On a minimal system of Aristotle's Syllogistic

Piotr Kulicki

February 9, 2011

Abstract

The system of Syllogistic presented by J. Słupecki is a minimal, Łukasiewicz style system that includes all the theses present in Aristotle's writings. The axiomatic system is quite simple but it has no straightforward semantic counterpart. In the paper the semantics of the Słupecki's system is investigated: two approaches are used which lead to its two semantic characteristics. One is based on typically defined models, the other is a model-based decision procedure, using the notion of a Horn formula.

Keywords: Aristotle's Syllogistic, calculus of names, axiomatic rejection

Introduction

For many centuries Aristotle's Syllogistic was a dominant part of logic. Achievements in the field of mathematical logic in the 19th and the beginning of the 20th century changed that situation and pushed it into the margin of the discipline. The situation changed again when J. Łukasiewicz [4] presented Syllogistic as an axiomatic system built on classical propositional calculus (PL). The logical work of Aristotle was thus introduced into the mainstream of contemporary formal logic. The idea of investigating Syllogistic from the standpoint of modern formal logic was commonly accepted but the details of Łukasiewicz's axiomatisation became a subject of a discussion. The discussion on Syllogistic is still open and new ideas appear from time to time, e.g. recently computer science oriented logicians began to take part (see [5]).

One of the alternative proposals was given by J. Słupecki [8]. He accepted the general idea of constructing Syllogistic as a quantifier free theory based on PL, used the same language with the same primitive symbols, but changed the content of the theory by changing the axioms. His intention was to axiomatise a set of those formulae of the language of Łukasiewicz's Syllogistic that are true in S. Leśniewski's ontology. According to Słupecki, the main difference between the systems concerns empty names. In contrast to the system of Łukasiewicz, which is commonly interpreted as describing relations between non-empty names, Leśniewski's ontology accepts the use of empty names. Słupecki used the so called strong interpretation of universal sentences, in which one of the conditions for such a sentence is that both names are non-empty. As a result, Słupecki defined a system that is weaker then the one of Łukasiewicz.

The later works [6, 7] proved, however, that Słupecki's axiomatisation is not adequate with respect to the intended semantics, being sound but not complete with respect to it.

The system, failing to fulfil its intended specification, seems to be interesting for a different reason, namely, it is the smallest system built in the style of Łukasiewicz that includes all the theses present in the works of Aristotle. The conversion syllogisms of four figures, laws of the logical square and laws of conversion are theses of the system. On the other hand, all axioms of the system were present in the original works of Aristotle, so no weaker system can have this property. In this sense Słupecki's system is a minimal system of Aristotle's Syllogistic. Thus, it may be seen as a system that is closest to Aristotle's original logic as it appears in his writings.

Surprisingly, Słupecki's system is almost absent in the literature. This is probably a result of the fact that the majority of works on the subject are strongly connected with the intuitions built on interpretations of Syllogistic sentences in set theory. Słupecki's system, although clear on the syntactic level, does not fit into any of those interpretations.

The motivation behind the present paper is to draw attention to Słupecki's system. The main contribution of the paper is the semantic characterisation of the system, which has not been constructed so far. Apart from that, model-based decision procedures are presented for the system, similar to the procedure defined in [3] for Łukasiewicz's system.

1 The axiomatic system of Słupecki

We start with defining the language of Syllogistic. The alphabet contains name variables $S, P, M, \ldots$, two-argument operators forming universal and particular sentences of Syllogistic: a, i for affirmative and e, o for negative, used with infix notation (sentences SaP and SiP can be read respectively *all S are P* and *some S are P*, sentences SeP and SoP – *no S are P* and *some S are not P*) and operators of PL: $\neg, \wedge, \vee, \rightarrow, \equiv$ with the standard meaning. We will also use metalanguage symbols (possibly with subscripts): $\mathcal{X}, \mathcal{Y}, \ldots$ for names, $\alpha, \beta, \ldots$ for propositions, $\vdash$ for assertion and $\dashv$ for rejection of a proposition. The language of Syllogistic can be defined (in Backus-Naur notation) as follows:

$$\alpha \quad ::= \quad \mathcal{X}a\mathcal{X} \mid \mathcal{X}i\mathcal{X} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \alpha \equiv \alpha,$$

where $\mathcal{X}$ represents the category of name variables. Operators e and o are introduced by the following abbreviations:

$$SeP \triangleq \neg SiP,$$

$$SoP \triangleq \neg SiP.$$

Formulae built from operators a and i and their arguments will be called *atomic* formulae. Any atomic formula and any implication $\alpha \rightarrow \beta$, where α is a conjunction of atomic formulae and β is an atom, will be called a *Horn* formula.

The axiomatic system presented by Słupecki (further referred to as **S**) is defined by the rules of Modus Ponens (MP) and substitution for name variables (Sub) of the usual schemata. The axioms of the system are all the substitutions of the theses of classical propositional calculus in the language and the following specific axioms:

$$SaP \to SiP, \tag{1}$$

$$SiP \to PiS, \tag{2}$$

$$MaP \wedge SaM \to SaP, \tag{3}$$

$$MaP \wedge SiM \to SiP. \tag{4}$$

All the axioms are present in Aristotle's writings and traditional logic being its continuation. Axiom (1) is one of the laws of the logical square, axiom (2) is the law of convertion for particular affirmative sentences, axioms (3) and (4) are respectively syllogisms *Barbara* and *Datisi*. Because of that, no proper subsystem of **S** can be strong enough to contain Aristotle's logic. On the other hand, all the syllogisms of four figures, laws of the logical square and conversion laws can be proved in the system.

In particular, the following formulae are theses of **S**:

$$PaS \to SiP \tag{5}$$

derived from (1) and (2);

$$MiP \wedge MaS \to SiP, \tag{6}$$

$$PiM \wedge MaS \to SiP \tag{7}$$

and

$$MaP \wedge MiS \to SiP \tag{8}$$

all derived from (4) and (2);

$$MaP \wedge MaS \to SiP \tag{9}$$

derived from (8) and (1).

