The EPR experiment: A paradox-free definition of reality

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The paradoxes of the EPR experiment with two particles are shown to originate in the implicit assumption that the particles are always located in the classical space. There exists a substitute for this assumption that yields a new definition of reality and offers a resolution of the paradoxes.

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The Bohr-Einstein debate on the meaning of quantum theory culminated in the famous EPR paper [1] followed by Bohr's reply [2]. At this time the issue of completeness of the theory was at stake. The completeness turned out to be dependent on the physical quantities designated to be *real* in the theory. In [1] EPR consider a pair of non-interacting entangled particles. The state function of the pair is such that, given the position or momentum of the first particle one can predict the position or momentum of the second. Because the particles do not interact, EPR argue that both position q and momentum p of the second particle must be real. On the other hand, quantum mechanics denies that q and p can be simultaneously determined. Accordingly, EPR conclude that the quantum-mechanical description of reality is incomplete.

In his reply [2], Bohr denies that position and momentum of a particle may be simultaneously real. He argues that the measuring instrument itself *defines* the reality of *either* position *or* momentum of the particle and that the quantum-mechanical description is complete. The two points of view can be summarized as follows.

(A) *Both, the position and the momentum of the particle in the example are real. Quantum-mechanical description of reality is incomplete.*

(B) *The position and momentum of the particle cannot be simultaneously real. The measuring device itself defines the reality of one or the other. Quantum-mechanical description of reality is complete.*

Bohr's position is generally adhered within the quantum community. It comes at a price of accepting that the reality of either position or momentum of the second (possibly distant) particle may be decided instantaneously by a measurement performed on the first particle. The resulting "spooky action at a distance" or quantum non-locality has never been acknowledged by Einstein and remains a mystery of the theory.

In this Letter a definition of reality that is capable of resolving the paradox of quantum non-locality will be proposed. This definition is consistent with Bohr's conclusion that either q or p but not both are real in the EPR example. At the same time it disagrees with the Bohr's positivist's statement that the measuring device *defines* the reality. In this it is in line with the EPR realist's attitude and the statement of incompleteness of quantum description. The Letter is a continuation of [3] where a similar approach was used to address the paradoxes of the double-slit experiment.

Recall that a single spinless particle found at a point $\mathbf{u} \in R^3$ is described in quantum mechanics by the eigenfunction $\delta^3_{\mathbf{u}}(\mathbf{x}) \equiv \delta^3(\mathbf{x}-\mathbf{u})$ of the position operator $\widehat{\mathbf{x}}$. Moreover, there is an obvious one-to-one correspondence ω between R^3 and the set M_3 of all delta functions $\delta^3_{\mathbf{u}}$ via $\omega : \mathbf{u} \longrightarrow \delta^3_{\mathbf{u}}$. Hence, ω maps points in the classical space to states of the particle located at these points. As shown in the Letter, for an appropriate Hilbert space H the map ω is an *isometric embedding*, which means that R^3 and M_3 are identical manifolds with a metric. In other words, the classical Euclidean space R^3 can be identified in a physically meaningful way with a submanifold of the Hilbert space of states H.

Consider now a pair of distinguishable particles such that the first particle is located at a point $\mathbf{u}$ and the second at a point $\mathbf{v}$ in R^3. In classical mechanics such a pair is described by a single point $(\mathbf{u},\mathbf{v})$ in the configuration space $R^6 = R^3 \times R^3$. In quantum mechanics the pair is described by the point $\delta^3_{\mathbf{u}} \otimes \delta^3_{\mathbf{v}}(\mathbf{x_1},\mathbf{x_2}) \equiv \delta^3_{\mathbf{u}}(\mathbf{x_1})\delta^3_{\mathbf{v}}(\mathbf{x_2})$ in the tensor product space $H \otimes H$. Here H is the space of states of one of the particles, which is assumed to be the same for both particles. Given the right H, the one-to-one map $\omega \otimes \omega : (\mathbf{u},\mathbf{v}) \longrightarrow \delta^3_{\mathbf{u}} \otimes \delta^3_{\mathbf{v}}$ identifies the configuration space R^6 with the six dimensional submanifold M_6 of $H \otimes H$ consisting of the state functions $\delta^3_{\mathbf{u}}(\mathbf{x_1})\delta^3_{\mathbf{v}}(\mathbf{x_2})$. As before, the map $\omega \otimes \omega$ is physically meaningful as it identifies each pair of points in the classical space R^3 with the state of the pair of particles located at these points.

