(L \setminus \beta)$ and $\zeta \in r_B$.

Assume (b). Since $L \subseteq \beta^*$ and ζ exists, $\gamma < \beta^*$, and hence $\gamma < \beta$. So $\gamma \in M'$. We claim that $\beta_{L,M} = \beta_{L,M'}$. Otherwise $\beta_{L,M'} > \beta$, so $\gamma \in M' \cap \beta_{L,M'} \subseteq L$. So $\gamma \in (L \cap M) \setminus \beta_{L,M}$, which is impossible. So $\beta_{L,M} = \beta_{L,M'}$. Then $\gamma \in M' \setminus \beta_{L,M'}$, so $\zeta \in R_{M'}(L)$.

Now consider ζ in $R_L(M)$. Then ζ is equal to either (a) $\min(M \setminus \beta_{L,M})$, or (b) $\min(M \setminus \gamma)$ for some $\gamma \in L \setminus \beta_{L,M}$. If $\zeta \ge \beta^*$, then since $\beta_{L,M} \le \beta^*$ and $L \subseteq \beta^*$, $\zeta = \min(M \setminus \beta^*)$. So $\zeta \in r_A$. Otherwise $\zeta < \beta$. Hence $\zeta \in M'$. Since $M' \le L$ and ζ is not in L, $\beta_{L,M} = \beta_{L,M'}$. Hence in either case (a) or (b), $\zeta \in R_L(M')$.

Suppose that $L < M'$. Then $L \cap \beta_{L,M'} \in Sk(M')$. Since $\beta_{L,M} \le \beta_{L,M'}, \beta$, $L \cap \beta_{L,M} \in Sk(M') \cap Sk(\beta) = Sk(M' \cap \beta) \subseteq Sk(M)$. So $L < M$.

Let $\zeta \in R_L(M)$. Then there is $\gamma \in L \setminus \beta_{L,M}$ such that $\zeta = \min(M \setminus \gamma)$. If $\zeta \ge \beta^*$, then since $\gamma < \beta^*$, $\zeta = \min(M \setminus \beta^*)$ and so $\zeta \in r_A$. Otherwise $\zeta < \beta$ and $\zeta \in M'$. Then $\gamma < \beta$. If $\beta < \beta_{L,M'}$, then since $L < M'$, $\gamma \in M' \cap \beta \subseteq M$, which is impossible since $\gamma \ge \beta_{L,M}$. So $\beta_{L,M} = \beta_{L,M'}$. Hence $\zeta \in R_L(M')$.

Finally, consider $\zeta \in R_M(L)$. Then ζ is equal to either (a) $\min(L \setminus \beta_{L,M})$, or (b) $\min(L \setminus \gamma)$ for some $\gamma \in M \setminus \beta_{L,M}$. Assume (a). If $\zeta \ge \beta$, then since $\beta_{L,M} \le \beta$, $\zeta = \min(L \setminus \beta)$ and hence $\zeta \in r_B$. Otherwise $\zeta < \beta$. Since $\zeta \notin M$, $\zeta \notin M'$. As $L < M'$, this implies that $\beta_{L,M} = \beta_{L,M'}$. Hence $\zeta \in R_{M'}(L)$.

Assume (b). Since $L \subseteq \beta^*$ and ζ exists, clearly $\gamma < \beta$. Hence $\gamma \in M'$. If $\beta_{L,M} = \beta_{L,M'}$, then $\gamma \in M' \setminus \beta_{L,M'}$, so $\zeta \in R_{M'}(L)$. Otherwise $\beta_{L,M'} > \beta$. Since $L < M'$, if $\zeta < \beta$ then $\zeta \in M' \cap \beta \subseteq M$, which contradicts that ζ is not in M. So $\zeta \ge \beta$. But $\gamma < \beta$. So $\zeta = \min(L \setminus \beta)$ and hence $\zeta \in r_B$. $\square$

3. Adding a club

Let S be a fat stationary subset of ω_2. That means that for every club $D \subseteq \omega_2$, $S \cap D$ contains a closed subset of order type $\omega_1 + 1$. We will define a forcing poset with finite conditions which preserves cardinals and adds a club subset of S.