Moreover,

$$MiN \wedge MaS \wedge NaP \to SiP \tag{10}$$

and

$$NiM \wedge MaS \wedge NaP \to SiP \tag{11}$$

can be derived from (4) in combination with respectively (7) and (6).

The following notions of chain and connection are taken from [9].

Definition 1 *A chain from one variable to another is defined recursively as follows.*
(i) $\mathcal{X}a\mathcal{Y}$ *is a chain from* $\mathcal{X}$ *to* $\mathcal{Y}$*, for any* $\mathcal{X}$ *and* $\mathcal{Y}$*.*
(ii) If α *is a chain from* $\mathcal{X}$ *to* $\mathcal{Z}$*, then* $\alpha \wedge \mathcal{Z}a\mathcal{Y}$ *is a chain from* $\mathcal{X}$ *to* $\mathcal{Y}$*, for any* $\mathcal{X}$*,* $\mathcal{Y}$ *and* $\mathcal{Z}$*.*

For example the formulae $SaM \wedge MaP$ and $SaM \wedge MaN \wedge NaP$ are both chains from S to P.

Definition 2 *A conjunction of atoms α connects variable $\mathcal{X}$ with variable $\mathcal{Y}$ if and only if there exists a chain β from $\mathcal{X}$ to $\mathcal{Y}$ such that α contains every atom occurring in β as a conjunct.*

Lemma 1 *For any $\mathcal{X}$ and $\mathcal{Y}$ if a conjunction of atoms α connects $\mathcal{X}$ with $\mathcal{Y}$, then the formula $\alpha \to \mathcal{X}a\mathcal{Y}$ is a thesis of* **S**.

Proof The result can be obtained by a straightforward induction on the length of the chain from $\mathcal{X}$ to $\mathcal{Y}$ with the use of axiom (3). $\square$

The considerations concerning Słupecki's system will be based on some results concerning Łukasiewicz's system (further referred to as Ł), so we have to introduce formally the later system as well. Ł is defined in the same manner as **S**, by the rules MP and Sub with all substitutions of the theses of classical propositional calculus as axioms. The specific axioms are (3), (8) and the following:

$$SaS, \tag{12}$$

$$SiS. \tag{13}$$

It is easy to see that (1) can be derived from (13) and (8); (2) – from (12) and (8); and (4) – from (1) and (8). Thus, **S** is contained in Ł.

Furthermore, the following lemma can be derived from the investigations presented in [9].

Lemma 2 *For any formula α of the form $\beta \to \mathcal{X}a\mathcal{Y}$ or $\beta \to \mathcal{X}i\mathcal{Y}$, where $\mathcal{X} \neq \mathcal{Y}$ and β is a conjunction of atoms, the following two conditions are equivalent:*
(i) α is a thesis of Ł;
(ii) α is a thesis **S**.

Proof (ii) $\Rightarrow$ (i) imediatelly follows from the fact that **S** is contained in Ł.

For (i) $\Rightarrow$ (ii) proofs of lemmas XV and XVI from [9] will be used. In these proofs it is shown that the formula $\beta \to \mathcal{X}a\mathcal{Y}$ (where $\mathcal{X}$ and $\mathcal{Y}$ are different variables) is a theorem of Ł iff β connects $\mathcal{X}$ with $\mathcal{Y}$ and the formula $\beta \to \mathcal{X}a\mathcal{Y}$ (where $\mathcal{X}$ and $\mathcal{Y}$ are different variables) is a theorem of Ł iff one of the following conditions holds: (a) β contains $\mathcal{X}i\mathcal{Y}$ or $\mathcal{Y}i\mathcal{X}$ as a conjunct; (b) β connects $\mathcal{X}$ with $\mathcal{Y}$ or $\mathcal{Y}$ with $\mathcal{X}$; (c) there exists a variable $\mathcal{Z}$ such that β connects $\mathcal{Z}$ with $\mathcal{X}$ and contains $\mathcal{Y}i\mathcal{Z}$ or $\mathcal{Z}i\mathcal{Y}$ as a conjunct; (d) there exists a variable $\mathcal{Z}$ such that β connects $\mathcal{Z}$ with $\mathcal{Y}$ and contains $\mathcal{X}i\mathcal{Z}$ or $\mathcal{Z}i\mathcal{X}$ as a conjunct; (e) there exists a variable $\mathcal{Z}$ such that β connects $\mathcal{Z}$ with $\mathcal{X}$ and $\mathcal{Z}$ with $\mathcal{Y}$; (f) there exist variables $\mathcal{Z}$ and $\mathcal{V}$ such that β connects $\mathcal{Z}$ with $\mathcal{X}$ and $\mathcal{V}$ with $\mathcal{Y}$ and contains $\mathcal{Z}i\mathcal{V}$ or $\mathcal{V}i\mathcal{Z}$ as a conjunct.

To complete the proof it is enough to show that for all of these cases the considered formulae are also theses of **S**. For the formulae of the form $\beta \to \mathcal{X}a\mathcal{Y}$ that fact is guaranteed by Lemma 1. For the formulae of the form $\beta \to \mathcal{X}i\mathcal{Y}$ it is a consequence of the fact that the following formulae are axioms or theses of **S**: (1) for case (a), (2) and (5) for case (b), (6) and (7) for case (c), (4) and (8) for case (d), (9) for case (e), (10) and (11) for case (f) and, for cases (b) – (f), Lemma 1. □

Thus, when Horn formulae of the language are concerned, the difference between Łukasiewicz's and Słupecki's systems is limited to the formulae in the consequent of which the same name variable appears twice. In Ł all such formulae are theses, in **S** we only have:

$$SaP \to PiP. \tag{14}$$

On the other hand, in the intended interpretation, any Horn formula with the consequence of the form $\mathcal{X}a\mathcal{X}$ or $\mathcal{X}i\mathcal{X}$ in which variable $\mathcal{X}$ occurs in the ancedent is valid. Many of them are not theses of **S**, for example:

$$SiP \to SaS, \tag{15}$$

$$SiP \to SiS, \tag{16}$$

$$SaP \to SiS, \tag{17}$$

$$SaP \to SaS, \tag{18}$$

$$SaP \to PaP. \tag{19}$$

A. Pietruszczak in [7] shows that adding formula (15) to **S** leads to a system that is complete with respect to the semantics given by Słupecki.