Recall that a single particle with momentum $\mathbf{p}$ is given in quantum mechanics by the eigenstate $e^{i\mathbf{px}}$ of the momentum operator $\widehat{\mathbf{p}}$. Consider the subset $\widetilde{M}_3$ of the space of states H consisting of the functions $e^{i\mathbf{px}}$ with $\mathbf{p} \in R^3$. Once again, for an appropriate realization of the space of states H the map $\rho : \mathbf{p} \longrightarrow e^{i\mathbf{px}}$ is an isometric embedding of the classical momentum space R^3 into the space of states. One can similarly consider the space R^6 of pairs $(\mathbf{p},\mathbf{q})$ of momenta of two particles. The map $\rho \otimes \rho : (\mathbf{p},\mathbf{q}) \longrightarrow e^{i\mathbf{px_1}}e^{i\mathbf{qx_2}}$ identifies R^6 with the submanifold $\widetilde{M}_6$ of $H \otimes H$ consisting of the state functions $e^{i\mathbf{px_1}}e^{i\mathbf{qx_2}}$. The embeddings ρ and $\rho \otimes \rho$ are both physically meaningful as they identify the momentum of each particle with the corresponding state. Note that the classical phase space of the pair cannot be embedded in such a way into the space of states. This is because there is no state in H for which both position and momentum of a particle are defined. Geometrically speaking, the intersections $M_3 \cap \widetilde{M}_3$ and $M_6 \cap \widetilde{M}_6$ are empty.

The maps ω, ρ, $\omega \otimes \omega$, $\rho \otimes \rho$ allows one to identify the physical quantities of position and momentum of a particle or a pair of particles with the variable φ taking values in one of the manifolds M_3, $\widetilde{M}_3$, M_6, $\widetilde{M}_6$. Suppose that the state function φ itself is the most appropriate way of describing the reality. Note in particular that: *(1) the state function yields the most complete description of quantum system and its evolution; (2) the state function is the "smallest" object that provides such a complete description in a sense that it contains only the experimentally verifiable information; (3) in special cases the knowledge of state function is equivalent to the knowledge of precise position or momentum of particles in the system.* In fact, (1) is known to be true in quantum mechanics, (3) was already discussed. As for (2), note that in principle, given sufficiently many copies of the system, one can experimentally determine the modulus and the phase (up to a constant initial phase) of the state function as precisely and one wishes. So, (2) is an accurate statement as well. The following alternative to the above positions (A), (B) is then proposed:

(C) *Physical reality of a pair of particles is most appropriately described by the state variable φ of the pair. The evolution of the pair in time is a path φ_t in the space of states. The variable φ generalizes the classical positions and momenta of the particles and reduces to those in special cases. Neither positions nor momenta of the particles are generally defined. The positions are defined if and only if φ takes values in the submanifold M_6 of the space of states $H \otimes H$ of the pair. The momenta are defined if and only if φ takes values in the submanifold $\widetilde{M}_6$ of $H \otimes H$. Because the intersection $M_6 \cap \widetilde{M}_6$ is empty, the positions and momenta cannot be simultaneously defined. The process of measurement does not create a reality: the state exists before and after the measurement. Rather, similarly to any interaction, a measurement simply moves the state. In particular, a measuring device that measures positions of the particles brings the initial state φ to a point of M_6. Similarly, a device that measures momenta of the particles forces the state onto $\widetilde{M}_6$.*

How does statement (C) help understand the EPR experiment? EPR consider a pair of particles in one dimension in an entangled state given by the state function $\varphi(x_1, x_2) = \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(x_1 - x_2 + x_0)} dp$, where x_0 is a constant. From the form of φ one can see that whenever the position of the first particle is known to be u, the position of the second must be $x_0 + u$. Similarly, whenever the momentum of the first particle is p, the momentum of the second must be $-p$ (see [1]). Let H be the Hilbert space of states of each particle so that φ is in $H \otimes H$. Note that neither the position nor the momentum of the particles in this state is defined. In the geometric terms that means, once again, that φ does not belong to the submanifolds M_6 or $\widetilde{M}_6$ of $H \otimes H$. After the measurement of position of the first particle, the state φ moves to a point of M_6. Similarly, the measurement of momentum of the first particle brings the state to $\widetilde{M}_6$. So, the system moves from the state in which neither position nor momentum of the particles is real to a state in which either position or momentum (but not both) of the particles is real. This is of course consistent with the Bohr's interpretation. However, for Bohr the act of measurement defines the reality. In particular, no physical description of collapse is possible. Here on the other hand, the reality is defined by the state. Because the state exists before and after the measurement, it becomes possible to analyze the collapse both mathematically and physically. In particular, it becomes possible to address the paradoxes of the EPR experiment.

