On observation of position in quantum theory

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Newtonian and Scrödinger dynamics can be formulated in a physically meaningful way within the same Hilbert space framework. This fact was recently used in [1] to discover an unexpected relation between classical and quantum motions that goes beyond the results provided by the Ehrenfest theorem. A formula relating the normal probability distribution and the Born rule was also found. Here the dynamical mechanism responsible for the latter formula is proposed and applied to measurements of macroscopic and microscopic systems. A relationship between the classical Brownian motion and the diffusion of state on the space of states is discovered. The role of measuring devices in quantum theory is investigated in the new framework. It is shown that the so-called collapse of the wave function is not measurement specific and does not require a "concentration" near the eigenstates of the measured observable. Instead, it is explained by the common diffusion of state over the space of states under interaction with the apparatus and the environment. This in turn provides us with a basic reason for the definite position of macroscopic bodies in space.

INTRODUCTION

In a recent paper [1] that serves a foundation for the analysis presented here an important new connection between Newtonian and Schrödinger dynamics was derived. The starting point was a realization of classical and quantum mechanics within the same Hilbert space framework and identification of observables with vector fields on the sphere of normalized states. This resulted in a physically meaningful interpretation of components of the velocity of state that surpassed the Ehrenfest results on the motion of averages. Newtonian dynamics was shown to be the Schrödinger dynamics of a system whose state is constrained to the classical phase space submanifold in the Hilbert space of states. This also resulted in a formula relating the normal probability distribution and the Born rule.

In this paper we continue exploring the implications of the proposed framework. First we show that there is a unique extension of Newtonian dynamics on the classical phase space submanifold to a unitary theory on the entire space of states. This allows us to find a connection between the Brownian motion of a macroscopic particle, the diffusion of state on the projective space CP^{L_2} and the Born rule. It also allows us to make progress in understanding the process of measurement in quantum theory, the meaning of collapse of the wave function, the cause of the classicality of macroscopic bodies and to clarify the role of decoherence in this. To make the paper somewhat self-contained, we begin with a brief review of the results reported in [1].

NEWTONIAN AND SCHRÖDINGER DYNAMICS IN HILBERT SPACE

Macroscopic bodies have well-defined position in space at any time. In quantum mechanics the state of a spinless particle with a known position $\mathbf{a} \in \mathbb{R}^3$ is described by the Dirac delta function $\delta^3_{\mathbf{a}}(\mathbf{x}) = \delta^3(\mathbf{x}-\mathbf{a})$. The map $\omega : \mathbf{a} \longrightarrow \delta^3_{\mathbf{a}}$ provides a one-to-one correspondence between points $\mathbf{a} \in \mathbb{R}^3$ and state "functions" $\delta^3_{\mathbf{a}}$. The set $\mathbb{R}^3$ can be then identified with the set M_3 of all delta functions in the space of state functions of the particle.

The common Hilbert space $L_2(\mathbb{R}^3)$ of state functions of a particle does not contain delta functions. By writing the inner product of functions $\varphi, \psi \in L_2(\mathbb{R}^3)$ as

$$(\varphi, \psi)_{L_2} = \int \delta^3(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\overline{\psi}(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y} \tag{1}$$

and approximating the kernel $\delta^3(\mathbf{x}-\mathbf{y})$ with the Gaussian function one obtains the inner product

$$(\varphi, \psi)_{\mathbf{H}} = \int e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}\varphi(\mathbf{x})\overline{\psi}(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y}. \tag{2}$$

The Hilbert space $\mathbf{H}$ with this inner product contains delta functions and their derivatives. In particular,

$$\int e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}\delta^3(\mathbf{x}-\mathbf{a})\delta^3(\mathbf{y}-\mathbf{a})d^3\mathbf{x}d^3\mathbf{y} = 1. \tag{3}$$

It follows that the set M_3 of all delta functions $\delta^3_{\mathbf{a}}(\mathbf{x})$ with $\mathbf{a} \in \mathbb{R}^3$ form a submanifold of the unit sphere in the Hilbert space $\mathbf{H}$, diffeomorphic to $\mathbb{R}^3$.

The map $\rho_\sigma : \mathbf{H} \longrightarrow L_2(\mathbb{R}^3)$ that relates L_2 and $\mathbf{H}$-representations is given by the Gaussian kernel

$$\rho_\sigma(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\sigma^2}}. \tag{4}$$

In fact,

$$G(\mathbf{x}, \mathbf{y}) = (\rho^*_\sigma\rho_\sigma)(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}, \tag{5}$$

which is consistent with (2). The map ρ_σ transforms delta functions $\delta^3_{\mathbf{a}}$ to Gaussian functions $\widetilde{\delta}^3_{\mathbf{a}} = \rho_\sigma(\delta^3_{\mathbf{a}})$, centered at $\mathbf{a}$. The image M^σ_3 of M_3 under ρ_σ is an embedded submanifold of the unit sphere in $L_2(\mathbb{R}^3)$ made of the functions $\widetilde{\delta}^3_{\mathbf{a}}$. The map $\omega_\sigma = \rho_\sigma \circ \omega : \mathbb{R}^3 \longrightarrow M_3^\sigma$ is a diffeomorphism. In what follows, the obtained realizations will be used interchangeably.

Let $\mathbf{r} = \mathbf{a}(t)$ be a path with values in $\mathbb{R}^3$ and let $\varphi = \delta^3_{\mathbf{a}(t)}$ be the corresponding path in M_3. It is easy to see that the norm $\left\| \frac{d\varphi}{dt} \right\|_H^2$ of the velocity in the space $\mathbf{H}$ is given by

$$\left\| \frac{d\varphi}{dt} \right\|_H^2 = \left. \frac{\partial^2 k(\mathbf{x},\mathbf{y})}{\partial x^i \partial y^k} \right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}} \frac{d\mathbf{a}^i}{dt} \frac{d\mathbf{a}^k}{dt}. \tag{6}$$

Here $k(\mathbf{x},\mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}$, so that

$$\left. \frac{\partial^2 k(\mathbf{x},\mathbf{y})}{\partial x^i \partial y^k} \right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}} = \frac{1}{4\sigma^2} \delta_{ik}, \tag{7}$$

where δ_{ik} is the Kronecker delta symbol. Assuming now that the distance in $\mathbb{R}^3$ is measured in the units of 2σ, we obtain

$$\left\| \frac{d\varphi}{dt} \right\|_H = \left\| \frac{d\mathbf{a}}{dt} \right\|_{\mathbb{R}^3}. \tag{8}$$

It follows that the map $\omega : \mathbb{R}^3 \longrightarrow \mathbf{H}$ is an isometric embedding. Furthermore, the set M_3 is complete in $\mathbf{H}$ so that there is no vector in $\mathbf{H}$ orthogonal to all of M_3. By defining the operations of addition $\oplus$ and multiplication by a scalar $\lambda\odot$ via $\omega(\mathbf{a}) \oplus \omega(\mathbf{b}) = \omega(\mathbf{a}+\mathbf{b})$ and $\lambda \odot \omega(\mathbf{a}) = \omega(\lambda \mathbf{a})$ with ω as before, we obtain M_3 as a vector space isomorphic to the Euclidean space $\mathbb{R}^3$.