Since the system of Pietruszczak lies clearly between **S** and Ł, the result from Lemma 2 also applies to it. Thus, we may notice that including or excluding empty names in models have influence only on Horn formulae with consequents of the form $\mathcal{X}a\mathcal{X}$ or $\mathcal{X}i\mathcal{X}$.

The rejected counterpart of **S** was introduced by B. Iwanuś in [1]. The following rejection rules: rejection by Modus Ponens (MP^{-1}), rejection by substitution (Sub^{-1}), rejection by composition ($Comp^{-1}$) are used (the rule ($Comp^{-1}$) used in the present paper is equivalent to the one introduced by Słupecki for rejection for Łukasiewicz's system, which was also used by Iwanuś).

Rule MP^{-1} takes the form:

$$\frac{\vdash \alpha \to \beta; \dashv \beta}{\dashv \alpha}.$$

Rule Sub^{-1} takes the form:

$$\frac{\dashv e(\alpha)}{\dashv \alpha},$$

where e is a substitution.

Rule $Comp^{-1}$ takes the form:

$$\frac{\dashv \alpha \to \beta_1; \ldots; \dashv \alpha \to \beta_n}{\dashv \alpha \to \beta_1 \vee \ldots \vee \beta_n}, n \geq 1,$$

where α is a conjunction of atoms and $\beta_i (1 \leq i \leq n)$ are atoms.

The following formulae are rejected axioms:

$$SaM \wedge MaM \rightarrow SiS, \tag{20}$$

$$SaM \wedge MaP \wedge SaS \wedge PaP \rightarrow MaM, \tag{21}$$

$$SaM \wedge PaM \wedge SaS \wedge PaP \wedge MaM \rightarrow SiP. \tag{22}$$

The intuitive meaning of the rejected axioms of the system is not quite clear. Thus, we shall treat them technically, as sufficient for completeness results.

The following lemmas proved in [1] state the fact that the refutation counterpart for Słupecki's system is adequate.

Lemma 3 *Every formula of the language of Syllogistic is either a thesis or a rejected formula of* **S** *and no formula is both a thesis and a rejected formula.*

Lemma 4 *If a Horn formula* α *is not a thesis of* **S**, *then the Słupecki's system plus* α *entails one of the rejected axioms.*

Lemma 3 will be used in the completeness proof in the next section, while Lemma 4 will be useful for designing the decision procedures in Section 3.

2 Models for Słupecki's system

Let us first recall the semantic results about Ł. The following structure is a model for the system: $\mathcal{M}^{\text{Ł}} = \langle \mathcal{B}, f, \mathcal{I}^{\text{Ł}} \rangle$, where $\mathcal{B}$ is a non-empty family of non-empty sets, f is a function from the set of name variables to $\mathcal{B}$ and $\mathcal{I}^{\text{Ł}}$ is a function whose argument is function f and value is a set of atomic formulae, such that:

$$\begin{array}{lcl} \mathcal{X}a\mathcal{Y} \in \mathcal{I}^{\text{Ł}}(f) & \Longleftrightarrow & f(\mathcal{X}) \subset f(\mathcal{Y}) \\ \mathcal{X}i\mathcal{Y} \in \mathcal{I}^{\text{Ł}}(f) & \Longleftrightarrow & f(\mathcal{X}) \cap f(\mathcal{Y}) \neq \emptyset \end{array}$$

Informally we think of the function $\mathcal{I}^{\text{Ł}}$ as an interpretation of atomic formulae and of the set $\mathcal{I}^{\text{Ł}}(f)$ as the set of atomic formulae that are true in model $\mathcal{M}^{\text{Ł}}$. Next we inductively define the notion of *truth* in the model $\mathcal{M}^{\text{Ł}}$ for arbitrary formula α $\mathcal{M}^{\text{Ł}}$ (written: $\mathcal{M}^{\text{Ł}} \models \alpha$) as follows:
(i) for any atomic α, $\mathcal{M}^{\text{Ł}} \models \alpha$ iff $\alpha \in \mathcal{I}^{\text{Ł}}(f)$;
(ii) the notion of truth in $\mathcal{M}^{\text{Ł}}$ preserves classical truth conditions for PL operators.

Let **Ł** be the set of all the models $\mathcal{M}^{\text{Ł}}$ (models based on interpretation function $\mathcal{I}^{\text{Ł}}$ with different sets $\mathcal{B}$ and functions f). We shall understand that a formula is valid in **Ł** iff it is true in all models from **Ł**.

The following adequacy theorem comes from [9] (its proof using standard tools for completeness proofs insetead of axiomatic refutation used by Słupecki is given in [2]).

Theorem 1 *L is sound and complete with respect to the class* **Ł**.

Now we are ready to present a model suitable for **S**. The following structure is a model for the system: $\mathcal{M}^{\mathsf{S}} = \langle \mathcal{B}, f, g, \mathcal{I}^{\mathsf{S}} \rangle$, where, as in the case of Ł, $\mathcal{B}$ is a non-empty family of non-empty sets, f is a function from the set of name variables of the language to $\mathcal{B}$. The additional element g is a function from the set of name variables of the language to set $\{0, 1\}$. Thus, an ordered pair: $\langle f(\mathcal{X}), g(\mathcal{X}) \rangle$ is attached to every name variable $\mathcal{X}$.

