In memory of M.I. Kuligina (1914–1994)

The Electromagnetic Mass of a Charged Particle

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A solution of the of electromagnetic mass is obtained in the framework of Maxwell's equations. There is a mathematical proof that the electromagnetic mass possesses the standard properties of the inertial mass. We conclude that Maxwell's equations deal with two kinds of fields. These are local fields of charged particles and the propagating fields of the electromagnetic wave.

Introduction

Two aspects of the electromagnetic mass (EMM) are considered. The first aspect is the classical problem of the EMM. Shortly after the discovery of the law of energy conservation by Poynting, it turned out that the EMM did not meet the standard properties of an inertial Mass. Below we shall consider examples.

The second aspect is the problem of charged particle models. It is known that charged particle models are not stable because of Coulomb forces, which must break the charge. The hypothesis was advanced that the inertial mass of a particle was equal to the sum of the EMM and the non-electromagnetic mass (NEM), creating a stable state in the particle. Here a constraint was set by the EMM problem: the *bad* properties of EMM must be balanced by the other *bad* properties of NEM. Without a solution to the problem of the EMM, the search for a model cannot succeed (Ivanenko 1949; Feynman *et al.* 1964). It seemed that a solution might be provided by quantum theory. However, this did not occur. Moreover, we know that many difficulties with quantum theory have classical roots. The problem of the EMM is one such problem.

Thus we have a vicious circle. Is there a way out? Epistemology requires that internal contradictions must not be allowed in any scientific theory. They have to be resolved by changing the interpretation, transformation of the model or modifications to the mathematical formalism of the theory.

Our task is to analyze the problem of the EMM (first aspect).

1. The Electromagnetic Mass Problem

We begin the analysis with examples where the problem can be seen clearly. The authors wish to point out that the density of energy of electromagnetic waves is described well by Poynting's vector. However, Poynting's vector is not in agreement with mechanics. Newton's mechanics states that the connection between the mass m and the momentum $\mathbf{P}$ is

$$\mathbf{P} = m\mathbf{v}$$

In the same way, the relation between the density of mass w/c^2 and the density of energy flow $\mathbf{S}$ is as follows:

$$\mathbf{S} = w\mathbf{v} \tag{1.1}$$

Analogously, we may write the density of energy flow $\mathbf{S}_e$ of the charge.

$$\mathbf{S}_e = w_e\mathbf{v} \tag{1.2}$$

where $w_e = \frac{3}{2}\left(\mathrm{grad} f\right)^2$ —the density of electromagnetic energy of a charge.

We shall not consider relativistic examples, since SRT has epistemological errors (Kuligin *et al.* 1989, 1990, 1994).

Example 1. Let us assume a charge with uniform electrical density. The charge moves along the x-axis with constant velocity $\mathbf{v}$. For comparison, we select two points on

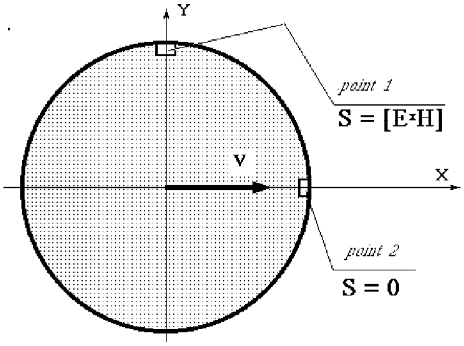

Figure 1

the surface of the charge, as shown in Figure 1. With Poynting's vector we can obtain the densities of the electromagnetic flux at the point.

$$\mathbf{S}_e = [\mathbf{E}\times\mathbf{H}] = \varepsilon(\mathrm{grad}\,\varphi)^2\,\mathbf{v} \qquad \text{(point 1)}$$

$$\mathbf{S}_e = [\mathbf{E}\times\mathbf{H}] = 0 \qquad \text{(point 2)}$$

The velocities and densities of the masses have equal values at the two points. At point 1, the flux density $\mathbf{S}_e$ is greater than expected by factor of 2. At point 2, the flux density $\mathbf{S}_e$ is equal to zero. What has happened?

If we consider relativistic velocities, then we have the problem of the "$4/3$" factor, which is discussed in many textbooks (*e.g.* Panofsky and Phillips 1962).