The projection of velocity and acceleration of the state $\delta^3_{\mathbf{a}(t)}$ onto the Euclidean space M_3 yields correct Newtonian velocity and acceleration of the classical particle:

$$\left(\frac{d}{dt} \delta^3_{\mathbf{a}}(\mathbf{x}), -\frac{\partial}{\partial x^i} \delta^3_{\mathbf{a}}(\mathbf{x}) \right)_{\mathbf{H}} = \frac{da^i}{dt} \tag{9}$$

and

$$\left(\frac{d^2}{dt^2} \delta^3_{\mathbf{a}}(\mathbf{x}), -\frac{\partial}{\partial x^i} \delta^3_{\mathbf{a}}(\mathbf{x}) \right)_{\mathbf{H}} = \frac{d^2a^i}{dt^2}. \tag{10}$$

The Newtonian dynamics of the classical particle can be now derived from the principle of least action for the action functional S on paths in $\mathbf{H}$, defined by

$$\int k(\mathbf{x},\mathbf{y}) \left[\frac{m}{2} \frac{d\varphi_t(\mathbf{x})}{dt} \frac{d\overline{\varphi}_t(\mathbf{y})}{dt} - V(\mathbf{x}) \varphi_t(\mathbf{x}) \overline{\varphi}_t(\mathbf{y}) \right] d^3\mathbf{x} d^3\mathbf{y} dt, \tag{11}$$

where m is the mass of the particle, V is the potential and $k(\mathbf{x},\mathbf{y}) = e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}$. Namely, under the constraint $\varphi_t(\mathbf{x}) = \delta^3(\mathbf{x} - \mathbf{a}(t))$ the action (11) becomes

$$S = \int \left[\frac{m}{2} \left(\frac{d\mathbf{a}}{dt} \right)^2 - V(\mathbf{a}) \right] dt, \tag{12}$$

which is the classical action functional for the particle. This shows that a classical particle can be considered a constrained dynamical system with the state φ of the particle and the velocity $\frac{d\varphi}{dt}$ of the state as dynamical variables. As shown in [1], a similar realization exists for mechanical systems consisting of any number of classical particles.

Now that Newtonian dynamics is embedded in the framework of Hilbert spaces, let's work from the opposite end and develop a vector representation in quantum theory. This representation will allow us to consider Newtonian and Schrödinger dynamics on an equal footing. The starting point is an identification of quantum observables with vector fields on the space of states. Given a self-adjoint operator $\widehat{A}$ on a Hilbert space L_2 of square-integrable functions (it could in particular be the tensor product space of a many body problem) one can introduce the associated linear vector field A_φ on L_2 by

$$A_\varphi = -i\widehat{A}\varphi. \tag{13}$$

The commutator of observables and the commutator (Lie bracket) of the corresponding vector fields are related in a simple way:

$$[A_\varphi, B_\varphi] = [\widehat{A}, \widehat{B}]\varphi. \tag{14}$$

The field A_φ associated with an observable, being restricted to the sphere S^{L_2} of unit normalized states, is tangent to the sphere.

Under the embedding, the inner product on the Hilbert space L_2 yields the induced Riemannian metric on the sphere S^{L_2}. The projection $\pi : S^{L_2} \longrightarrow CP^{L_2}$ yields the induced Riemannian (Fubini-Study) metric on CP^{L_2}. The resulting metrics can be used to find physically meaningful components of vector fields A_φ associated with observables. Since A_φ is tangent to S^{L_2}, it can be decomposed into components tangent and orthogonal to the fibre $\{\varphi\}$ of the fibre bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$. These components have a simple physical meaning. From

$$\overline{A} \equiv (\varphi, \widehat{A}\varphi) = (-i\varphi, -i\widehat{A}\varphi), \tag{15}$$

one can see that the expected value of an observable $\widehat{A}$ in state φ is the projection of the vector $-i\widehat{A}\varphi \in T_\varphi S^{L_2}$ onto the fibre $\{\varphi\}$. Because

$$(\varphi, \widehat{A}^2\varphi) = (\widehat{A}\varphi, \widehat{A}\varphi) = (-i\widehat{A}\varphi, -i\widehat{A}\varphi), \tag{16}$$

the term $(\varphi, \widehat{A}^2\varphi)$ is the norm of the vector $-i\widehat{A}\varphi$ squared. The vector $-i\widehat{A}_\perp\varphi = -i\widehat{A}\varphi - (-i\overline{A}\varphi)$ associated with the operator $\widehat{A} - \overline{A}I$ is orthogonal to the fibre $\{\varphi\}$. Accordingly, the variance

$$\Delta A^2 = (\varphi, (\widehat{A} - \overline{A}I)^2\varphi) = (\varphi, \widehat{A}_\perp^2\varphi) = (-i\widehat{A}_\perp\varphi, -i\widehat{A}_\perp\varphi) \tag{17}$$

is the norm squared of the component $-i\widehat{A}_\perp\varphi$.

From the Schrödinger equation using the decomposition of $-i\widehat{h}\varphi$ onto the components parallel and orthogonal to the fibre we get

$$\frac{d\varphi}{dt} = -i\overline{E}\varphi + \left(-i\widehat{h}\varphi + i\overline{E}\varphi \right) = -i\overline{E}\varphi - i\widehat{h}_\perp\varphi, \tag{18}$$

where $\overline{E}$ is the expected value of the Hamiltonian $\widehat{h}$ in the state φ. By projecting both sides of this equation by $d\pi$ we obtain

$$\frac{d\{\varphi\}}{dt} = -i\widehat{h}_{\perp}\varphi. \quad (19)$$

From this and the already derived equality $\| - i\widehat{h}_{\perp}\varphi\| = \Delta h$, it follows that the speed of evolution of state in the projective space is equal to the uncertainty of energy. This gives us two physically meaningful components of the velocity vector $\frac{d\varphi}{dt}$, corresponding to the expected value and uncertainty of the Hamiltonian.

It turns out that the orthogonal component $-i\widehat{h}_{\perp}\varphi$ of the velocity can also be decomposed into physically meaningful quantities. More importantly, the embedding $\omega_\sigma = \rho_\sigma \circ \omega$ of $\mathbb{R}^3$ into the space of states $L_2(\mathbb{R}^3)$ together with the vector representation of observables provide us with a bridge between Newtonian to Schrödinger dynamics. To demonstrate this, recall first that the basic relation between the classical and quantum physics is given by the Ehrenfest theorem

$$\frac{d}{dt}(\varphi, \widehat{A}\varphi) = -i(\varphi, [\widehat{A}, \widehat{h}]\varphi). \quad (20)$$

Here $\widehat{A}$ does not depend on t. Compare (20) to another equation that follows from the Schrödinger dynamics:

$$2\left(\frac{d\varphi}{dt}, -i\widehat{A}\varphi\right) = \left(\varphi, \{\widehat{A}, \widehat{h}\}\varphi\right) - \left(\varphi, [\widehat{A}, \widehat{h}]\varphi\right). \quad (21)$$

The Ehrenfest theorem (20) for a time-independent observable amounts to using the imaginary part of (21), i.e., the part with the commutator $[\widehat{A}, \widehat{h}]$. The left hand side of (21) is twice the projection of the velocity of state onto the vector field associated with the observable $\widehat{A}$. The real part of this projection (the term with the anti-commutator $\{\widehat{A}, \widehat{h}\}$) is twice the projection in the sense of Riemannian metric on S^{L_2}. This Riemannian projection will be now used to identify components of the velocity of state.