Note that since S is fat, $S \cap \mathrm{cof}(\omega_1)$ is stationary. Thinning out S if necessary using fatness, assume that $S \cap \mathrm{cof}(\omega_1) \subseteq \Lambda$ and for all $\alpha \in S \cap \mathrm{cof}(\omega_1)$, $S \cap \alpha$ contains a club subset of α. Let $\mathcal{Z}$ denote the set of N in $\mathcal{X}$ such that $\sup(N) \in S$

and for all $\alpha \in N \cap S$, $\sup(N \cap \alpha) \in S$. A straighforward argument shows that $\mathcal{Z}$ is a stationary subset of $P_{\omega_1}(\omega_2)$.

Pairs of ordinals $\langle \alpha, \alpha' \rangle$ and $\langle \gamma, \gamma' \rangle$ are said to be *overlapping* if either $\alpha < \gamma \le \alpha'$ or $\gamma < \alpha \le \gamma'$; otherwise they are *nonoverlapping*. A pair $\langle \alpha, \alpha' \rangle$ and an ordinal ζ are *overlapping* if $\alpha < \zeta \le \alpha'$, and otherwise are *nonoverlapping*.

Definition 3.1. *Let $\mathbb{P}$ be the forcing poset whose conditions are of the form $p = (x_p, A_p)$ satisfying:*

1. *x_p is a finite set of nonoverlapping pairs of the form $\langle \alpha, \alpha' \rangle$, where $\alpha \le \alpha' < \omega_2$ and $\alpha \in S$;*
2. *A_p is a finite S-adequate subset of $\mathcal{Z}$;*
3. *let $M \in A_p$ and $\langle \alpha, \alpha' \rangle \in x_p$; if $M \cap [\alpha, \alpha'] \neq \emptyset$, then α and α' are in M; if $M \cap [\alpha, \alpha'] = \emptyset$ and $M \setminus \alpha \neq \emptyset$, then $\min(M \setminus \alpha) \in S$;*
4. *if $\zeta \in R_{A_p}$ then ζ is nonoverlapping with any pair in x_p.*

Let $q \le p$ if $x_p \subseteq x_q$ and $A_p \subseteq A_q$.[1]

For a condition p, a pair $\langle \alpha, \alpha' \rangle$ *can be added to* p if $(x_p \cup \{\langle \alpha, \alpha' \rangle\}, A_p)$ is a condition (and in that case is obviously below p).

Let p be a condition and $\zeta \in S$. Then $\langle \zeta, \zeta \rangle$ can be added to p provided that there is no pair $\langle \alpha, \alpha' \rangle$ in x such that $\alpha < \zeta \le \alpha'$, and for any N in A_p such that $\zeta \notin N$ and $N \setminus \zeta \neq \emptyset$, $\min(N \setminus \zeta) \in S$.

In particular, suppose that $\zeta \in R_{A_p}$. Then $\zeta \in S$ and ζ does not overlap any pair in x_p. Also if $N \in A_p$, $\zeta \notin N$, and $N \setminus \zeta \neq \emptyset$, then by Lemma 2.1, $\min(N \setminus \zeta) \in R_{A_p}$, so $\min(N \setminus \zeta) \in S$. It follows that $\langle \zeta, \zeta \rangle$ can be added to p. Consequently there are densely many conditions p satisfying that for all $\zeta \in R_{A_p}$, $\langle \zeta, \zeta \rangle \in x_p$.

If (x, A) satisfies properties (1), (2), and (3), and for all $\zeta \in R_{A_p}$, $\langle \zeta, \zeta \rangle \in x_p$, then p is a condition. For in that case, property (4) follows from property (1).

Let $\dot{D}$ be a $\mathbb{P}$-name such that $\mathbb{P}$ forces

$$\dot{D} = \{\alpha : \exists p \in \dot{G} \; \exists \gamma \; \langle \alpha, \gamma \rangle \in x_p\}.$$

Clearly $\dot{D}$ is forced to be a subset of S. We will show that $\mathbb{P}$ preserves cardinals and forces that $\dot{D}$ is club in ω_2.