Example 2. Here we shall deal with a charged plane of infinite extent. The plane is plotted in Figure 2 If the plane moves upward with velocity $v_y\,(v_y \ll c)$, then the flux density is equal to:

$$\mathbf{S}_e = [\mathbf{E}\times\mathbf{H}] = \varepsilon\,(\mathrm{grad}\,\varphi)^2\,\mathbf{v} \qquad (1.3)$$

Here again we find a violation of the classical rule (1.2). In any part of the charged plane, flux density is twice as high as the flux density as in Equation (1.2). We have the alternative result if the plane is moved along the x-axis: because of the symmetry the magnetic field is absent. Consequently, the flux density is equal to zero.

$$\mathbf{S}_e = [\mathbf{E}\times\mathbf{H}] = \varepsilon(\mathrm{grad}\,\varphi)^2\,\mathbf{v} \qquad (1.4)$$

Once again, we find the paradox. In nature, inertial mass is a *scalar* quantity. Logically we must accept that it has to acquire *tensor* properties! What properties must NEM have so that the full mass of the particle possesses the standard inertial properties?

Moreover, any EMM of a charge which has the asymmetrical form (for example ellipsoidal or toroidal form), must have tensor properties. Any student can check this. But this is nonsense!

2. Umov's Vector

Now we shall solve the problem of EMM in the framework of the non-relativistic case only. Two considerations lead us to this approach.

1. Historically Maxwell's equations arose due to Coulomb's law, Ampere's law and Faraday's law. We must use experimental laws here.
2. We regard SRT as a questionable theory (Kuligin *et al*. 1989, 1990, 1994). We must therefore use the mathematical formalism of SRT (Lorentz transformation) with extreme care.

It is known that the EMM depends on interactions (Kuligin *et al*. 1986). For instance, if a charge is changed by factor 3 without any change in volume, then the EMM is changed by a factor 9, not by a factor 3. From this point of view we shall work out the problem for the free charge, where there are no interactions and the velocity is constant. First of all, we must ascertain the connection between Newton's mechanics and Maxwell's equations.

We write Maxwell's equations in Lorentz's gauge and obtain the non-relativistic equations, which are correct up to second order of v/c.

$$\Delta\mathbf{A} = -\mu\,\mathbf{j} \qquad (2.1)$$

$$\Delta\varphi = -\frac{\rho}{\varepsilon} \qquad (2.2)$$

$$\mathrm{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \qquad (2.3)$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

$$\mathbf{A} = \frac{\varphi\,\mathbf{v}}{c^2} \qquad (2.4)$$

and $\mathbf{j} = \rho\mathbf{v}$ (2.5) Additional Equations (2.4) and (2.5) are necessary for the analysis.

We must show that Equations (2.1), (2.2) and (2.3) are consistent with classical mechanics. For this purpose, we replace the vector potential $\mathbf{A}$ in Equation (2.1) by the scalar potential φ (using Equations (2.4) and (2.5)).

$$\begin{aligned}\Delta\mathbf{A} + \mu\,\mathbf{j} = {} & \frac{1}{c^2}\mathrm{rot}\left(-\mathrm{grad}\,\varphi \times \mathbf{v}\right) \\ & + \frac{\partial}{\partial t}\left(-\mathrm{grad}\,\varphi\right) + \mathbf{v}\,\mathrm{div}\left(-\mathrm{grad}\,\varphi\right) \\ = {} & 0\end{aligned} \qquad (2.6)$$

In the mechanics of continuous media we have the proof of the condition, when the vector $\mathbf{a}$ and the intensity of its field lines are conserved (Kochin 1965):

$$\mathrm{rot}[\mathbf{a}\times\mathbf{v}] + \frac{\partial\mathbf{a}}{\partial t} + \mathbf{v}\,\mathrm{div}\mathbf{a} = 0$$

If we replace the vector $\mathbf{a}/c^2$ by $\mathbf{E} = -\mathrm{grad}\,\varphi$, then we obtain Maxwell's equation (2.1) for the free charge.