Suppose that at $t = 0$ a microscopic particle is prepared in the state

$$\varphi_0(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{x}_0)^2}{4\sigma^2}} e^{i\frac{\mathbf{p}_0(\mathbf{x}-\mathbf{x}_0)}{\hbar}}, \quad (22)$$

where σ is the same as in (4) and $\mathbf{p}_0 = m\mathbf{v}_0$ with $\mathbf{v}_0$ being the initial group-velocity of the packet. The set of all initial states φ_0 given by (22) form a 6-dimensional embedded submanifold $M^{\sigma}_{3,3}$ in $L_2(\mathbb{R}^3)$. The map $\Omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow M^{\sigma}_{3,3}$,

$$\Omega(\mathbf{a}, \mathbf{p}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}} \quad (23)$$

is a diffeomorphism from the classical phase space of the particle onto the manifold $M^{\sigma}_{3,3}$. The vectors $\frac{\partial r}{\partial x^\alpha}e^{i\theta}$ and $i\frac{\partial \theta}{\partial p^\beta}re^{i\theta}$ are tangent to the manifold $M^{\sigma}_{3,3}$ at a point φ_0, orthogonal to each other in the induced Riemannian metric and form a basis in the tangent space $T_{\varphi_0}(M^{\sigma}_{3,3})$ at that point. For any path φ_τ with values in $M^{\sigma}_{3,3}$ the norm of velocity vector $\frac{d\varphi}{d\tau}$ is given by

$$\left\|\frac{d\varphi}{d\tau}\right\|^2_{L_2} = \frac{1}{4\sigma^2}\left\|\frac{d\mathbf{a}}{d\tau}\right\|^2_{\mathbb{R}^3} + \frac{\sigma^2}{\hbar^2}\left\|\frac{d\mathbf{p}}{d\tau}\right\|^2_{\mathbb{R}^3}. \quad (24)$$

It follows that under a proper choice of units, the map Ω is an isometry that identifies the Euclidean phase space $\mathbb{R}^3 \times \mathbb{R}^3$ of the particle with the embedded submanifold $M^{\sigma}_{3,3} \subset L_2(\mathbb{R}^3)$ furnished with the induced Riemannian metric. The map Ω is an extension to the phase space of the isometric embedding $\omega_\sigma = \rho_\sigma \circ \omega$ of the space $\mathbb{R}^3$, considered earlier in this section.

To decompose the orthogonal component $-\frac{i}{\hbar}\widehat{h}_{\perp}\varphi$ of the velocity $\frac{d\varphi}{dt}$, notice that the orthogonal vectors $\frac{\partial r}{\partial x^\alpha}e^{i\theta}$ and $i\frac{\partial\theta}{\partial p^\beta}re^{i\theta}$ tangent to $M^{\sigma}_{3,3}$ are also orthogonal to the fibre $\{\varphi\}$. Calculation of the projection of the velocity $\frac{d\varphi}{dt}$ onto the unit vector $-\widehat{\frac{\partial r}{\partial x^\alpha}}e^{i\theta}$ (i.e., the classical space component of $\frac{d\varphi}{dt}$) for an arbitrary Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$ yields

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial x^\alpha}}e^{i\theta}\right)\Bigg|_{t=0} = \left(\frac{dr}{dt}, -\widehat{\frac{\partial r}{\partial x^\alpha}}\right)\Bigg|_{t=0} = \frac{v_0^\alpha}{2\sigma}. \quad (25)$$

Calculation of the projection of velocity $\frac{d\varphi}{dt}$ onto the unit vector $i\widehat{\frac{\partial\theta}{\partial p^\alpha}}\varphi$ (momentum space component) gives

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial\theta}{\partial p^\alpha}}\varphi\right)\Bigg|_{t=0} = \frac{mw^\alpha\sigma}{\hbar}, \quad (26)$$

where

$$mw^\alpha = -\left.\frac{\partial V(\mathbf{x})}{\partial x^\alpha}\right|_{\mathbf{x}=\mathbf{x}_0} \quad (27)$$

and σ is assumed to be small enough for the linear approximation of $V(\mathbf{x})$ to be valid within intervals of length σ.

The velocity $\frac{d\varphi}{dt}$ also contains component due to the change in σ (spreading), which is orthogonal to the fibre $\{\varphi\}$ and the phase space $M^{\sigma}_{3,3}$, and is equal to

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{d\varphi}{d\sigma}}\right)\Bigg|_{t=0} = \frac{\sqrt{2}\hbar}{8\sigma^2 m}. \quad (28)$$

Calculation of the norm of $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ at $t = 0$ gives

$$\left\|\frac{d\varphi}{dt}\right\|^2 = \frac{\overline{E}^2}{\hbar^2} + \frac{\mathbf{v}_0^2}{4\sigma^2} + \frac{m^2\mathbf{w}^2\sigma^2}{\hbar^2} + \frac{\hbar^2}{32\sigma^4 m^2}, \quad (29)$$

which is the sum of squares of the found components. This completes a decomposition of the velocity of state at any point $\varphi_0 \in M^{\sigma}_{3,3}$.

From (25), (26) and a simple consistency check one can see that the phase space components of the velocity of state $\frac{d\varphi}{dt}$ assume correct classical values at any point $\varphi_0 \in M^{\sigma}_{3,3}$. This remains true for the time dependent potentials as well. The immediate consequence of this and the linear nature of the Schrödinger equation is that under the Schrödinger evolution with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x},t)$, the state constrained to $M^{\sigma}_{3,3}$ moves like a point in the phase space representing a particle in Newtonian dynamics. More generally, Newtonian dynamics of n particles is the Schrödinger dynamics of n-particle quantum system whose state is constrained to the phase-space submanifold $M^{\sigma}_{3n,3n}$ of the space $L_2(\mathbb{R}^3)\otimes \ ... \ \otimes L_2(\mathbb{R}^3)$ consisting of tensor product states $\varphi_1 \otimes \ ... \ \otimes \varphi_n$ with φ_k of the form (22).

To complete a review of [1], note that isometric embedding of the classical space M^{σ}_3 into the space of states $L_2(\mathbb{R}^3)$ results in a relationship between distances in $\mathbb{R}^3$ and the projective space CP^{L_2}. The distance between two points $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$ is $\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^3}$. Under the embedding of the classical space into the space of states, the variable $\mathbf{a}$ is represented by the state $\tilde{\delta}^3_{\mathbf{a}}$. The set of states $\tilde{\delta}^3_{\mathbf{a}}$ form a submanifold M^{σ}_3 in the Hilbert spaces of states $L_2(\mathbb{R}^3)$, which is "twisted" in $L_2(\mathbb{R}^3)$, it belongs to the sphere S^{L_2} and spans all dimensions of $L_2(\mathbb{R}^3)$. The distance between the states $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ on the sphere S^{L_2} or in the projective space CP^{L_2} is not equal to $\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^3}$. In fact, the former distance measures length of a geodesic between the states while the latter is obtained using the same metric on the space of states, but applied along a geodesic in the twisted manifold M^{σ}_3. The precise relation between the two distances is given by

$$e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{4\sigma^2}} = \cos^2\rho(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}), \tag{30}$$

where ρ is the Fubini-Study distance between states. This relation has an immediate implication onto the form of probability distributions of random variables over M^{σ}_3 and CP^{L_2}. In particular, consider a random variable ψ over CP^{L_2}. Suppose that the restricted random variable ψ defined over $M^{\sigma}_3 = \mathbb{R}^3$ is distributed normally on $\mathbb{R}^3$. (That is, the truncated distribution is normal.)Then, by (30), the isotropic (i.e., direction independent) probability distribution of ψ over CP^{L_2} must satisfy the Born rule. That is, the normal probability distribution of a position random variable for a particle in the classical space implies the Born rule for transitions between arbitrary quantum states of the particle and vice versa.

EXTENSION OF NEWTONIAN DYNAMICS TO THE SPACE CP^{L_2} OF QUANTUM STATES

Recall that the Schrödinger dynamics with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$ was used to find the classical and momentum space components of the velocity $\frac{d\varphi}{dt}$ for a particle. For convenience, these results (formulae (25), (26)) are reproduced here:

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial x^{\alpha}}}e^{i\theta}\right)\Bigg|_{t=0} = \left(\frac{dr}{dt}, -\widehat{\frac{\partial r}{\partial x^{\alpha}}}\right)\Bigg|_{t=0} = \frac{v^{\alpha}}{2\sigma}, \tag{31}$$

and

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial \theta}{\partial p^{\alpha}}}\varphi\right)\Bigg|_{t=0} = \frac{mw^{\alpha}\sigma}{\hbar}, \tag{32}$$

where

$$mw^{\alpha} = -\left.\frac{\partial V(\mathbf{x})}{\partial x^{\alpha}}\right|_{\mathbf{x}=\mathbf{a}} \tag{33}$$

and σ is sufficiently small for the linear approximation of $V(\mathbf{x})$ to be valid over intervals of length σ. Formulae (31), (32) were used to establish that Newtonian dynamics of a particle is the Schrödinger dynamics constrained to the classical phase space $M^{\sigma}_{3,3}$ of the particle.