$\mathcal{I}^{\mathsf{S}}$ is a function whose arguments are functions f and g and its value is a set of atomic formulae, such that:

$$
\begin{array}{lcl}
\mathcal{X}a\mathcal{Y} \in \mathcal{I}^{\mathsf{S}}(f, g) & \iff & f(\mathcal{X}) \subset f(\mathcal{Y}) \\
 & & \text{or } f(\mathcal{X}) = f(\mathcal{Y}) \text{ and } g(\mathcal{X}) = g(\mathcal{Y}) = 1 \\
\mathcal{X}i\mathcal{Y} \in \mathcal{I}^{\mathsf{S}}(f, g) & \iff & f(\mathcal{X}) \neq f(\mathcal{Y}) \text{ and } f(\mathcal{X}) \cap f(\mathcal{Y}) \neq \emptyset \\
 & & \text{or } f(\mathcal{X}) = f(\mathcal{Y}) \text{ and } |f(\mathcal{X})| \geq 2 \\
 & & \text{or } f(\mathcal{X}) = f(\mathcal{Y}) \text{ and } g(\mathcal{X}) = g(\mathcal{Y}) = 1
\end{array}
$$

Note that the inclusion present in the condition for $\mathcal{X}a\mathcal{Y}$ is proper.

The inductive definition of the notion of *truth* in the model $\mathcal{M}^{\mathsf{S}}$ for an arbitrary formula α is the same as for $\mathcal{M}^{\mathsf{Ł}}$.

Let **S** be the set of all models $\mathcal{M}^{\mathsf{S}}$ (models based on $\mathcal{I}^{\mathsf{S}}$ with different sets $\mathcal{B}$ and functions f and g). We shall understand that a formula is valid in **S** iff it is true in all models from **S**.

In order to match the axiomatic system the model structure does not follow directly Słupecki's initial motivation. To capture its intuitive meaning let us compare it with **Ł**.

For atomic formulae with two different name variables the interpretations $\mathcal{I}^{\mathsf{Ł}}(f)$ and $\mathcal{I}^{\mathsf{S}}(f, g)$ behave in the same way, i.e. the same atomic sentences of that type are true in both interpretations. On the other hand, for atomic formulae with two identical arguments (technically the same set in a model) an additional function g, independent from f, is being used. The function g determines which of the sentences of the form $\mathcal{X}a\mathcal{X}$ and $\mathcal{X}i\mathcal{X}$ are valid in **S**. The value of the function g can be seen as a separate property of a name. For some names g gives 1, for others 0, regardless of the extension of the name.

Moreover, the second condition for the formulae of the form $\mathcal{X}i\mathcal{Y}$ ($f(\mathcal{X}) = f(\mathcal{Y})$ and $|f(\mathcal{X})| \geq 2$), corresponds to thesis (14) of **S**.

The class of models **S** is based on non-empty sets, like the semantics for Ł. To get a little closer to Słupecki's idea we shall define another class of models **S'** in which empty sets are allowed. The structures are equivalent in the sense that the set of valid formulae of both is the same.

S' is the class of all models $\mathcal{M}^{\mathsf{S'}} = \langle \mathcal{B}, f, g, \mathcal{I}^{\mathsf{S'}} \rangle$, where $\mathcal{B}$ is a non-empty family of arbitrary sets, f and g are functions as in $\mathcal{M}^{\mathsf{S}}$, and $\mathcal{I}^{\mathsf{S'}}$ is a function analogous to $\mathcal{I}^{\mathsf{S}}$ defined as follows:

$$
\begin{array}{lcl}
\mathcal{X}a\mathcal{Y} \in \mathcal{I}^{\mathsf{S'}}(f, g) & \iff & \emptyset \neq f(\mathcal{X}) \subset f(\mathcal{Y}) \\
 & & \text{or } f(\mathcal{X}) = f(\mathcal{Y}) \text{ and } g(\mathcal{X}) = g(\mathcal{Y}) = 1 \\
\mathcal{X}i\mathcal{Y} \in \mathcal{I}^{\mathsf{S'}}(f, g) & \iff & f(\mathcal{X}) \neq f(\mathcal{Y}) \text{ and } f(\mathcal{X}) \cap f(\mathcal{Y}) \neq \emptyset \\
 & & \text{or } f(\mathcal{X}) = f(\mathcal{Y}) \text{ and } |f(\mathcal{X})| \geq 2 \\
 & & \text{or } f(\mathcal{X}) = f(\mathcal{Y}) \text{ and } g(\mathcal{X}) = g(\mathcal{Y}) = 1
\end{array}
$$

Theorem 2 *An arbitrary formula α of the language of Syllogistic is valid in* **S'** *if and only if it is valid in* **S**.

Proof Let us first observe that if non-empty sets are considered the interpretations $\mathcal{I}^{\mathbf{S'}}$ and $\mathcal{I}^{\mathbf{S}}$ they always give the same value.

If α is valid in **S'**, then it is true in all models from **S'**, including all models defined on non-empty sets. Consequently α is valid in **S**.

If, on the other hand, α is not valid in **S'**, then there exists a model $\mathcal{M}_1 = \langle \mathcal{B}_1, f_1, g_1, \mathcal{I}^{\mathbf{S'}} \rangle \in$ **S'**, such that $\mathcal{M}_1 \not\models \alpha$. If $\emptyset \notin \mathcal{B}_1$, for a model $\mathcal{M}_2 = \langle \mathcal{B}_1, f_1, g_1, \mathcal{I}^{\mathbf{S}} \rangle \in$ **S** we have $\mathcal{M}_2 \not\models \alpha$.

Otherwise we can construct a model $\mathcal{M}_3 = \langle \mathcal{B}_3, f_3, g_3, \mathcal{I}^{\mathbf{S}} \rangle \in$ **S**, such that $\mathcal{M}_3 \not\models \alpha$ in the following way. Let o be an object which is not an element of the set $\bigcup\{f_1(\mathcal{X}) : \mathcal{X}$ appears in $\alpha\}$. $\mathcal{B}_3, f_3, g_3$ are defined as follows:

$$\mathcal{B}_3 = \mathcal{B}_1 \cup \{o\}$$

$$f_3 = \begin{cases} \{o\} & \text{if } f_1(\mathcal{X}) = \emptyset \\ f_1(\mathcal{X}) & \text{otherwise} \end{cases}$$

$$g_3 = g_1$$

To complete the proof it is enough to show that for arbitrary atomic formula β, $\mathcal{M}_1 \models \beta$ iff $\mathcal{M}_3 \models \beta$.