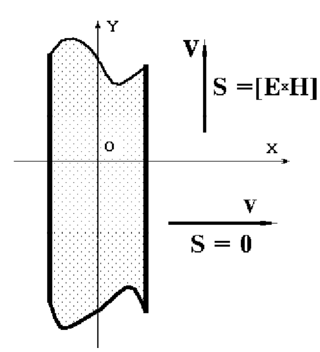

Figure 2

Similarly, we obtain Equation (2.7) from Equation (2.3). This is the continuity equation of the potential *in the mechanics of continuous media*.

$$\mathrm{div}\,\mathbf{v}\varphi + \frac{\partial \varphi}{\partial t} = 0 \qquad (2.7)$$

In Equation (2.8) the scalar potential is generated by the source ρ.

$$\Delta\varphi = \frac{-\rho}{\varepsilon} \qquad (2.8)$$

It can readily be seen that quasi-static electrodynamics and mechanics have similar equations. Earlier work (Kuligin *et al*. 1986) demonstrates this result. Now we begin the proof of the law of conservation of energy.

Proof

Let φ be the potential of the source ρ (Equation (2.8). We write the integral I.

$$I = \frac{1}{2}\int \rho \frac{\partial\varphi}{\partial t} d^3r = -\frac{\varepsilon}{2}\int \Delta\varphi \frac{\partial \varphi}{\partial t} d^3 r \qquad (2.9)$$

where d^3r is a volume element. With Gauss's formula we may write

$$I = -\frac{\varepsilon}{2}\oint \frac{\partial\varphi}{\partial t}\mathrm{grad}\varphi\;\mathbf{n}^o ds + \frac{\varepsilon}{4}\int\frac{\partial}{\partial t}(\mathrm{grad}\,\varphi)^2 dt \qquad (2.10)$$

where ds is the surface element and $\mathbf{n}^o$ is unit surface normal. On the other hand with Equation (2.6) and Equation (2.7), we may write Equation (2.9) in the following form

$$I = -\frac{\varepsilon}{2}\oint[\mathrm{grad}\varphi\times(\mathbf{v}\times\mathrm{grad}\varphi)]\mathbf{n}^o ds - \frac{\varepsilon}{4}\int\frac{\partial}{\partial t}(\mathrm{grad}\varphi)^2 dt \qquad (2.11)$$

Comparison of Equation (2.10) with Equation (2.11) yields

$$\oint \mathbf{S}_u\mathbf{n}^o ds + \int \frac{\partial}{\partial t} w_e dt = 0 \qquad (2.12)$$

where $\mathbf{S}_u$ is the density of electromagnetic flux or Umov's vector,

$$\mathbf{S}_u = \frac{\varepsilon}{2}\left\{-\frac{\partial\varphi}{\partial t}\mathrm{grad}\varphi + [\mathrm{grad}\varphi\times(\mathbf{v}\times\mathrm{grad}\varphi)]\right\} = \mathbf{v}w_e \qquad (2.13)$$

Here Equation (2.7) was used; w_e is the density of electromagnetic energy.

$$w_e = \frac{\varepsilon}{2}(\mathrm{grad}\varphi)^2 \qquad (2.14)$$

Equation (2.12) is Umov's law of energy conservation, which was proved by Umov (1874) for the mechanics of continuous media. A second proof of Umov's law was given by us (Kuligin *et al.* 1986).

It is clear that Equation (2.13) and Equation (2.14) correspond to the equations of Newton's mechanics (1.1) and (1.2). With this result, we can calculate the correct electromagnetic flux density in examples discussed previously. Now we calculate the EMM and the momentum of a charge of arbitrary form

$$m_e = \iiint w_e\, dx\, dy\, dz\,;\;\; \mathbf{P}_e = \iiint \mathbf{S}_u dx\, dy\, dz\,;\;\; \mathbf{P}_e = w_e\mathbf{v}$$

3. Kinetic Energy Equilibrium

Now we shall prove another important result: the kinetic energy equilibrium equation. We shall show that the EMM possesses kinetic energy. This fact is not particularity new. However, we must have the full picture of the phenomenon.

First we consider the physical model of the change of kinetic energy of the field. If external forces act on the charge, then the charge is accelerated and its kinetic energy is changed. The change is connected with the current density j and the vector potential $\mathbf{A}$.