Lemma 3.2. *Let p be a condition. Suppose that $\langle \alpha, \alpha' \rangle \in x_p$, $N \in A_p$, $N \cap [\alpha, \alpha'] = \emptyset$, and $N \setminus \alpha \neq \emptyset$. Let $\beta := \min(N \setminus \alpha)$. Then $\langle \beta, \beta \rangle$ can be added to p.*

Proof. Note that $\beta \in S$. Let $\langle \gamma, \gamma' \rangle$ be in x, and suppose for a contradiction that $\gamma < \beta \le \gamma'$. Since $\beta \in N$, $N \cap [\gamma, \gamma'] \neq \emptyset$. Hence γ and γ' are in N. Since $\gamma < \beta$ and $\beta = \min(N \setminus \alpha)$, $\gamma < \alpha$. But then $\gamma < \alpha \le \gamma'$, contradicting that p is a condition.

Suppose that $M \in A_p$, $\beta \notin M$, and $M \setminus \beta \neq \emptyset$. We will show that $\zeta := \min(M \setminus \beta)$ is in S. If $\beta_{M,N} \le \beta$, then since $\beta \in N$, ζ is in $R_N(M)$ and hence in S. Assume that $\beta_{M,N} > \beta$. Then as $\beta \in N \setminus M$, $M < N$. As $\alpha \le \alpha' < \beta < \beta_{M,N}$ and $M \cap \beta_{M,N} \subseteq N$, $M \cap [\alpha, \alpha'] = \emptyset$. So $\min(M \setminus \alpha) \in S$. But easily $\min(M \setminus \alpha) = \zeta$. $\square$

Proposition 3.3. *The forcing poset $\mathbb{P}$ preserves ω_1.*

Proof. Let $p \Vdash \dot{g} : \omega \to \omega_1$ is a function. Fix $\chi > \omega_2$ regular with $\dot{g} \in H(\chi)$. Let N^* be a countable elementary substructure of $H(\chi)$ such that $\mathbb{P}, p, \dot{g}, \pi^*, C^*, \Lambda, \mathcal{X}, S, \mathcal{Z} \in$

[1] The difference between this forcing poset and the one we defined in [3] is the additional requirement (4), and a slightly different definition of pairs overlapping.

N^* and $N := N^* \cap \omega_2 \in \mathcal{Z}$. This is possible as $\mathcal{Z}$ is stationary. Note that since $\pi^* \in N^*$, $N^* \cap H(\omega_2) = \pi^*[N] = Sk(N)$. In particular, $N^* \cap \mathbb{P} \subseteq Sk(N)$.

Let $q := (x_p, A_p \cup \{N\})$. We will prove that q is N^*-generic. It follows that q forces that the range of $\dot{g}$ is contained in N, so $\dot{g}$ does not collapse ω_1. Fix a dense set $D \in N^*$, and we will show that $N^* \cap D$ is predense below q.

Let $r \leq q$. Extending r if necessary using Lemma 3.2, assume that whenever $\langle \alpha, \alpha' \rangle \in x_r$, $M \in A_r$, $M \cap [\alpha, \alpha'] = \emptyset$, and $M \setminus \alpha$ is nonempty, then $\langle \min(M \setminus \alpha), \min(M \setminus \alpha) \rangle \in x_r$. Similarly assume that for all $\zeta \in R_{A_r}$, $\langle \zeta, \zeta \rangle \in x_r$.

Let $M_0, \ldots, M_k$ list the sets M in A_r with $M < N$. For each $i \leq k$, $\beta_{M_i,N} \in \Lambda_{M_i}$ implies that $\beta_{M_i,N} = \min(\Lambda \setminus (\sup(M_i \cap \beta_{M_i,N})))$, and hence $\beta_{M_i,N} \in N$ by elementarity.

The objects r, N, and $M_0, \ldots, M_k$ witness the following statement: there exists v, N', and $M_0', \ldots, M_k'$ satisfying:

(1) $v \in \mathbb{P}$;
(2) $x_r \cap Sk(N) \subseteq x_v$, $A_r \cap Sk(N) \subseteq A_v$, and $M_0', \ldots, M_k'$ and N' are in A_v;
(3) $R_{A_r}(N) = R_{A_v}(N')$;
(4) for all $i \leq k$, $M_i' < N'$, $M_i \cap \beta_{M_i,N} = M_i' \cap \beta_{M_i,N}$, and $\beta_{M_i,N} = \beta_{M_i',N'}$.