To see how this can be done, let's express the state function φ of the EPR-pair in the form

$$\varphi(x_1, x_2) = \int \delta_u(x_1)\delta_{x_0+u}(x_2)du. \tag{1}$$

In this form the state φ is a superposition of all states $\delta_u(x_1)\delta_{x_0+u}(x_2)$ that correspond to the first particle being at a point u and the second at the point $x_0 + u$. In discussing the EPR experiment one usually makes a tacit assumption that the two particles are always located in the classical space. The fact that the superposition in (1) contains various terms $\delta_u(x_1)\delta_{x_0+u}(x_2)$ signifies then that the particles are located at *all* pairs of points $(u, x_0 + u)$ at once. That is, the particles must somehow "split" between these points. This thinking leads one to the conclusion that measuring position of the first particle we somehow "collect" the particle into a single point and pass this information through the classical space to the second particle so that it could also "assemble" at a predetermined point. This is certainly paradoxical! So, *how could a measurement of position x_1 of the first particle instantaneously fix the position x_2 of the second, possibly distant particle?*

According to (C), the reality is not given by the components $\delta_u(x_1)\delta_{x_0+u}(x_2)$ of φ, but rather by the state φ itself. So the pair does not "split" between various points in the classical space but is given instead by a *single* point φ in the space of states $H \otimes H$, away from the submanifold M_6. The fact that φ is not a product $\xi(x_1)\chi(x_2)$ of two functions signifies that the reality before the collapse cannot be described in terms of individual particles. Instead, the state function φ of the *pair* provides the only adequate representation of reality. Furthermore, to measure position of the "first particle" is to bring the pair represented by φ to M_6. Indeed, by definition of φ, whenever the first particle is at the point $x_1 = a$, the second particle is at $x_2 = x_0 + a$. Consequently, the state function of the pair after the measurement is $\delta_a(x_1)\delta_{x_0+a}(x_2)$, which is a point in M_6. So instead of collecting pieces of the particles, spread over the classical space, the process of collapse moves the pair from the point φ onto the manifold M_6. Thus, the process of collapse is a path φ_t in the space of states that connects the point φ in $H \otimes H$ to the point $\delta_a \otimes \delta_{x_0+a}$ in M_6.

How could the collapse happen instantaneously even when the particles are far apart? Because the process of collapse is happening on the space of states $H \otimes H$ and not on the classical space, the spatial distance between the particles is irrelevant. What matters now is the distance between the states φ and $\delta_u \otimes \delta_{x_0+u}$ and the speed of evolution φ_t *in the space of states*. Importantly, the distance between the states may be small even when the distance between the particles is known to be large. Indeed, as shown below, for an appropriate space of states H the map $\omega \otimes \omega$ identifies the classical configuration space R^6 with a submanifold of an arbitrarily small sphere $S^{H\otimes H}$ in $H \otimes H$. Accordingly, the distance between any two states may be arbitrarily small. In this case a finite speed of the evolution φ_t on the space of states may be perceived as an instantaneous process on the classical space. Namely, it is claimed that

(S) An apparently discontinuous, nonlocal process of collapse on the classical space can be modeled by a continuous, local process on the space of states.

Before proving (S), let's address yet another mystery of the EPR pair: *How could the reality of either position or momentum of the second particle be instantaneously determined by the observer's decision to measure position or momentum of the first particle?* Once again, the key to resolving this mystery is to observe that before the measurement, the pair of the particles is *not* located in the classical configuration space M_6 or the classical momentum space $\widetilde{M}_6$. In particular, the particles are *not* spread over all possible positions or momenta. (This by itself would be contradictory. Indeed, should the particles be spread over possible positions or possible momenta? If that depends on a measurement, then how would a particular spreading be created?) Under the position measurement "on the first particle", the entire pair moves along a path φ_t from the point φ to a point $\delta_a(x_1)\delta_{x_0+a}(x_2)$ on the submanifold M_6 of $H \otimes H$. Likewise, the momentum measurement on the first particle brings the pair along a different path $\widetilde{\varphi}_t$ from φ to a point $e^{iqx_1}e^{-iqx_2}$ on the submanifold $\widetilde{M}_6$ of $H \otimes H$. So a particular measuring device (either a man-made instrument or a natural phenomenon) moves the pair to either M_6 or $\widetilde{M}_6$. The discontinuous, nonlocal nature of the collapse can be now explained via statement (S).