The accelerated motion of the charge can be treated as the jump from one instantaneously co-moving inertial frame to the next frame. The instantaneously co-moving inertial frame and the non-inertial frame have equal velocities at one instant. The field $\mathbf{e} = \mathrm{grad}\varphi$ is not time-dependent and the vector potential A is equal to zero in the instantaneously co-moving frame. The accelerated motion of the charge induces the additional electrical field $\mathbf{E}'$, which is caused by the change of vector potential A over time (see Appendix 1). The field $\mathbf{E}'$ cannot be considered as a negligible quantity. In the instantaneously co-moving frame the field is equal to

$$\mathbf{E}' = -\frac{1}{2}\frac{\partial\mathbf{A}}{\partial t} = -\frac{\varphi}{2c^2}\frac{\partial\mathbf{v}}{\partial t} \qquad (3.1)$$

The density of the power which is generated by the charge is equal to

$$p_k = \rho\mathbf{E}\mathbf{v} = \frac{\partial}{\partial t}m_e^*\frac{\mathbf{v}^2}{2} = -\frac{\partial}{\partial t}\left(\frac{\mathbf{jA}}{4}\right) \qquad (3.2)$$

The power density does not depend on the inertial frame in Newton's mechanics.

Now we shall describe this model mathematically. To prove the equation we use Green's formula of vector potential

$$\int \mathbf{E}\,\Delta\mathbf{M}\,dt = \int(\mathrm{div}\mathbf{E}\,\mathrm{div}\mathbf{M} + \mathrm{rot}\mathbf{E}\,\mathit{rot}\mathbf{M})\mathbf{n}^o dt$$

where $\mathbf{E}$ and $\mathbf{M}$ are the vector potentials of two arbitrary fields.

Let $\mathbf{E} = \frac{1}{2}(\partial\mathbf{A}/\partial t)$ be the field which is generated by the accelerated charge and $\mathbf{M} = \mathbf{A}/\mu$ be the vector potential of the field divided by μ. In this case we obtain full kinetic energy equilibrium equation, and we can write the differential form of this equation :

$$\mathrm{div}\mathbf{S}_k + \frac{\partial w_k}{\partial t} + p_k = 0 \qquad (3.3)$$

where:

a) $$p_k = -\frac{1}{2}\mathbf{j}\frac{\partial\mathbf{A}}{\partial t} = -\left(\frac{\mathbf{jA}}{4}\right) \qquad (3.4)$$

is the density of power which changes kinetic energy ; and

b) $$w_k = \frac{1}{4\mu}\left[(\mathit{ivB}\mathbf{A})^2 + (\mathrm{rot}\mathbf{A})^2\right] \qquad (3.5)$$

is the kinetic energy density. With Equation (2.4) we have Newton's result

$$w_k = \frac{\mathbf{v}^2}{2c^2}\frac{\varepsilon}{2}\left(\operatorname{grad}\varphi\right)^2 = \frac{w_e \mathbf{v}^2}{2c^2} = m_e^* \frac{\mathbf{v}^2}{2}$$

c) $$\mathbf{S}_k = -\frac{1}{2\mu}\left(\frac{\partial \mathbf{A}}{\partial t}\operatorname{div}\mathbf{A} + \frac{\partial \mathbf{A}}{\partial t}\times \operatorname{rot}\mathbf{A}\right) \qquad (3.6)$$

i.e. the kinetic energy flux density. We now illustrate this kinetic energy equilibrium equation with a simple example.

4. Change of Energy of a Current Element

In quasi-static electrodynamics the vector potential of a current element is equal to :

$$\mathrm{d}\mathbf{A} = \mu\frac{I(t)\mathrm{d}\mathbf{l}}{4\pi r} \qquad (4.1)$$

Substituting Equation (4.1) into Equation (3.6) and Equation (3.8) we have the following results.