Let $\beta = \mathcal{X}a\mathcal{Y}$. If $\mathcal{M}_1 \models \beta$, then (i) $\emptyset \neq f_1(\mathcal{X}) \subset f_1(\mathcal{Y})$ or (ii) $f_1(\mathcal{X}) = f_1(\mathcal{Y})$ and $g_1(\mathcal{X}) = g_1(\mathcal{Y}) = 1$. In case (i) since both $f_1(\mathcal{X})$ and $f_1(\mathcal{Y})$ are non-empty $f_3(\mathcal{X}) = f_1(\mathcal{X})$ and $f_3(\mathcal{Y}) = f_1(\mathcal{Y})$ and consequently $f_3(\mathcal{X}) \subset f_3(\mathcal{Y})$ and $\mathcal{M}_3 \models \beta$. In case (ii) if $f_1(\mathcal{X}) = f_1(\mathcal{Y}) = \emptyset$, then $f_3(\mathcal{X}) = f_3(\mathcal{Y}) = \{o\}$ and if $f_1(\mathcal{X}) = f_1(\mathcal{Y}) \neq \emptyset$, then $f_3(\mathcal{X}) = f_1(\mathcal{X}) = f_1(\mathcal{Y}) = f_3(\mathcal{Y})$. Since $g_3(\mathcal{X}) = g_1(\mathcal{X}) = g_1(\mathcal{Y}) = g_3(\mathcal{Y}) = 1$, $\mathcal{M}_3 \models \beta$.

If $\mathcal{M}_3 \models \beta$, then (i) $f_3(\mathcal{X}) \subset f_3(\mathcal{Y})$ or (ii) $f_3(\mathcal{X}) = f_3(\mathcal{Y})$ and $g_3(\mathcal{X}) = g_3(\mathcal{Y}) = 1$. In case (i), since, to fulfil the condition, none of $f_3(\mathcal{X})$, $f_3(\mathcal{Y})$ may equal $\{o\}$, $\emptyset \neq f_1(\mathcal{X}) \subset f_1(\mathcal{Y})$ and $\mathcal{M}_1 \models \beta$. In case (ii) either $f_3(\mathcal{X}) = f_3(\mathcal{Y}) = \{o\}$ and $f_1(\mathcal{X}) = f_1(\mathcal{Y}) = \emptyset$ or $f_3(\mathcal{X}) = f_3(\mathcal{Y}) \neq \{o\}$ and $f_1(\mathcal{X}) = f_3(\mathcal{X}) = f_3(\mathcal{Y}) = f_1(\mathcal{Y})$. Since $g_1(\mathcal{X}) = g_3(\mathcal{X}) = g_3(\mathcal{Y}) = g_1(\mathcal{Y}) = 1$, $\mathcal{M}_1 \models \beta$.

Now, let $\beta = \mathcal{X}i\mathcal{Y}$. If $\mathcal{M}_1 \models \beta$, then (i) $f_1(\mathcal{X}) \neq f_1(\mathcal{Y})$ and $f_1(\mathcal{X}) \neq f_1(\mathcal{Y}) \neq \emptyset$ or (ii) $f_1(\mathcal{X}) = f_1(\mathcal{Y})$ and $|f_1(\mathcal{X})| \geq 2$ or (iii) $f_1(\mathcal{X}) = f_1(\mathcal{Y})$ and $g_1(\mathcal{X}) = g_1(\mathcal{Y}) = 1$. In cases (i) and (ii) $f_3(\mathcal{X}) = f_1(\mathcal{X})$ and $f_3(\mathcal{Y}) = f_1(\mathcal{Y})$ and obviously $\mathcal{M}_3 \models \beta$. In case (iii) if $f_1(\mathcal{X}) = f_1(\mathcal{Y}) = \emptyset$, then $f_3(\mathcal{X}) = f_3(\mathcal{Y}) = \{o\}$ and if $f_1(\mathcal{X}) = f_1(\mathcal{Y}) \neq \emptyset$, then $f_3(\mathcal{X}) = f_1(\mathcal{X}) = f_1(\mathcal{Y}) = f_3(\mathcal{Y})$. Since $g_3(\mathcal{X}) = g_1(\mathcal{X}) = g_1(\mathcal{Y}) = g_3(\mathcal{Y}) = 1$, $\mathcal{M}_3 \models \beta$.

If $\mathcal{M}_3 \models \beta$, then (i) $f_3(\mathcal{X}) \neq f_3(\mathcal{Y})$ and $f_3(\mathcal{X}) \neq f_3(\mathcal{Y}) \neq \emptyset$ or (ii) $f_3(\mathcal{X}) = f_3(\mathcal{Y})$ and $|f_3(\mathcal{X})| \geq 2$ or (iii) $f_3(\mathcal{X}) = f_3(\mathcal{Y})$ and $g_3(\mathcal{X}) = g_3(\mathcal{Y}) = 1$. In cases (i) and (ii), since to fulfil the condition, none of $f_3(\mathcal{X})$, $f_3(\mathcal{Y})$ may equal $\{o\}$, $f_3(\mathcal{X}) = f_1(\mathcal{X})$ and $f_3(\mathcal{Y}) = f_1(\mathcal{Y})$ and obviously $\mathcal{M}_1 \models \beta$. In case (iii), like in case (ii) for $\beta = \mathcal{X}a\mathcal{Y}$, $f_1(\mathcal{X}) = f_1(\mathcal{Y})$ $g_1(\mathcal{X}) = g_1(\mathcal{Y}) = 1$ and consequently $\mathcal{M}_1 \models \beta$. $\square$

To construct a completeness proof for **S** we shall use the following two lemmas.