To prove (S) it suffices to provide a specific model satisfying the statement. For this, consider the Hilbert space obtained by completing the space $L_2(R^3)$ of complex-valued square-integrable functions on R^3 in the norm defined by the inner product

$$(\varphi, \psi)_H = \int e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2} \varphi(\mathbf{x})\overline{\psi}(\mathbf{y}) d^3\mathbf{x} d^3\mathbf{y}. \qquad (2)$$

By plugging in $\varphi = \psi = \delta^3_{\mathbf{a}}$ one concludes that H contains the delta functions and the norm of any delta-function is one. So ω identifies R^3 with a submanifold M_3 of a unit sphere S^H in H. Note that for finitely many different vectors $\mathbf{a_k}$ the functions $\delta^3_{\mathbf{a_k}}$ are linearly independent. Also, no element of H is orthogonal to all delta functions. It follows that the manifold M_3 "spirals through" all available dimensions forming a *complete set* in H (see Fig. 1). The induced metric on M_3 is given by the components $g_{ik} = \left.\frac{\partial^2 k(x,y)}{\partial x^i \partial y^k}\right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}}$, where $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}$ is the kernel of the metric (2). Differentiation yields the ordinary Euclidean metric, so that M_3 is identical (i.e., *isometric*) to the Euclidean space R^3 (see [4]).

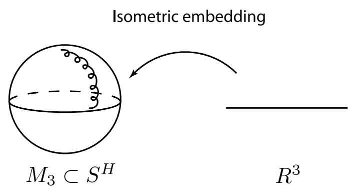

FIG. 1: R^3 as a submanifold of the sphere S^H

Because the classical space R^3 is isometrically embedded into H, the distances on S^H can be measured in the ordinary units of length. To make the distance between any two states on S^H small, the unit sphere itself must be small. Accordingly, the unit of length must be small. For instance, in the Planck system of units the radius of the unit sphere S^H is one Planck length ($\approx 1.6 \cdot 10^{-35}$m). In this case the distance between any two states on the sphere (which is equal to the angle θ between the corresponding vectors in H) does not exceed π Planck lengths. For example, the distance between $\delta^3_{\mathbf{a}}$ and $\delta^3_{\mathbf{b}}$ increases monotonically with $\|\mathbf{a}-\mathbf{b}\|_{R^3}$ and tends to $\pi/2$ Planck lengths as $\|\mathbf{a}-\mathbf{b}\|_{R^3}$ tends to infinity. Of course, when this distance is measured along the classical space "spiral" M_3 (rather than the great circle connecting the states), it takes arbitrarily large values, equal to the norm $\|\mathbf{a}-\mathbf{b}\|_{R^3}$.

Note that when the Planck system of units is used, the kernel $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}$ of the metric (2) is an extremely sharp, practically point-supported function. Indeed, $k(\mathbf{x}, \mathbf{y})$ falls off to almost zero within the first few Planck lengths of $\|\mathbf{x}-\mathbf{y}\|_{R^3}$. That means that for the usual in applications functions, $k(\mathbf{x}, \mathbf{y})$ behaves like the delta function $\delta^3(\mathbf{x}-\mathbf{y})$. By replacing the kernel $k(\mathbf{x}, \mathbf{y})$ in (2) with $\delta^3(\mathbf{x}-\mathbf{y})$ one obtains the ordinary L_2-inner product: $\int \delta^3(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\overline{\psi}(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y} = \int \varphi(\mathbf{x})\overline{\psi}(\mathbf{x})d^3\mathbf{x}$. It follows that the H-norms of the usual square-integrable functions are extremely close to their L_2-norms. That verifies that the Hilbert space H is *physical*, i.e., it can be consistently used in quantum mechanics in place of the ordinary space $L_2(R^3)$.