Suppose on the contrary that for any initial state $\varphi_{\mathbf{a}}$ of the form

$$\varphi_{\mathbf{a}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}} \tag{34}$$

there exists a path $\varphi = \varphi_t$ in $L_2(\mathbb{R}^3)$, passing at $t = 0$ through the point $\varphi_{\mathbf{a}}$, and such that (31), (32) are satisfied. Suppose further that the evolution $\varphi = \varphi_t$ is unitary and, therefore, by Stone's theorem,

$$\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{H}\varphi \tag{35}$$

for a Hermitian operator $\widehat{H}$ on $L_2(\mathbb{R}^3)$. The claim is that this implies the Ehrenfest theorem on states (34) and, by linear extension, on the entire space $L_2(\mathbb{R}^3)$. It then follows that the operator $\widehat{H}$ is uniquely defined and is equal to $-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$. More precisely,

There is a unique unitary evolution on $L_2(\mathbb{R}^3)$, which, being constrained to the classical phase space $M^{\sigma}_{3,3}$, satisfy Newtonian equations of motion for the particle. This evolution obeys the Schrödinger equation of motion with the usual Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$.

Let's first prove that (31) and (32) imply the Ehrenfest theorem on states $\varphi \in M^{\sigma}_{3,3}$. As discussed, the Ehrenfest theorem can be written in the following form:

$$2\mathrm{Re}\left(\frac{d\varphi}{dt}, \widehat{x}\varphi\right) = \left(\varphi, \frac{\widehat{p}}{m}\varphi\right) \tag{36}$$

and

$$2\mathrm{Re}\left(\frac{d\varphi}{dt}, \widehat{p}\varphi\right) = \left(\varphi, -\nabla V(\mathbf{x})\varphi\right). \tag{37}$$

From (31) and (34) we have at $t = 0$

$$\frac{v^\alpha}{2\sigma} = \text{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial x^\alpha}} e^{i\theta}\right) = \frac{1}{\sigma}\text{Re}\left(\frac{d\varphi}{dt}, (x-a)^\alpha \varphi\right). \tag{38}$$

Because of the unitary condition, we have $\text{Re}\left(\frac{d\varphi}{dt}, \varphi\right) = 0$ and so (38) yields

$$2\text{Re}\left(\frac{d\varphi}{dt}, x^\alpha \varphi\right) = v^\alpha = \frac{p^\alpha}{m}. \tag{39}$$

Together with $(\varphi, \widehat{p}\varphi) = (\varphi, \mathbf{p}\varphi) = \mathbf{p}$ this gives the first Ehrenfest theorem (36) on states $\varphi \in M^\sigma_{3,3}$.

Similarly, from (32) and (34) we have at $t = 0$

$$\frac{mw^\alpha \sigma}{\hbar} = \text{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial \theta}{\partial p^\alpha}}\varphi\right) = \frac{\hbar}{\sigma}\text{Re}\left(\frac{d\varphi}{dt}, \frac{i(x-a)^\alpha}{\hbar}\varphi\right), \tag{40}$$

with

$$mw^\alpha = -\left.\frac{\partial V(\mathbf{x})}{\partial x^\alpha}\right|_{\mathbf{x}=\mathbf{a}}. \tag{41}$$

On the other hand,

$$\widehat{p}\varphi = -i\hbar\nabla\varphi = -i\hbar\left(-\frac{\mathbf{x}-\mathbf{a}}{2\sigma^2} + \frac{i\mathbf{p}}{\hbar}\right)\varphi. \tag{42}$$

Again, from the unitary condition we have $\text{Re}\left(\frac{d\varphi}{dt}, \varphi\right) = 0$ and so we can rewrite (40) as

$$\frac{mw^\alpha \sigma}{\hbar} = \frac{\sigma}{\hbar}\text{Re}\left(\frac{d\varphi}{dt}, \widehat{p}^\alpha \varphi\right), \tag{43}$$

or,

$$2\text{Re}\left(\frac{d\varphi}{dt}, \widehat{p}^\alpha \varphi\right) = mw^\alpha. \tag{44}$$

From this and (33) we get the second Ehrenfest theorem (37) on states $\varphi \in M^\sigma_{3,3}$. From (38) and (40) one can also see that velocity and acceleration terms are the real and imaginary parts of a complex vector, tangent to $M^\sigma_{3,3}$.

Now, from the derived Ehrenfest theorems and the Stone's theorem for unitary evolution

$$\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{H}\varphi, \tag{45}$$

we get the following equations for the unknown Hermitian operator $\widehat{H}$, valid for all functions φ in $M^\sigma_{3,3}$:

$$\left(\varphi, i[\widehat{H}, \widehat{x}]\varphi\right) = \frac{\hbar}{m}(\varphi, \widehat{p}\varphi) \tag{46}$$

and

$$\left(\varphi, i[\widehat{H}, \widehat{p}]\varphi\right) = \hbar\,(\varphi, -\nabla V(\mathbf{x})\varphi). \tag{47}$$

Because the operators on the right hand sides of (46), (47) are known and defined on $L_2(\mathbb{R}^3)$, there is a unique extension of expressions on the right from $M^\sigma_{3,3}$ to all linear combinations of functions φ in $M^\sigma_{3,3}$. That is, the right hand sides of (46), (47) become quadratic forms on $L_2(\mathbb{R}^3)$. Let us show that there is a unique operator $\widehat{H}$ for which the equations (46), (47) remain true for these extensions. That is, there exists a unique operator $\widehat{H}$ for which

$$\left(f, i[\widehat{H}, \widehat{x}]f\right) = \frac{\hbar}{m}(f, \widehat{p}f) \tag{48}$$

and

$$\left(f, i[\widehat{H}, \widehat{p}]f\right) = \hbar\,(f, -\nabla V(\mathbf{x})f) \tag{49}$$

for all functions f in the dense subset D of $L_2(\mathbb{R}^3)$, which is the common domain of all involved operators. In fact, by choosing an orthonormal basis $\{e_j\}$ in D and considering (48), (49) on functions $f = e_k + e_l$ and $f = e_k + ie_l$ we conclude that all matrix elements of the operators on the left and right of the equations (48), (49) must be equal. So the equations can be written in the operator form

$$i[\widehat{H}, \widehat{x}] = \frac{\hbar}{m}\widehat{p} \tag{50}$$

and

$$i[\widehat{H}, \widehat{p}] = -\hbar\nabla V(\mathbf{x}). \tag{51}$$

From (50) and (51) it then follows that, up to an irrelevant constant, $\widehat{H} = \frac{\widehat{p}^2}{2m} + V(\mathbf{x})$.

OBSERVATION OF POSITION OF MACROSCOPIC AND MICROSCOPIC PARTICLES

The goal here is to understand the relationship of measurements in classical and quantum mechanics. This will be done by using measurement of position as a (rather general) example. Recall that under the isotropy condition the normal probability distribution on $\mathbb{R}^3$ has a unique extension to CP^{L_2} and yields the Born rule. We also saw that the Schrödinger dynamics restricted to the classical phase space induces the Newtonian dynamics. Likewise, Newtonian dynamics of a particle in a potential $V(\mathbf{x})$ extends in a unique way to the Schrödinger dynamics on the space of states. To relate the measurements we are therefore entitled to use Newtonian mechanics in modeling the process of measuring position of a macroscopic particle as a first step. We will then attempt to extend this model and to describe measurements in quantum mechanics in a way consistent with the Schrödinger dynamics.