Lemma 5 *For any formula α of a form $\beta \to \mathcal{X}a\mathcal{Y}$ or $\beta \to \mathcal{X}i\mathcal{Y}$, where $\mathcal{X} \neq \mathcal{Y}$ and β is a conjunction of atoms, if α is not a thesis of* **S**, *then there exists a model $\mathcal{M} = \langle \mathcal{B}, f, g, \mathcal{I}^{\mathsf{S}} \rangle \in \mathsf{S}$ such that $\mathcal{M} \not\models \alpha$ and $f(\mathcal{X}) \neq f(\mathcal{Y})$.*

Proof By virtue of Lemma 2, if α is not a thesis of **S**, then it is not a thesis of Ł. Thus, there exists a model $\mathcal{M}' = \langle \mathcal{B}', f', \mathcal{I}^{\text{Ł}} \rangle \in \text{Ł}$, such that $\mathcal{M}' \not\models \alpha$. Obviously, $f'(\mathcal{X}) \neq f'(\mathcal{Y})$. Now it is enough to put $\mathcal{B} = \mathcal{B}'$, $f = f'$ and $g(\mathcal{X}) = 1$, for every $\mathcal{X}$, to get $\mathcal{M} \not\models \alpha$. □

Lemma 6 *Let $\mathcal{M}_1 = \langle \mathcal{B}_1, f_1, g_1, \mathcal{I}^{\mathsf{S}} \rangle$ and $\mathcal{M}_2 = \langle \mathcal{B}_2, f_2, g_2, \mathcal{I}^{\mathsf{S}} \rangle$ be models from S. Let further $\mathcal{M}_3 = \langle \mathcal{B}_1 \times \mathcal{B}_2, g_3, f_3, \mathcal{I}^{\mathsf{S}} \rangle$, where $f_3(\mathcal{X}) = f_1(\mathcal{X}) \times f_2(\mathcal{X})$ and $g_3(\mathcal{X}) = min(g_1(\mathcal{X}), g_2(\mathcal{X}))$.*
(i) If $\mathcal{M}_1 \models \mathcal{X}a\mathcal{Y}$ and $\mathcal{M}_2 \models \mathcal{X}a\mathcal{Y}$, then $\mathcal{M}_3 \models \mathcal{X}a\mathcal{Y}$.
(ii) If $\mathcal{M}_1 \models \mathcal{X}i\mathcal{Y}$ and $\mathcal{M}_2 \models \mathcal{X}i\mathcal{Y}$, then $\mathcal{M}_3 \models \mathcal{X}i\mathcal{Y}$.

Proof In the proof two cases have to be considered: (a) $f_1(\mathcal{X}) \neq f_1(\mathcal{Y})$ or $f_2(\mathcal{X}) \neq f_2(\mathcal{Y})$ and (b) $f_1(\mathcal{X}) = f_1(\mathcal{Y})$ and $f_2(\mathcal{X}) = f_2(\mathcal{Y})$.

(i) In case (a) $f_3(\mathcal{X}) = (f_1(\mathcal{X}) \times f_2(\mathcal{X})) \neq f_3(\mathcal{Y}) = (f_1(\mathcal{Y}) \times f_2(\mathcal{Y}))$ and $f_3(\mathcal{X}) \subset f_3(\mathcal{Y})$. Thus $\mathcal{M}_3 \models \mathcal{X}a\mathcal{Y}$. In case (b) $f_3(\mathcal{X}) = (f_1(\mathcal{X}) \times f_2(\mathcal{X})) = f_3(\mathcal{Y}) = (f_1(\mathcal{Y}) \times f_2(\mathcal{Y}))$. Since $\mathcal{M}_1 \models \mathcal{X}a\mathcal{Y}$ and $\mathcal{M}_2 \models \mathcal{X}a\mathcal{Y}$, $g_1(\mathcal{X}) = g_1(\mathcal{Y}) = g_2(\mathcal{X}) = g_2(\mathcal{Y}) = 1$. Thus $g_3(\mathcal{X}) = g_3(\mathcal{Y}) = 1$ and consequently $\mathcal{M}_3 \models \mathcal{X}a\mathcal{Y}$.

(ii) In case (a), since $f_1(\mathcal{X}) \cap f_1(\mathcal{Y}) \neq \emptyset$ and $f_2(\mathcal{X}) \cap f_2(\mathcal{Y}) \neq \emptyset$ also $f_3(\mathcal{X}) \cap f_3(\mathcal{Y}) \neq \emptyset$ and consequently $\mathcal{M}_3 \models \mathcal{X}i\mathcal{Y}$. In case (b) $f_3(\mathcal{X}) = f_3(\mathcal{Y})$. If $|f_1(\mathcal{X})| \geq 2$ or $|f_2(\mathcal{X})| \geq 2$, then also $|f_3(\mathcal{X})| \geq 2$ and consequently $\mathcal{M}_3 \models \mathcal{X}i\mathcal{Y}$. Otherwise, $g_1(\mathcal{X}) = g_1(\mathcal{Y}) = g_2(\mathcal{X}) = g_2(\mathcal{Y}) = 1$. Thus $g_3(\mathcal{X}) = g_3(\mathcal{Y}) = 1$ and consequently $\mathcal{M}_3 \models \mathcal{X}i\mathcal{Y}$. □

Theorem 3 *System* **S** *is sound and complete with respect to the class S.*

Proof Because of Lemma 3 it is enough to show that all the theses are valid and all the rejected formulae are not. For all the axioms it is a usual routine to check that if the antecedent of an axiom is true in a model, then the consequent of the axiom is true in the model as well. Rejected axioms are false in the following models: for axiom (20) $\mathcal{B} = \{1,2\}, f(S) = \{1\}, f(M) = \{1,2\}, g(S) = 0, g(M) = 1$, for axiom (21) $\mathcal{B} = \{1,2,3\}, f(S) = \{1\}, f(M) = \{1,2\}, f(P) = \{1,2,3\}, g(S) = g(P) = 1, g(M) = 0$, $\mathcal{B} = \{1,2\}, f(S) = \{1\}, f(P) = \{2\}, f(M) = \{1,2\}, g(S) = g(P) = g(M) = 1$. Rule MP preserves the truth and MP^{-1} preserves falsehood, because the system is based on PL. Analogical facts hold for Sub and Sub^{-1} because, if a formula is satisfied in all models, a substitution for name variables cannot change that fact.