Consider first the collapse of a single particle state under a measurement of the particle's position. Assume that collapse is the motion along a geodesic φ_t connecting the initial and the terminal states on S^H [6]. Because geodesics are continuous curves, the path φ_t is *continuous*. Also, because the equation of geodesics is a differential equation, the metric on a small neighborhood of a point is sufficient to find the path φ_t near that point. In other words, the collapse is in this case a *continuous local process* on the sphere of states. Suppose now that the speed of collapse on the sphere of states is equal to the speed of light. Recall that the distance between any two states on S^H does not exceed π Planck lengths. It follows that the collapse of an *arbitrary* initial state onto an *arbitrary* terminal state happens in less than 10^{-43}s. For instance, the collapse from the superposition $c_1\delta^3_{\mathbf{x_1}} + c_2\delta^3_{\mathbf{x_2}}$ of two position eigenstates of a particle onto the state $\delta^3_{\mathbf{x_1}}$ of the particle found at the point $\mathbf{x_1}$ happens in less than this time interval for *all* values of $\mathbf{x_1}$ and $\mathbf{x_2}$ at once! On the other hand, if the process of collapse is supposed to propagate in the classical space from x_1 to x_2 at a constant speed, then that speed must be infinite. Even with the limitation that the distance between x_1 and x_2 does not exceed the size of the universe ($\approx 10^{27}$m), the above time interval would still require the collapse to have a ridiculous speed of $\approx 10^{70}$m/s! By all standards the resulting process is a discontinuous action at a distance.

In the case of a position measurement on a pair of particles consider the tensor product $H \otimes H$ with the above space H. The norm of the state $\delta^3_{\mathbf{a}} \otimes \delta^3_{\mathbf{b}}$ in $H \otimes H$ is the product of the H-norms of each delta-function, so it is equal to one. Accordingly, the set M_6 forms a submanifold of the unit sphere $S^{H\otimes H}$ in $H \otimes H$. As before, this sphere can be made small by using the Planck scale, in which case the previous consideration applies. To include the collapse due to a measurement of momentum of a particle or a pair of particles, one must change H so that to include the eigenstates of the momentum operator. In particular, by changing the kernel of the metric (2) to $e^{-\alpha\mathbf{x}^2}e^{-(\mathbf{x}-\mathbf{y})^2}e^{-\alpha\mathbf{y}^2}$, where $\alpha > 0$, one obtains a possible such space. At the same time, for a sufficiently small coefficient α other earlier discussed properties remain valid. In particular, the new Hilbert space $\widetilde{H}$ remains physical and the metric induced on the submanifolds M_3 and $\widetilde{M}_3$ is arbitrarily close to the Euclidean metric. The spaces $\widetilde{H}$ and $\widetilde{H} \otimes \widetilde{H}$ are appropriate for modeling collapse processes involving position and momentum measurements on a single particle or a pair of particles. In all these cases the previous model applies making collapse a continuous local process on the sphere of states that looks like an instantaneous process on the classical space. This completes the proof of statement (S).

It is generally accepted that quantum mechanical description of reality is more meager than the classical description. For instance, position and momentum of a particle in quantum mechanics do not have a simultaneous meaning. However, if reality is to be described by the state function of the system, the situation is reversed. For example, a complete classical description of a system of N particles at a given time requires $6N$ numbers (positions and momenta of the particles). On the other hand, to identify the state of a single particle at a given time one needs in general infinitely many numbers (components of the state function in a basis). So the state gives a much richer (although different) information about the system than the classical mechanical physical quantities.

One may wonder how could the state function description of *reality* be richer if the outcomes of our experiments are specific values of the physical quantities? The answer is simple: the state function contains information about *all* outcomes of the experiments on the system at once. It follows that quantum mechanics contains more information about reality than it normally gets credit for. Because the state is available to experimental determination, one should not insist that reality can only be associated with a specific outcome of a measurement. Rather, *all* possible outcomes of measurements on copies of a system identify a *single* reality of the system before measurement. Namely, these outcomes are *projections* of the reality that identify the reality itself (i.e., the state or the *position* of the system in the space of states).

The new definition of reality opens a way of investigating what happens to quantum system before, during, and after it has been measured. The fact that this can be done suggests that the current quantum mechanics is indeed incomplete. This incompleteness is not due to the lack of classicality in the quantum description. On the contrary, it originates in the lack of a consistent eradication of classicality from the basic tenets of quantum theory. By identifying the outcomes of measurements with the special cases of the reality associated with the state, one obtains a tool for embedding the classical into the quantum. The very first steps in this direction demonstrate that by properly completing the theory one can successfully resolve the mysteries of quantum mechanics and provide a much richer description of reality than the classical physics could ever hope for.

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