The parameters of the above statement, namely $\mathbb{P}$, $x_r \cap Sk(N)$, $A_r \cap Sk(N)$, $R_{A_r}(N)$, and $M_i \cap \beta_{M_i,N}$ and $\beta_{M_i,N}$ for $i \leq k$, are all members of N^*. By the elementarity of N^*, fix v, N', and $M_0', \ldots, M_k'$ in N^* which satisfy the same statement.

Fix $w \leq v$ in $N^* \cap D$. Extending w if necessary, assume that for all $\zeta \in R_{A_w}$, $\langle \zeta, \zeta \rangle \in x_w$. We will prove that w is compatible with r, which finishes the proof. Define t by letting $x_t := x_r \cup x_w$ and $A_t := A_r \cup A_w$. We will show that t is a condition. Then clearly $t \leq r, w$ and we are done.

(1)–(4) imply that the hypotheses of Proposition 2.2 hold for $A = A_r$ and $B = A_w$. It follows that A_t is S-adequate and $R_t = R_{A_r} \cup R_{A_w}$. So by the choice of r and w, if $\zeta \in R_{A_t}$ then $\langle \zeta, \zeta \rangle \in x_t$. Thus t is a condition provided that requirements (1), (2), and (3) in the definition of $\mathbb{P}$ are satisfied. We already know that (2) is true.

(1) Let $\langle \alpha, \alpha' \rangle \in x_r$ and $\langle \gamma, \gamma' \rangle \in x_w$ be given. Suppose for a contradiction that $\alpha < \gamma \leq \alpha'$. Since $\gamma \in N$, $N \cap [\alpha, \alpha'] \neq \emptyset$. So α and α' are in N, and hence $\langle \alpha, \alpha' \rangle \in x_w$. This contradicts that w is a condition.

Now assume for a contradiction that $\gamma < \alpha \leq \gamma'$. If $N \cap [\alpha, \alpha'] \neq \emptyset$, then α and α' are in N and $\langle \alpha, \alpha' \rangle \in x_w$, which contradicts that w is a condition. Assume that $N \cap [\alpha, \alpha'] = \emptyset$. Let $\zeta := \min(N \setminus \alpha)$. Then by the choice of r, $\langle \zeta, \zeta \rangle \in x_r \cap Sk(N) \subseteq x_w$. But $\gamma < \zeta \leq \gamma'$, which contradicts that w is a condition.

(3, 4) Let $M \in A_w$ and $\langle \alpha, \alpha' \rangle \in x_r$ be given. Assume that $N \cap [\alpha, \alpha'] \neq \emptyset$. Then α and α' are in N, and hence $\langle \alpha, \alpha' \rangle \in x_w$, and we are done since w is a condition. Assume that $N \cap [\alpha, \alpha'] = \emptyset$. As $M \in Sk(N)$, $M \cap [\alpha, \alpha'] = \emptyset$. Suppose that $M \setminus \alpha \neq \emptyset$. Let $\zeta := \min(N \setminus \alpha)$. Then $\zeta \in S$, and by the choice of r, $\langle \zeta, \zeta \rangle \in x_r \cap Sk(N) \subseteq x_w$. If $\zeta \in M$ then $\min(M \setminus \alpha) = \zeta$, which is in S. Otherwise $\zeta \notin M$, so $\min(M \setminus \zeta) \in S$ since w is a condition. But $\min(M \setminus \zeta) = \min(M \setminus \alpha)$.

Now let $M \in A_r$ and $\langle \alpha, \alpha' \rangle \in x_w$ be given. First suppose that $M \cap [\alpha, \alpha'] = \emptyset$ and $M \setminus \alpha$ is nonempty. Let $\zeta := \min(M \setminus \alpha)$. Note that $\zeta = \min(M \setminus \alpha')$. If $\beta_{M,N} \leq \alpha'$, then since $\alpha' \in N$, $\zeta \in R_N(M)$ and hence $\zeta \in S$. Suppose that $\alpha' < \beta_{M,N}$. Then since $\alpha' \in N \setminus M$, $M < N$. If $\beta_{M,N} \leq \zeta$, then $\zeta = \min(M \setminus \beta_{M,N})$. So ζ is in $R_N(M)$ and hence in S. Finally suppose that $\beta_{M,N} > \zeta$. Then $M' \cap [\alpha, \alpha'] = M \cap [\alpha, \alpha'] = \emptyset$ and $\zeta = \min(M' \setminus \alpha)$. So $\zeta \in S$ since w is a condition.