Under measurement of position of a macroscopic particle the position random variable satisfy generically the normal distribution law. This is consistent with the central limit theorem and indicates that a specific way in which position is measured may not be important. One common way of finding the position of a macroscopic particle (assumed to be a rigid body) is to expose it to light of sufficiently short wavelength and to observe the scattered photons. In many cases, due to the unknown path of the incident photons, multiple scattering events on the particle, random change in position of the particle, etc., the process of observation can be described by the diffusion equation with the observed position of the particle experiencing a Brownian-like motion during the time of observation.

Let's detail this picture of measurement. For this consider the density of states functional $\rho_t[\varphi]$, which measures the number of states of particles in the space of states on a neighborhood of a point $\varphi \in CP^{L_2}$. The functional $\rho_t[\varphi]$ is supposed to be an extension of the density of particles function in the usual diffusion process on $\mathbb{R}^3$. Its precise meaning for the measurements resulting in a specific position of a microscopic particle will be soon explained. From Einstein argument, assuming the number of states (particles) is preserved, we have:

$$\rho_{t+\tau}[\varphi] = \int \rho_t[\varphi + \eta]\gamma[\eta]D\eta, \tag{52}$$

where $\gamma[\eta]$ is the probability functional of the variation η in the state φ and integration goes over all possible variations.

Let's first obtain from this the usual diffusion equation on $\mathbb{R}^3 = M_3^\sigma$. The idea is that if (52) induces the usual diffusion on $\mathbb{R}^3$, then, as we already know, it must also predict the Born rule on CP^{L_2} and we may learn something in the process. The restriction of (52) to M_3^σ means that $\varphi = \tilde{\delta}^3_{\mathbf{a}}$ and $\eta = \tilde{\delta}^3_{\mathbf{a}+\epsilon} - \tilde{\delta}^3_{\mathbf{a}}$, where ϵ is a displacement vector in $\mathbb{R}^3$. Also, the function $\rho_t(\mathbf{a}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}}]$ must be the usual density of particles in space for which (52) must result in the diffusion equation on $\mathbb{R}^3$. The power expansion for the functional $\rho_t[\varphi]$ for real valued functions φ has the form

$$\rho_t[\varphi] = c_0 + \int c_1(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} + \int c_2(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})\varphi(\mathbf{y})d\mathbf{x}d\mathbf{y} + ..., \tag{53}$$

where $c_0, c_1, c_2, ...$ are functions of t. To obtain the diffusion equation for $\rho_t(\mathbf{a})$ out of (52), the terms of order higher than two in (53) must vanish. (Otherwise the resulting second order terms for $\rho_t(\mathbf{a})$ do not yield the needed term $\Delta\rho_t(\mathbf{a})$.) Likewise, the first two terms in the expansion (53) can only make a linear in $\mathbf{a}$ and constant in t contribution to $\rho_t(\mathbf{a})$ and can therefore be dropped without affecting the equation. Then, without loss of generality, the expression for the desired functional $\rho_t[\varphi]$ can be written as the quadratic form

$$\rho_t[\varphi] = \int c_t(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})\varphi(\mathbf{y})d\mathbf{x}d\mathbf{y}. \tag{54}$$

To ensure that the functional is defined on complex valued functions in CP^{L_2} we need Hermiticity:

$$\rho_t[\varphi] = \int c_t(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})\overline{\varphi}(\mathbf{y})d\mathbf{x}d\mathbf{y}, \tag{55}$$

with $c_t(\mathbf{x},\mathbf{y}) = \overline{c}_t(\mathbf{y},\mathbf{x})$.

Let's substitute $\varphi = \tilde{\delta}^3_{\mathbf{a}}$ and $\eta = \tilde{\delta}^3_{\mathbf{a}+\epsilon} - \tilde{\delta}^3_{\mathbf{a}}$ into (52), use (55), replace $\gamma[\eta]$ with the corresponding probability density function $\gamma(\epsilon)$ and integrate over the space $\mathbb{R}^3$ of all possible vectors ϵ. Let us also make the usual in the classical diffusion theory assumption that $\gamma(\epsilon)$ is independent of the direction of ϵ (space symmetry). Then the terms $\int \epsilon^k\gamma(\epsilon)d\epsilon$ and $\int \epsilon^k\epsilon^l\gamma(\epsilon)d\epsilon$ with $k \neq l$ vanish. It follows that

$$\frac{\partial\rho_t(\mathbf{a})}{\partial t} = k\Delta\rho_t(\mathbf{a}), \tag{56}$$

where $\rho_t(\mathbf{a}) = c_t(\mathbf{a},\mathbf{a})$ and $k = \frac{1}{2\tau}\int \epsilon^2\gamma(\epsilon)d\epsilon$.

Now, coming back to (55), we need to figure out the meaning of $c_t(\mathbf{x},\mathbf{y})$. According to (56), $c_0(\mathbf{a},\mathbf{a})$ is the initial probability distribution. Suppose that initial probability distribution is given by the delta function and that by the time τ (typical time of measurement) the distribution is Gaussian, corresponding to the state $\tilde{\delta}^3_{\mathbf{b}} \in M_3^\sigma$: $\rho_\tau(\mathbf{a}) = |\tilde{\delta}^3_{\mathbf{b}}(\mathbf{a})|^2 = \frac{1}{\sigma^3}|(\tilde{\delta}^3_{\mathbf{b}}, \tilde{\delta}^3_{\mathbf{a}})|^2$ (see [1]). We must have in this case

$$c_\tau(\mathbf{x},\mathbf{y}) = \frac{1}{\sigma^3}\tilde{\delta}^3_{\mathbf{b}}(\mathbf{x})\tilde{\delta}^3_{\mathbf{b}}(\mathbf{y}). \tag{57}$$

From (55), by requiring that the probability of transition from state ψ to state φ is the same as the probability of transition from φ to ψ (or just by the sesquilinear extension in (57)), we get for a general state $\psi(\mathbf{x})$ at the time of measurement that

$$c_\tau(\mathbf{x},\mathbf{y}) = \frac{1}{\sigma^3}\overline{\psi}(\mathbf{x})\psi(\mathbf{y}). \tag{58}$$

It follows that $c_\tau(\mathbf{x},\mathbf{y})$ is the density matrix for the particle. For $\varphi(\mathbf{x}) = \tilde{\delta}^3_{\mathbf{a}}(\mathbf{x}) = \sigma^{\frac{3}{2}}\delta^3_{\mathbf{a}}(\mathbf{x})$ formula (55) yields then

$$\rho_\tau(\mathbf{a}) = \int \overline{\psi}(\mathbf{x})\psi(\mathbf{y})\delta^3_{\mathbf{a}}(\mathbf{x})\delta^3_{\mathbf{a}}(\mathbf{y})d\mathbf{x}d\mathbf{y} = |\psi(\mathbf{a})|^2, \tag{59}$$

which is the Born rule.

To summarize the derivation, we assumed first that the number of particles is conserved. This conservation is expressed in (52) in terms of the particles' states (the states can only move around the sphere of states, they don't disappear or get created). We also required that restriction of (52) to the classical space M_3^σ must induce the usual diffusion equation on $\mathbb{R}^3$ with the delta function as the initial condition (and normal distribution of width σ by the time τ). The outcome of these assumptions is that the density functional $\rho_\tau[\varphi]$ must be a sesquilinear form given by (55) and (58), and that therefore the resulting probability to find the "measured" state ψ at the point φ depends only on the distance between the states in the Fubini-Study metric and is given by the Born rule.