To complete the proof we have to show that rule $Comp^{-1}$ leads from false formulae to a formula that is also false. In order to do that it is enough to prove that for a conjunction of atoms α and atoms β_1 and β_2 if $\alpha \to \beta_1$ and $\alpha \to \beta_2$ are false, then $\alpha \to \beta_1 \vee \beta_2$ is also false. Extending it for rule $Comp^{-1}$ for an arbitrary n, can be done by a straightforward induction. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the families of sets for which there exist functions f_1, g_1, f_2 and g_2 such that $\alpha \to \beta_1$ is falsified by $\mathcal{M}_1 = \langle \mathcal{B}_1, f_1, g_1, \mathcal{I}^{\mathsf{S}} \rangle$ and $\alpha \to \beta_2$ is falsified by $\mathcal{M}_2 = \langle \mathcal{B}_2, f_2, g_2, \mathcal{I}^{\mathsf{S}} \rangle$.

By Lemma 5 if β_i ($i \in \{1,2\}$) is built with the use of two different variables $\mathcal{X}$ and $\mathcal{Y}$, than f_i can be defined in such a way that $f_i(\mathcal{X}) \neq f_i(\mathcal{Y})$. If, on the other hand, β_i ($i \in \{1,2\}$) is of the form $\mathcal{X}i\mathcal{X}$, then functions f_1 and f_2 can be defined in such a way, that $|f_1(\mathcal{X})| = |f_2(\mathcal{X})| = 1$ — otherwise $\mathcal{M}_i \not\models \alpha \to \beta_i$ would not hold.

We will now show that for $\mathcal{M}_3$=$\langle \mathcal{B}_1 \times \mathcal{B}_2, g_3, f_3, \mathcal{I}^{\mathbf{S}} \rangle$, where $f_3(\mathcal{X}) = f_1(\mathcal{X}) \times f_2(\mathcal{X})$ and $g_3(\mathcal{X}) = \min(g_1(\mathcal{X}), g_2(\mathcal{X}))$ $\mathcal{M}_3 \not\models \alpha \to \beta_1 \vee \beta_2$. By Lemma 6 $\mathcal{M}_3 \models \alpha$. We have to show that $\mathcal{M}_3 \not\models \beta_1$ and $\mathcal{M}_3 \not\models \beta_2$. If β_i ($i \in \{1,2\}$) is of the form $\mathcal{X}a\mathcal{Y}$ or $\mathcal{X}i\mathcal{Y}$ ($\mathcal{X} \neq \mathcal{Y}$), then $f_3(\mathcal{X}) \neq f_3(\mathcal{Y})$. Furthermore, in the case of $\mathcal{X}a\mathcal{Y}$ we have $f_3(\mathcal{X}) \not\subset f_3(\mathcal{Y})$ and in the case of $\mathcal{X}i\mathcal{Y}$ we have $f_3(\mathcal{X}) \cap f_3(\mathcal{Y}) = \emptyset$. Thus $\mathcal{M}_3 \not\models \beta_i$. Otherwise $\beta_i = \mathcal{X}a\mathcal{X}$ or $\beta_i = \mathcal{X}i\mathcal{X}$. In both cases $g_3(\mathcal{X}) = 0$. Morover, in the case of $\mathcal{X}i\mathcal{X}$ $f_3(X) = f_1 \times f_2$ is a singleton. Consequently $\mathcal{M}_3 \not\models \beta_i$. $\square$

3 Matrices for Horn formulae

In this section we construct a different semantic characterisation of **S**. The presence of rule $Comp^{-1}$ allows us to state the following disjunction property. Let α be a conjunction of atoms and $\beta_i (1 \leq i \leq n)$ be atoms.

$\alpha \to \beta_1 \vee \ldots \vee \beta_n$ is a thesis of **S**
if and only if one of the formulae
$\alpha \to \beta_1$ or ... or $\alpha \to \beta_n, n \geq 1$, is a thesis of the system.

That property in axiomatic systems based on PL (in which any formula has an equivalent formula in a conjunctive normal form) reduces the problem of the truth of arbitrary formulae to Horn formulae.

The following matrices a_1 and i_1, combined with the classical interpretation for conjunction and implication, give the interpretation for Horn formulae of **S**.

a	x	y	z	v	w
x	0	0	0	1	1
y	0	1	0	1	1
z	0	0	1	0	1
v	0	0	0	0	1
w	0	0	0	0	1

i	x	y	z	v	w
x	0	0	0	1	1
y	0	1	0	1	1
z	0	0	1	0	1
v	1	1	0	1	1
w	1	1	1	1	1

Table 1: Matrices a_1 and i_1

Theorem 4 *A Horn formula of the language of Syllogistic is a thesis of* **S** *iff value 1 is obtained for all the substitutions of values from set* $\{x, y, z, v, w\}$ *for name variables, using matrices* a_1*,* i_1 *and classical matrices for conjunction and implication.*

Proof By virtue of Lemma 3 (ii) it is enough to show that rules MP and Sub are preserved (which, in the case of MP, is ensured through the use of classical matrix for implication, which is normal and in the case of Sub is obvious), axioms are true for all substitutions of values for name variables (routine checking is left to the Reader) and there exist substitutions for which the rejected axioms are false. The required substitutions are as follows: S/x and M/w in axiom (20); S/y, P/w and M/v in axiom (21); S/y, P/z and M/w in axiom (22). □

Theorem 4 shows that to decide whether a Horn formula (indeed, as mentioned above, the procedure can be syntactically extended for arbitrary formulae) is a thesis of **S**, it is enough to consider its substitutions in a domain of 5 constant names, no matter how long the formula is. This property can be interesting from the computational point of view.