Now suppose that $M \cap [\alpha, \alpha'] \neq \emptyset$, and we will show that α and α' are in M. First assume that there is $\xi \in M \cap [\alpha, \alpha']$ such that $\beta_{M,N} \leq \xi$. Since $\xi \in M$, $\zeta := \min(N \setminus \xi)$ is in $R_M(N)$ and $\alpha < \zeta \leq \alpha'$. Since $\zeta \in R_M(N)$, $\zeta \in R_{A_w}$. But ζ and $\langle \alpha, \alpha' \rangle$ overlap, which contradicts that w is a condition.

Otherwise $M \cap [\alpha, \alpha'] \subseteq \beta_{M,N}$. In particular, $\alpha < \beta_{M,N}$. Suppose that $N \leq M$. Then $\alpha \in M$. If $\alpha' < \beta_{M,N}$, then $\alpha' \in M$ as well. Assume that $\alpha < \beta_{M,N} \leq \alpha'$. Since $N \leq M$, $\zeta := \min(N \setminus \beta_{M,N})$ is in $R_M(N)$ and hence in R_{A_w}. But $\alpha < \zeta \leq \alpha'$, which contradicts that w is a condition.

Finally, assume that $M < N$. Then $M \cap \beta_{M,N} = M' \cap \beta_{M,N}$, so $M' \cap [\alpha, \alpha'] \neq \emptyset$. It follows that α and α' are in M'. So $\alpha \in M$. If $\alpha' < \beta_{M,N}$, then $\alpha' \in M$ as well. Otherwise $\beta_{M,N} \leq \alpha'$. But $\beta_{M,N} = \beta_{M',N'}$ and $M' < N'$. Since $\alpha < \beta_{M,N}$, $\alpha \in M' \cap \beta_{M',N'}$ and hence $\alpha \in N'$. So $N' \cap [\alpha, \alpha'] \neq \emptyset$, which implies that $\alpha' \in N'$. So $\alpha' \in (M' \cap N') \setminus \beta_{M',N'}$, which is impossible. $\square$

Proposition 3.4. *The forcing poset $\mathbb{P}$ preserves ω_2.*

Proof. Let $p \Vdash \dot{g} : \omega_1 \to \omega_2$ is a function. Fix $\chi > \omega_2$ regular such that $\dot{g} \in H(\chi)$. Let $N^* \prec H(\chi)$ be of size ω_1 such that $\mathbb{P}, p, \dot{g}, \pi^*, C^*, \Lambda, \mathcal{X}, S, \mathcal{Z} \in N^*$ and $\beta^* := N^* \cap \omega_2 \in S \cap \mathrm{cof}(\omega_1)$. This is possible since $S \cap \mathrm{cof}(\omega_1)$ is stationary. Note that since $\pi^* \in N^*$, $Sk(\beta^*) = \pi[\beta^*] = N^* \cap H(\omega_2)$. In particular, $N^* \cap \mathbb{P} \subseteq Sk(\beta^*)$.

Let $q := (x_p \cup \{\langle \beta^*, \beta^* \rangle\}, A_p)$. We will show that q is N^*-generic. It follows that q forces that N^* is closed under $\dot{g}$, and hence $\dot{g}$ does not collapse ω_2. So fix a dense open set $D \in N^*$, and we will show that $N^* \cap D$ is predense below q.

Let $r \leq q$ be given. We will find a condition w in $N^* \cap D$ which is compatible with r. Extending r if necessary, assume that for all $\zeta \in R_{A_r}$, $\langle \zeta, \zeta \rangle \in x_r$. Also by Lemma 3.2 assume that whenever $M \in A_r$, $M \setminus \beta^* \neq \emptyset$, and $\xi = \min(M \setminus \beta^*)$, then $\langle \xi, \xi \rangle \in x_r$. Note that if $M \in A_r$ and $M \setminus \beta^* \neq \emptyset$, then $\min(M \setminus \beta^*) \in S$. Also note that for all $\langle \alpha, \alpha' \rangle \in x_r$, if $\alpha < \beta^*$, then $\alpha' < \beta^*$.