The derivation means that the classical Brownian motion of a particle is a restriction to the classical space M_3^σ of a motion of state, in which the state of the particle has equal probability to be pushed in any direction in CP^{L_2}. Moreover, according to the previous section, the equation for the state can be obtained by the sesquilinear extension from the classical phase space $M_{3,3}^\sigma$ onto the space of states of the appropriate classical equations of motion, expressed in terms of the Ehrenfest theorem (39), (44).

Without pretending to be complete in any way, let us just see how the Schrödinger evolution of state for a particle interacting with the surroundings could in fact lead to an equal probability of any direction of displacement of the state in the space of states. For this, let's try to extend the Brownian motion of the components $\tilde{\delta}^3_{\mathbf{b}}$ of ψ on the space M_3^σ (equivalently, on $M_{3,3}^\sigma$) to the motion of ψ itself in a way consistent with the Schrödinger evolution. A general initial state ψ of a microscopic particle under observation can be approximated as well as needed by a linear combination of states (22) in the classical phase space $M_{3,3}^\sigma$:

$$\psi(\mathbf{x}) = \sum_{\mathbf{b}} C_{\mathbf{b}} \tilde{\delta}^3_{\mathbf{b}}(\mathbf{x}) e^{i\frac{\mathbf{p}_{\mathbf{b}}(\mathbf{x}-\mathbf{b})}{\hbar}}. \tag{60}$$

Interaction with measuring particles in the apparatus can be modeled by the perturbation

$$\widehat{V} = \sum_{\mathbf{b}} V_{\mathbf{b}}(\mathbf{x}) \cos(\omega t + \gamma_{\mathbf{b}}), \tag{61}$$

where $V_{\mathbf{b}}$ is a non-vanishing linear potential on a neighborhood of $\mathbf{b}$ (containing only one of the $\mathbf{b}$'s) and $\gamma_{\mathbf{b}}$ is a phase that depends on $\mathbf{b}$. The potential $\widehat{V}$ is a sum of potentials commonly used in studying the emission and absorption of radiation in Schrödinger mechanics. The change η in the state (60) due to the potential $\widehat{V}$ acting during time interval τ is given in the first order approximation in τ by

$$\eta(\mathbf{x}) = -\frac{i}{\hbar}\widehat{V}\psi(\mathbf{x})\tau. \tag{62}$$

Substituting ψ from (60), neglecting the impact of the kinetic energy term in the Hamiltonian during the time τ of interaction, except the change in phase, and using the near-orthogonality of the components for different values of $\mathbf{b}$, we obtain

$$\eta(\mathbf{x}) = -\frac{i\tau}{\hbar}\sum_{\mathbf{b}} C_{\mathbf{b}} V_{\mathbf{b}}(\mathbf{x}) \cos(\omega t + \gamma_{\mathbf{b}}) \tilde{\delta}^3_{\mathbf{b}}(\mathbf{x}) e^{i\theta_{\mathbf{b}}}, \tag{63}$$

where $e^{i\theta_{\mathbf{b}}}$ is the overall phase factor.

It is possible to see now why all directions of displacement η in the tangent space $T_\psi(CP^{L_2})$ are equally likely. Note first of all that, because the evolution is unitary, vector η is in the tangent space $T_\psi(S^{L_2})$ to the sphere states. Also, the involved functions of $\mathbf{x}$ considered for all $\mathbf{b}$ form a complete set, so that any tangent vector is a linear combination of these functions. The phase $\gamma_{\mathbf{b}}$ changes randomly in time, every time interval τ of a particular encounter with a measuring particle. So it is reasonable to assume that $\gamma_{\mathbf{b}}$ form a sequence of independent random variables, indexed by the time interval of interaction, with all variables distributed uniformly on the interval $[0, 2\pi)$. Furthermore, assume that the random variables $\gamma_{\mathbf{b}}$ for different values of $\mathbf{b}$ are independent. It then follows that for any particular time interval of interaction and for all non-vanishing coefficients $C_{\mathbf{b}}$, the ratios

$$\frac{|C_{\mathbf{b_1}} \cos(\omega t + \gamma_{\mathbf{b_1}})|}{|C_{\mathbf{b_2}} \cos(\omega t + \gamma_{\mathbf{b_2}})|} \tag{64}$$

with $\mathbf{b_1} \neq \mathbf{b_2}$ are equally likely to take any value between 0 and infinity. Also, assume that for different time intervals the phase factors $e^{i\theta_{\mathbf{b}}}$ in (63) evaluated at $\mathbf{b}$ are independent random variables distributed uniformly on the circle, and independent for different values of $\mathbf{b}$. Then, for an arbitrary time interval all complex values of the ratio

$$\frac{C_{\mathbf{b_1}} \cos(\omega t + \gamma_{\mathbf{b_1}}) e^{i\theta_{\mathbf{b}_1}}}{C_{\mathbf{b_2}} \cos(\omega t + \gamma_{\mathbf{b_2}}) e^{i\theta_{\mathbf{b}_2}}} \tag{65}$$

are equally likely. Finally, since the ratios (65) define the direction of η up to a constant phase factor, all obtained directions of η in $T_\psi(CP^{L_2})$ are equally likely. Note that vanishing of a particular coefficient $C_{\mathbf{b}}$ in the decomposition of the initial state ψ indicates that ψ is orthogonal to the corresponding state $\tilde{\delta}^3_{\mathbf{b}}$, not that the state cannot get displaced in the direction of $\tilde{\delta}^3_{\mathbf{b}}$. (Although the probability of it reaching $\tilde{\delta}^3_{\mathbf{b}}$ is zero, by the derived Born rule.)

One may question the model that led to the conclusion. However, what was really used was a random change in the modulus and the phase of the coefficients $C_{\mathbf{b}}$. The change in modulus is readily produced by a "noise" in space. Acquiring random phases is also typical when a quantum system interacts with the environment. The conditions imposed on random variables are physically sound and quite generic stochastic conditions. This seems to suggest that the result is rather general and robust, although this would need to be proved, of course. At any rate, the goal here was to see in an independent way the conclusion forced upon us by the need to derive the classical Brownian motion of a particle from the motion of its state constrained to the manifold $M_{3,3}^\sigma$.

It seems at first that not much has been achieved. The fact that a random noise may lead to random fluctuation of state is rather simple and goes against of what one normally tries to achieve when explaining collapse under measurement. The collapse models utilize various ad hoc additions to Schrödinger equation with the goal of explaining why the state under a random walk "concentrates" to an eigenstate of the measured observable. Instead, we see a clear indication that the state under a generic measurement have equal probability of moving in any direction and "diffuses" isotropically into the space of states. Surprisingly, this diffusion on the space of states, being in agreement with the Schrödinger evolution, is capable of addressing the major issues of measurement in quantum theory.

In fact, under the diffusion, the probability of transition of ψ to any other state $\varphi \in CP^{L_2}$ was shown to depend only on the distance between the states and to satisfy the Born rule. The role of the measuring device can be then seen in initializing the diffusion(creating a "noise") and in registering a particular location of the diffused state. For instance, the "noise" in the position measuring device under consideration is due to the stream of photons. The device then registers the state reaching a point in M_3^σ. In a similar way, a momentum measuring device registers the diffused states that reach the eigen-manifold of the momentum operator (which is the image of M_3^σ under the Fourier transformation). It follows, in particular, that the measuring device in quantum mechanics is not responsible for creating a basis for the state to be expanded into. If several measuring devices are present they are not "fighting" for the basis. When the eigen-manifolds of the corresponding observables don't overlap, only one of them can "click" for the measured particle as the state can reach only one of the eigen-manifolds. Note finally the similarity in the role of measuring devices in quantum and classical mechanics: in both cases the devices are designed to measure a particular physical quantity and inadvertedly create a "noise", which results in a distribution of values of the measured quantity.