In fact, these matrices define one of many possibilities of relations between the five constant names that can be used for such a procedure. With the classical interpretation of conjunction and implication there is no smaller domain that fulfils such a condition.

However, the number of names can be reduced to 4 by applying a different interpretation for propositional operators. On the basis of Łukasiewicz's 3-valued propositional logic, defined by matrices in Table 2, matrices a_2 and i_2 can be used to obtain an interpretation for Horn formulae of **S**.

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

Table 2: 3-valued Łukasiewicz logic tables for conjunction and implication

a	x	y	z	v
x	1	1	0	1
y	0	0	0	1
z	0	0	$\frac{1}{2}$	1
v	0	0	0	1

i	x	y	z	v
x	1	1	0	1
y	1	1	0	1
z	0	0	$\frac{1}{2}$	1
v	1	1	1	1

Table 3: Matrices a_2 and i_2

Theorem 5 *A Horn formula of the language of Syllogistic is a thesis of* **S** *iff value 1 is obtained for all the substitutions of values from set* $\{x, y, z, v\}$ *for name variables, using matrices* a_2, i_2 *and Łukasiewicz's 3-valued matrices for conjunction and implication.*

Proof Łukasiewicz's 3-valued matrix for implication is normal, so the rule MP is preserved. Checking that the axioms are true in the matrix system is, just as in the proof of Theorem 4, left to the Reader. Substitutions in which rejected axioms are false are as follows: S/z and M/v in axiom (20); S/x, P/v and M/y in axiom (21); S/x, P/z and M/v in axiom (22). □

The number of names used for the decision procedure can be further reduced by using yet another interpretation of propositional operators, defined by the 4-valued matrices for conjunction and implication presented in Table 4. The matrix for conjunction is Łukasiewicz's 4-valued logic and the one for implication is his 4-valued modal logic from [4].

$\wedge$	0	n_1	n_2	1
0	0	0	0	0
n_1	0	n_1	n_1	n_1
n_2	0	n_1	n_2	n_2
1	0	n_1	n_2	1

$\rightarrow$	0	n_1	n_2	1
0	1	1	1	1
n_1	n_2	1	n_2	1
n_2	n_1	n_1	1	1
1	0	n_1	n_2	1

Table 4: 4-valued interpretation of conjunction and implication

In that case the following 3-valued matrices a_3, i_3 for predicates are adequate.

a	y	z	v
y	n_1	0	1
z	0	n_2	1
v	0	0	n_1

i	y	z	v
y	n_1	0	1
z	0	n_2	1
v	1	1	1

Table 5: Matrices a_3 and i_3

Theorem 6 *A Horn formula of the language of Syllogistic is a thesis of* **S** *iff value 1 is obtained for all the substitutions of values from set* $\{y, z, v\}$ *for name variables, using matrices* a_3, i_3 *and the presented in Table 5, 4-valued matrices for conjunction and implication.*

Proof Again, the matrix for the implication is normal, so rule MP is preserved. Checking that the axioms are true in the matrix system is a usual routine. Substitutions in which rejected axioms are false are as follows: S/z and M/v in axiom (20); S/z, P/v and M/z in axiom (21); S/y, P/z and M/v in axiom (22). □

It is interesting to compare the above matrices with analogous matrices for Ł shown in Table 6, introduced in [3].

Matrices coincide when two different name constants appear in the same atomic formula, and the difference lies in the formulae with the same name appearing twice. For **S** in such situations values n_1 and n_2, which can be interpreted as *undetermined* values, are used. The value of the formula *viv* in matrix i_3 is an exception stemming from the fact that formula (14) is a thesis of the system.

a	y	z	v
y	1	0	1
z	0	1	1
v	0	0	1

i	y	z	v
y	1	0	1
z	0	1	1
v	1	1	1

Table 6: Matrices for Ł

Conclusions

Słupecki's system of Syllogistic considered in the paper is the minimal quantifier free system of Syllogistic based on classical propositional calculus (built in the style of Łukasiewicz) including the laws of Aristotle's logic. The system defines the meaning of general and particular affirmative sentences of Syllogistic, which do not have a direct interpretation in set theory. The semantics for the calculus was introduced. Decision procedures for the system, based on models, were also presented.

Acknowledgements

I wish to thank prof. Andrzej Pietruszczak for his valuable advice and helpful comments on earlier versions of this paper.

References

[1] B. Iwanuś. O sylogistyce Arystotelesa. *Zeszyty Naukowe WSP w Opolu, Matematyka*, XXVIII:41–56, 1972.

[2] Shepherdson J.C. On the interpretation of Aristotelian syllogistic. *The Journal of Symbolic Logic*, 21:137–147, 1956.

[3] P. Kulicki. Remarks on axiomatic rejection in Aristotle's syllogistic. *Studies in Logic and Theory of Knowledge*, 5:231–236, 2002.

[4] J. Łukasiewicz. *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic.* Clarendon Press, Oxford, 1957.

[5] L.S. Moss. Completeness theorems for syllogistic fragments. In S. Kepser F. Hamm, editor, *Logics for Linguistic Structures*, pages 143–174. Mouton de Gruyter, Berlin, New York, 2008.

[6] A. Pietruszczak. O logice tradycyjnej i rachunku nazw dopuszczającym podstawienia nazw pustych. *Ruch Filozoficzny*, 44:158–166, 1987.

[7] A. Pietruszczak. O pewnym ujęciu logiki tradycyjnej. *Acta Universitatis Nicolai Copernici, Logika*, I:31–41, 1991.

[8] J. Słupecki. Uwagi o sylogistyce Arystotelesa. *Annales UMCS*, I:187–191, 1946.

[9] J. Słupecki. *Z badań nad sylogistyką Arystotelesa*. Wroclaw, 1948.

Piotr Kulicki
Chair of Foundation of Computer Science
Faculty of Philosophy
The John Paul II Catholic University of Lublin
Al. Raclawickie 14, Lublin, Poland,
kulicki@kul.pl