Let $M_0, \ldots, M_k$ enumerate A_r. The objects r, β^*, and $M_0, \ldots, M_k$ witness the following statement: there exists v, β, and $M_0', \ldots, M_k'$ satisfying:

(1) $v \in \mathbb{P}$;
(2) $\beta \in S \cap \mathrm{cof}(\omega_1)$ and $\langle \beta, \beta \rangle \in x_v$;
(3) $x_r \cap N^* \subseteq x_v$, $A_r \cap N^* \subseteq A_v$, and $M_0', \ldots, M_k' \in A_v$;
(4) for all $i \leq k$, $M_i \cap \beta^* = M_i' \cap \beta$.

The parameters $\mathbb{P}$, S, $x_r \cap N^*$, $A_r \cap N^*$, and $M_i \cap \beta^*$ for $i \leq k$ are in N^*. By elementarity, fix v, β, and $M_0', \ldots, M_k'$ in N^* which satisfy the same properties.

Extend v to w in $D \cap N^*$. Extending w if necessary, assume that for all $\zeta \in R_{A_w}$, $\langle \zeta, \zeta \rangle \in x_w$, and for all $M \in A_w$, if $\xi = \min(M \setminus \beta)$ then $\langle \xi, \xi \rangle \in x_w$. Let $r_0 = \{\min(M \setminus \beta^*) : M \in A_r\}$ and $r_1 = \{\min(M \setminus \beta) : M \in A_w\}$. Then r_0 and r_1 are subsets of S. So all the hypotheses of Proposition 2.3 are satisfied. It follows that $A_r \cup A_w$ is S-adequate and $R_{A_r \cup A_s} \subseteq R_{A_r} \cup R_{A_w} \cup r_0 \cup r_1$.

Define t by letting $x_t = x_r \cup x_w$ and $A_t = A_r \cup A_w$. We will prove that t is a condition. Then clearly $t \leq r, w$ and we are done. By the choice of r and w, for every $\zeta \in R_{A_t}$, $\langle \zeta, \zeta \rangle \in x_t$. So it suffices to show that t satisfies properties (1), (2), and (3). We already know that (2) holds. For (1) let $\langle \alpha, \alpha' \rangle \in x_w$ and $\langle \gamma, \gamma' \rangle \in x_r$. Then either γ and γ' are both below β^* and $\langle \gamma, \gamma' \rangle \in x_w$, or $\beta^* \leq \gamma$. In either case, the pairs do not overlap.

(3,4) Let $M \in A_w$ and $\langle \alpha, \alpha' \rangle \in x_r \setminus x_w$. Then $\beta^* \leq \alpha$. So $M \cap [\alpha, \alpha'] = \emptyset$ and $\min(M \setminus \alpha)$ does not exist. Now let $M \in A_r \setminus A_w$ and $\langle \alpha, \alpha' \rangle \in x_w$. Then α and

α' are below β^*. First assume that $\beta \leq \alpha$. Since $M \cap \beta^* \subseteq \beta$, $M \cap [\alpha, \alpha'] = \emptyset$ and $\min(M \setminus \alpha) = \min(M \setminus \beta^*)$, which is in S.

Now assume that $\alpha < \beta$. Then since $\langle \beta, \beta \rangle \in x_w$, $\alpha' < \beta$. Suppose that $M \cap [\alpha, \alpha'] = \emptyset$. Since $M \cap \beta = M' \cap \beta$, $M' \cap [\alpha, \alpha'] = \emptyset$. Let $\zeta := \min(M \setminus \alpha)$. If $\zeta < \beta$, then $\zeta = \min(M' \setminus \alpha)$ and hence $\zeta \in S$. Otherwise $\zeta = \min(M \setminus \beta^*)$, which is in S. Now assume that $M \cap [\alpha, \alpha'] \neq \emptyset$. Then $M' \cap [\alpha, \alpha'] \neq \emptyset$. So α and α' are in $M' \cap \beta$ and hence in M. $\square$

Proposition 3.5. *The forcing poset $\mathbb{P}$ forces that $\dot{D}$ is a club.*

Proof. It is easy to see that $\mathbb{P}$ forces that $\dot{D}$ is unbounded. Suppose that p forces that α is a limit point of $\dot{D}$. Let $A_0 := \{K \in A_p : \sup(K \cap \alpha) < \alpha\}$ and $A_1 := \{M \in A_p : \sup(M \cap \alpha) = \alpha\}$. Note that for all M and N in A_1, α is a limit point of both M and N and hence $\beta_{M,N} > \alpha$.