Coming back to the derivation of the diffusion equation (56) and the Born rule (59), we see how the Brownian motion experienced by the components $\tilde{\delta}^3_{\mathbf{b}}$ of state ψ of the measured particle results in a "Born-like" motion of ψ itself. The probability density to find the state ψ at a point $\tilde{\delta}^3_{\mathbf{a}}$ (particle at a point $\mathbf{a}$) is given by (59). From (59) and (60) it also follows that

$$\rho_0(\mathbf{a}) = \sum_{\mathbf{b}} |C_{\mathbf{b}}|^2 \, |\tilde{\delta}^3_{\mathbf{b}}(\mathbf{a})|^2, \tag{66}$$

which is a weighted sum of the normal probability distributions for each components. Furthermore, due to linearity of the diffusion equation (56), the initial "cloud" $\rho_0(\mathbf{a}) = |\psi(\mathbf{a})|^2$ evolves in time as if it was in fact a cloud "diffusing" in $\mathbb{R}^3$, rather than a single state (point) moving stochastically in CP^{L_2}. The reason for it is clear: since we restrict the outcomes of measurements to only those in the space $M_3^\sigma = \mathbb{R}^3$, the probability density is the probability of getting a particular position value in $\mathbb{R}^3$. The fact that the original state does not belong to M_3^σ is not explicit in the density function $\rho_t(\mathbf{a})$, giving us the cloud interpretation in $\mathbb{R}^3$.

So what does it all say about measurement of position of macroscopic and microscopic particles? During the period of observation of position of a macroscopic particle in the model, the position random variable experiences a Brownian motion. Normally observation happens during a short enough interval of time so that the particle does not get displaced much and the spread of the probability density is sufficiently small. A particular value of position variable during the observation is simply a realization of one of the possible outcomes. The Brownian motion of macroscopic particle can be equivalently thought of as either a stochastic process $\mathbf{b}_t$ with values in $\mathbb{R}^3$ or a process $\tilde{\delta}^3_{\mathbf{b},t}$ with values in M_3^σ. The advantage of the latter representation is that the position random variable $\tilde{\delta}^3_{\mathbf{b}}$ gives both, the position of the particle in $M_3^\sigma = \mathbb{R}^3$ and the probability density to find it in a different location $\mathbf{a}$ (in the state $\tilde{\delta}^3_{\mathbf{a}}$), due to uncontrollable interactions with the surroundings under observation and the resulting Brownian motion.

Measuring position of a microscopic particle has, in essence, a very similar nature. Under observation each component of the state ψ of the particle in (60) experiences the usual Brownian motion on M_3^σ (or the phase space $M_{3,3}^\sigma$). As a result, the state ψ itself becomes a random variable, taking values in the space of states CP^{L_2}. To measure position is to observe the state on the submanifold M_3^σ (or $M_{3,3}^\sigma$) in CP^{L_2}. In this case the random variable ψ assumes one of the values $\tilde{\delta}^3_{\mathbf{a}}$, with the uniquely defined probability density compatible with the normal density in the space $\mathbb{R}^3$. This probability density (associated with the conditional probability to find the state ψ at $\tilde{\delta}^3_{\mathbf{a}}$ given that ψ has reached M_3^σ) is exactly the one given by the Born rule. Here too the random variable ψ gives both, the position of the state of the particle in CP^{L_2} and the probability density to find the particle in a different state $\tilde{\delta}^3_{\mathbf{a}}$.

So the difference between the measurements is twofold. First, under a measurement the state ψ of a microscopic particle is a random variable over the entire space of states CP^{L_2} and not just over the submanifold M_3^σ. Second, unless ψ is already constrained to M_3^σ (which case would mimic measurement of position of a macroscopic particle), to measure position is to observe the state that "diffused" enough to reach the submanifold M_3^σ. To put it differently, the measuring device is not where the initial state was. Assuming the state has reached M_3^σ, the probability density of reaching a particular point in M_3^σ is given, as we saw, by the Born rule.

We don't use the term collapse of position random variable when measuring position of a macroscopic particle.

Likewise, there is no physics in the term collapse of the state of a microscopic particle. Instead, due to the diffusion of state there is a probability density to find the particle in various locations on CP^{L_2}. In particular, the state may reach the space manifold $M_3^\sigma = \mathbb{R}^3$. If that happens and we have detectors spread over the space, then one of them clicks. If the detector at a point $\mathbf{a} \in \mathbb{R}^3$ clicks, that means the state is at the point $\tilde{\delta}^3_{\mathbf{a}} \in CP^{L_2}$ (that is, the state *is* $\tilde{\delta}^3_{\mathbf{a}}$). The number of clicks at different points $\mathbf{a}$ is given by the Born rule. The state is not a "cloud" in $\mathbb{R}^3$ that shrinks to a point under observation. Rather, the state is a point in CP^{L_2} which may or may not be on $\mathbb{R}^3 = M_3^\sigma$. When the detector clicks we know that the state is on M_3^σ.

Note once again that there is no need in any new mechanism of "collapse". There is no "concentration" of state involved and the stochastic process is in agreement with the conventional Schrödinger equation with a randomly fluctuating potential ("noise"). The origin of the potential depends on the type of measuring device or properties of the environment, capable of "measuring" the system. Fluctuation of the potential can be traced back to thermal motion of molecules, atomic vibrations in solids, vibrational and rotational molecular motion, and the surrounding fields. Transition from individual effect of a "kick" on a spatial component of the state ψ in (60) to their combined effect on ψ and the resulting stochastic process require a change in description. The linear equation for the state results in a linear equation for the probability density, i.e., the diffusion equation, which, of course, is not linear in ψ.

GENERALITIES OF THE CLASSICAL BEHAVIOR OF MACROSCOPIC BODIES

It was shown that the Schrödinger evolution of state constrained to the classical phase space $M_{3,3}^\sigma$ results in the Newtonian motion of the particle. Similar results hold true for systems of particles. To reconcile the laws of quantum and classical physics one must also explain the nature of this constraint. Why microscopic particles are free to leave the classical space, while macroscopic particles are bound to it? What is the role of decoherence in this, if any?

Suppose first that the macroscopic particle under consideration is a crystalline solid. Position of one cell in the solid defines the position of the entire solid. If one of the cells was observed at a certain point at rest, the state of the solid immediately after the observation (in one dimension) is the product

$$\psi = \tilde{\delta}_a \otimes \tilde{\delta}_{a+\Delta} \otimes \ \dots \ \otimes \tilde{\delta}_{a+N\Delta}, \tag{67}$$

where Δ is the lattice length parameter. The general quantum-mechanical state of the solid is then a superposition of states (67) for different values of a in space:

$$\psi = \sum_a C_a \tilde{\delta}_a \otimes \tilde{\delta}_{a+\Delta} \otimes \ \dots \ \otimes \tilde{\delta}_{a+N\Delta}. \tag{68}$$

Why would non-trivial superpositions of this sort be absent in nature?

By now it is a well established and experimentally confirmed fact that macroscopic bodies experience an unavoidable interaction with the surroundings. As with the measurement of position of a microscopic particle, the "cells" of a macroscopic solid body (either crystalline or not) are pushed in all possible directions by the surrounding particles. In the previous section it was demonstrated that this can lead quite generally to a diffusion-like process on the space of states, in place of the free Schrödinger evolution. However, the solid can only be in the state similar to (68), which means that the the state of the solid can only be pushed within the submanifold formed by these states. But the probability of such a push formed by random and independent fluctuations of states of each cell is vanishingly small. (This statement is easy to make precise in finite dimensional spaces of states, but is more subtle in the infinite-dimensional case, when Lebesgue measure is not available.) This may explain how the state of a solid initially on the manifold M_{3N}^σ remains on that manifold.