Extending p if necessary, we may assume the following: (1) for all $\zeta \in R_{A_p}$, $\langle \zeta, \zeta \rangle \in x_p$; (2) whenever $\langle \beta, \beta' \rangle \in x_p$, $M \in A_p$, $M \cap [\beta, \beta'] = \emptyset$, and $\xi = \min(M \setminus \beta)$, then $\langle \xi, \xi \rangle \in x_p$; (3) let γ be the largest ordinal such that $\gamma < \alpha$ and $\langle \gamma, \gamma' \rangle \in x_p$ for some γ'; then γ is larger than $\sup(K \cap \alpha)$ for all $K \in A_0$.

If $\langle \alpha, \alpha' \rangle \in x_p$ for some α', then p forces that $\alpha \in \dot{D}$ and we are done. So assume not. Then for all $\langle \xi, \xi' \rangle$ in x_p, ξ and ξ' are either both below or both strictly above α. (3) implies that for all $K \in A_0$ with $K \setminus \alpha \neq \emptyset$, $\min(K \setminus \alpha) = \min(K \setminus \gamma) \in S$. As a result of these observations, if $\alpha \in S$ but we cannot add $\langle \alpha, \alpha \rangle$ to p, then there is $N \in A_1$ such that $\alpha \notin N$, $N \setminus \alpha \neq \emptyset$, and $\min(N \setminus \alpha) \notin S$. Note that if $M \in A_1$ then $\gamma \in M$. For otherwise $\gamma < \min(M \setminus \gamma) < \alpha$ and (2) implies that $\langle \min(M \setminus \gamma), \min(M \setminus \gamma) \rangle$ is in x_p, contradicting the maximality of γ.

Suppose that there is $M \in A_p$ with $\sup(M) = \alpha$. We claim that $\langle \alpha, \alpha \rangle$ can be added to p. By definition of $\mathcal{Z}$, $\alpha \in S$. So if this pair cannot be added, then by the comments above there is $N \in A_1$ such that $\alpha \notin N$ and $\xi := \min(N \setminus \alpha) \notin S$. Since $\beta_{M,N} > \alpha$ and α is not in M nor N, $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Since $\xi \notin M$, $\beta_{M,N} \leq \xi$, so $\xi = \min(N \setminus \beta_{M,N})$. Then ξ is in $R_M(N)$ and hence in S, which is a contradiction. So we may assume that for all $M \in A_1$, $M \setminus \alpha \neq \emptyset$.

Suppose that there is $K \in A_0$ such that $\alpha \in K$. Then $\alpha = \min(K \setminus \alpha)$ is in S as discussed above. We claim that we can add $\langle \alpha, \alpha \rangle$ to p. Otherwise there is $N \in A_1$ with $\alpha \notin N$ and $\min(N \setminus \alpha) \notin S$. Note that $\beta_{K,N} > \alpha$. So as $\alpha \in K \setminus N$, $N < K$. But this is impossible since $\sup(K \cap \alpha) < \alpha = \sup(N \cap \alpha)$. Hence we may assume that for all $K \in A_0$, $\alpha \notin K$.

Suppose that $A_1 = \emptyset$. Then it is easy to see that $\langle \gamma, \alpha \rangle$ can be added to p, which contradicts that p forces that α is a limit point of $\dot{D}$. Namely, this pair does not overlap any pair in x_p by the maximality of γ. And it does not conflict with any $K \in A_0$ with $K \setminus \alpha \neq \emptyset$, since $K \cap [\gamma, \alpha] = \emptyset$ and $\min(K \setminus \alpha) \in S$.

Assume that $A_1 \neq \emptyset$. Let M be a minimal set in A_1 such that the ordinal $\sigma := \min(M \setminus \alpha)$ is minimal amongst all minimal sets in A_1. Suppose first that $\sigma \in S$. By definition of $\mathcal{Z}$, $\alpha = \sup(M \cap \sigma) \in S$. We claim that $\langle \alpha, \alpha \rangle$ can be added to p. If not, then there is $N \in A_1$ such that $\alpha \notin N$ and $\tau := \min(N \setminus \alpha) \notin S$. In particular, $\tau \neq \sigma$. Since $\beta_{M,N} > \alpha$ and α is not in N, $N \leq M$. So by the minimality of M, $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Hence N is also minimal in A_1. By the minimality of σ, $\sigma < \tau$. Since $\sigma \notin N$, $\beta_{M,N} \leq \sigma$. So $\tau = \min(N \setminus \sigma)$. Therefore τ is in $R_M(N)$ and hence in S, which is a contradiction.

Finally, assume that $\sigma \notin S$. We will show that we can add $\langle \gamma, \sigma \rangle$ to p, which contradicts that p forces that α is a limit point of $\dot{D}$. We claim that for all $K \in A_0$ with $K \setminus \alpha \neq \emptyset$, $\tau := \min(K \setminus \alpha) > \sigma$. Since $\tau \in S$ and $\sigma \notin S$, $\tau \neq \sigma$. Assume for a contradiction that $\tau < \sigma$. Then $\sigma = \min(M \setminus \tau)$. So if $\beta_{K,M} \leq \tau$, then σ is in $R_K(M)$ and hence in S which is false. Suppose that $\tau < \beta_{K,M}$. Then since $\tau \in K \setminus M$, $M < K$. But this is impossible since $\sup(K \cap \alpha) < \alpha = \sup(M \cap \alpha)$.

Let us show that $\langle \gamma, \sigma \rangle$ has no conflict with models in A_p. Let $K \in A_0$. If $\sup(K) < \alpha$, then $K \cap [\gamma, \sigma] = \emptyset$ and $\min(K \setminus \gamma)$ does not exist. Otherwise by the last paragraph, $\min(K \setminus \gamma) = \min(K \setminus \alpha) > \sigma$. Hence $K \cap [\gamma, \sigma] = \emptyset$ and $\min(K \setminus \gamma) \in S$. Now let $N \in A_1$. We already observed that $\gamma \in N$. To prove that $\sigma \in N$, by the minimality of M it suffices to show that $\sigma < \beta_{M,N}$. Assume for a contradiction that $\beta_{M,N} \leq \sigma$. Then $\alpha \leq \beta_{M,N} \leq \sigma$, so $\sigma = \min(M \setminus \beta_{M,N})$. Hence σ is in $R_N(M)$ and therefore in S, which is a contradiction.

Let $\langle \beta, \beta' \rangle \in x_p$. Since $\langle \gamma, \gamma' \rangle \in x_p$, it is false that $\beta < \gamma \leq \beta'$. Suppose that $\gamma < \beta \leq \sigma$. Then by the maximality of γ, $\alpha < \beta$. Since $\beta \in S$ and $\sigma \notin S$, $\beta < \sigma$. Then $\beta \notin M$, which implies that $\min(M \setminus \beta) \in S$. But $\min(M \setminus \beta) = \sigma$, which contradicts that σ is not in S. $\square$

References

[1] S.D. Friedman. Forcing with finite conditions. In *Set Theory: Centre de Recerca Matemàtica, Barcelona, 2003-2004, Trends in Mathematics*, pages 285–295. Birkhäuser Verlag, 2006.

[2] J. Krueger. Forcing with adequate sets of models as side conditions. Submitted.

[3] J. Krueger. Strongly adequate sets and adding a club with finite conditions. *Arch. Math. Logic*, 53(1-2):119–136, 2014.

[4] J. Krueger and M.A. Mota. Coherent adequate forcing and preserving CH. Submitted.

[5] W. Mitchell. $I[\omega_2]$ can be the nonstationary ideal on $\mathrm{Cof}(\omega_1)$. *Trans. Amer. Math. Soc.*, 361(2):561601, 2009.

[6] I. Neeman. Forcing with sequences of models of two types. *Notre Dame J. Form. Log.*, 55(2):265–298, 2014.

[7] S. Todorčević. A note on the proper forcing axiom. In *Axiomatic set theory (Boulder, Colo., 1983)*, volume 31 of *Contemp. Math.*, pages 209–218. Amer. Math. Soc., Providence, RI, 1984.

Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203

E-mail address: jkrueger@unt.edu