The situation is surprisingly similar to that of pollen grains and a ship in still water. While under the kicks from the molecules of water the pollen grains experience a Brownian motion, the ship in still water will not move at all. Because of the established relation of Newtonian and Schrödinger dynamics this is more than analogy. Namely, when the state is constrained to the classical phase space submanifold, the "pushes" experienced by the state become the classical kicks in the space that lead to Brownian motion of the system. Note also that this does not seem to preclude the motion of state of a solid as a whole. The motion of the ship under an appropriate force provides an example in classical physics. In principle, the motion of state of a solid away from the classical space seems also possible, although not because of its interaction with the random environment.

So far the state of the system by itself was used as a dynamical variable, available during a measurement. This was shown to be possible because the constructed model of measurement is an extension of the classical model, where position variable is available. It was also shown that the obtained extension is consistent with the Schrödinger dynamics. However, it is well known that interaction with the measuring device and the environment creates an entangled state of the system and the surroundings. It is therefore impossible to talk about the state of the system by itself, contradicting the previous assertion.

To understand the situation, let's begin once again with a measurement of position of a macroscopic particle (i.e., the particle whose state is constrained to the classical phase space $M^{\sigma}_{3,3}$) by observing scattered photons or other particles. When the measured particle is observed this way, the incident particles and the measured particle exchange energy and momentum creating what could be called a "classical entanglement" of the particle with the measuring device and environment. In particular, position, energy and momentum of the measured particle are not known until a measurement on the outgoing particle is made. Here too, the information about the measured particle "leaks out" into the environment, affecting potentially the entire universe. However, entanglement in the usual sense is absent. The state of the observed particle and the surroundings has the form

$$\tilde{\delta}^3_{\mathbf{a}} \otimes E_{\mathbf{a}}, \tag{69}$$

where $E_{\mathbf{a}}$ represent the state of the apparatus and the environment. The state of the system belongs to the submanifold $M^{\sigma}_{3,3,E}$ of the tensor product of Hilbert spaces of the particle and the surroundings that consists of the product states (69). The position of the particle is defined and can be found at any time, at least in principle. In some cases the surroundings can be modeled by a potential and position of the particle is found by solving Newton's equations of motion. In some cases to predict position of the particle we have to consider a system consisting of the observed particle and particles in the surroundings. When many particles of the surroundings are involved, the position is best described in terms of the probability. In the model considered here the diffusion equation was used to find the probability density of the position random variable.

Suppose now the position of a microscopic particle is measured in the same way. In some cases interaction with the surroundings can be modeled by a potential. In some cases we have to deal with a many particle system and attempt solving the Schrödinger equation for the system. For a large number of particles in the surroundings the Schrödinger equation may yield a stochastic equation. In the model considered here it is the diffusion of state on the space of states CP^{L_2} of the particle. However, in the most general case all we can claim is that the state Ψ of the system consisting of the particle and the surroundings is a sum of terms in (69). In this case the state of the system at any time is a point on the sphere of states in the Hilbert space, which is the tensor product of Hilbert spaces of the particle and the surroundings. However, the proposed model of measurement remains the same. As before, the random nature of interaction between the involved particles result in a random fluctuation of Ψ along the sphere with equal probability of all directions of displacement along the sphere. The state Ψ undergoes a diffusion on the sphere. In particular, Ψ can reach the submanifold $M^{\sigma}_{3,3,E}$ of the sphere, consisting of the product states $\tilde{\delta}^3_{\mathbf{a}} \otimes E_{\mathbf{a}}$. If that happens, the position of the state in $M^{\sigma}_{3,3}$ becomes defined. That is, the position of the particle in the classical sense is defined and can be recorded by the measuring device.

So far decoherence was not present in the discussion. The decoherence is a mathematical expression of the fact that a quantum system interacting with the environment behaves like a probabilistic mixture and needs to be described by the probability and not by the state. The "pure" form of decoherence is the entanglement with the environment, which does not use any dynamical change in the components and is, therefore, rather formal. The theory is centered around the issue of entanglement and the resulting loss of coherence and the fact that the state of the measured system is still present in the theory is often overlooked. In fact, the probability of a system exposed to interaction is the probability of a certain state of the system. Decoherence theory does not usually go beyond recognition of the loss of coherence and the resulting need in probabilistic description of the system. It does not describe the way in which specific measurement results are obtained and does not derive the Born rule. At the same time, decoherence theory consists of an array of very useful models. These models testify to the universal character of the loss of coherence and transition to classical probability resulting from interaction with the environment. The results provide an additional support for a stochastic description of measurement and interaction with the environment. This is despite the characterization of diffusion models such as the one used here as "fake" decoherence in the decoherence theory, due to their microscopically unitary character.

One could think that interaction with the environment is the only reason for the classical behavior of macroscopic bodies. Note however that interactions between atoms and cells of the solid that keep it rigid are implicitly present in (67). Furthermore, there may be a completely "internal" way of explaining the absence of superpositions of the states in (68). For instance, consider the solid as a collection of independent two-level systems - one for each cell in the solid. Suppose the lower level corresponds to the state $\tilde{\delta}_a$ of the cell with a fixed position a, while the upper level is a superposition of such states for different values of a. For instance, the ground state of a harmonic oscillator is Gaussian and in case of a strong enough potential can be thought of as the state of a cell with a fixed position. Any of the higher level states in the oscillator can be written as a superposition of the ground states of oscillators with different positions.

Now, from (68) it is clear that for the state ψ of the entire solid to be a non-trivial superposition, each cell would have to be in the upper state. In fact, if one of the cells is in the lower state for some value of a, then only the term with that value of a for one of the cells would be present in (68), which means that the position of every cell in the solid would then be fixed. However, for a large collection of cells it would be next to impossible to put each cell in the upper state. More importantly, even if this was somehow achieved, spontaneous and induced emissions would immediately bring some of the cells to the lower level, "collapsing" the entire solid.

More realistically, by the very meaning of a solid, a large percentage of its cells must have a fixed position, when position of at least one is observed. To obtain a superposition of states of different position of the solid in the described way it would be necessary to "free" at least a large part of the cells in the solid. To do this we would have to ensure a transition of each such cell to an upper energy level. In the nature, global transitions of this sort can only be achieved by a dramatic increase in temperature of the solid. To free a body would mean to transform it from the solid state of matter to gas or plasma state.

SUMMARY

The dynamics of a classical n-particle mechanical system is identified with the Schrödinger dynamics constrained to the classical phase space submanifold in the space of states. Conversely, there is a unique unitary time evolution on the space of states of a quantum system that yields Newtonian dynamics when constrained to the classical phase space. This results in a tight, previously unnoticed relationship between classical and quantum physics. In particular, under a measurement of position of a macroscopic particle the position random variable obeys generically the normal distribution law. This predicts the Born rule for transition between quantum states. Therefore, any classical (i.e., based on Newtonian dynamics) model of measurement of a macroscopic particle that predicts the normal distribution law of the position random variable extends in a unique way to the corresponding quantum model (i.e., satisfying Schrödinger dynamics) that satisfy the Born rule. The model used in the paper was based on the diffusion equation. Other models are, of course, possible. The central limit theorem makes it easy for a system experiencing interaction with the surroundings to satisfy the normal distribution law and therefore to imply the Born rule. We see that macroscopic and microscopic particles are not so fundamentally different after all. The only important difference is that microscopic systems live in the space of states while their macroscopic counterparts line on the classical phase space submanifold of thereof. Since our own life happens primarily in the macro-world, it is hard for us to understand the infinite-dimensional quantum world around us. As soon as the classical space centered point of view is extended to its Hilbert space centered counterpart, the new, clearer view of the classical-quantum relationship emerges.

[1] A. Kryukov, *J. Math. Phys.* **52** 082103 (2017)