1. The kinetic energy density is equal to :

$$\mathrm{d}^2 w_k = \frac{\mu}{2}\left(\frac{I(t)\mathrm{d}\mathbf{l}}{4\pi r}\right)^2 \qquad (4.2)$$

   The distribution of the kinetic energy density is radially symmetric.
2. The kinetic energy flux density is

$$d^2\mathbf{S}_k = \mathbf{r}\frac{\partial}{\partial t}\mathrm{d}^2 w_k \qquad (4.3)$$

   Now we discuss the peculiar properties of the flux density $\mathrm{d}^2\mathbf{S}_k$
   a) The change of $\mathrm{d}^2 w_k$ is associated with $\mathrm{d}^2\mathbf{S}_k$. The flux density $\mathrm{d}^2\mathbf{S}_k$ depends on the change of squared current I in time. If the current increases, then the flux density $\mathrm{d}^2\mathbf{S}_k$ is positive and $\mathrm{d}^2\mathbf{S}_k$ is directed toward the radius. This flux increases the kinetic energy of the electrical field. If the current I decreases, then the flux comes back toward the current without loss. The flux tends to conserve the previous current in time. The flux density $\mathrm{d}^2\mathbf{S}_k$ decreases as $1/r^3$ in space.
   b) If the current changes, then the kinetic energy flux appears simultaneously throughout space.
   c) Contrary to Umov's vector, which deals with the transfer of energy with velocity $\mathbf{v}$, the kinetic energy flux is connected only with the acceleration of the charge.
   The electrical field is

$$\mathbf{E}' = -\frac{1}{2}\frac{\partial \mathbf{A}}{\partial t}$$

   We can regard this as integral EMF (self-induction) of a current element. This analogy is given for illustration.

Conclusions

1. We have investigated the problem of the EMM of a free charge. Note that in the proof *no hypotheses were used*. The EMM has Newton's momentum and classical kinetic energy within the framework of Maxwell's equations.
2. The inertial mass m_o of the charged particle is equal to

$$m_o = m_e + m_n$$

   where: m_e is the electromagnetic mass and m_n is the non-electromagnetic mass.

   Using the induction method we can prove that the NEM has the standard properties of the inertial mass. The thesis can be extended to the general case. *Any inertial mass must have standard mechanical properties, which do not depend on nature of the mass.* This is a very important result.
3. We conclude that Maxwell's equations deal with two kinds of fields :
   a) the fields of charges (Coulomb's potentials and Umov's vectors; the rest EMM of the charge *is not equal to zero*);
   b) the fields of the electromagnetic waves (retarded potentials and Poynting's vectors; the rest EMM of the electromagnetic wave *is equal to zero*). If we use only retarded potentials in our research, then we cannot give a full and correct picture of nature. It is also possible that the quantum properties of particles may be explained by classical methods.

   The problem of a classical model (or structure) of charges is now of prime importance.

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Appendix

We write the integral variable of the charge which interact with the potential forces. The charge density is constant and rotation of the charge is absent. All points of the charge move with the same velocity.

$$S = \iint \left\{ -m^*\left(1 - \frac{\mathbf{v}^2}{2c^2}\right) + \Lambda \mathrm{d}t\, \mathrm{d}t \right\} \qquad (\mathrm{A}.1)$$

where: $m^* = m_e^* + m_n^*$; m_e^* is the electromagnetic mass density; m_n^* is density of non-electromagnetic mass.

The ponderomotive equation follows from Equation (A.1).

$$\frac{\partial}{\partial t}\left(m^*\mathbf{v}\right) + \mathbf{v}\times\operatorname{rot}\left(m^*\mathbf{v}\right) - \operatorname{grad}\left(m^* c^2\right) + \operatorname{grad}\Lambda = 0 \qquad (\mathrm{A}.2)$$

a) Suppose that external forces are absent ($\Lambda = 0$). The particle is stable if the following condition is met:

$$\operatorname{grad} m^* = \operatorname{grad} m_e^* + \operatorname{grad} m_n^* = 0 \qquad (\mathrm{A}.3)$$

b) If external forces exist ($\Lambda \neq 0$), then we must suppose that the structure of the particle is conserved and, hence, the condition (Equation (A.3)) applies. Now Equation (A.2) is multiplied by $\mathbf{v}$. With Equation (A.3) we can write the product.

$$-\mathbf{v}\frac{\partial}{\partial t}\left(m_e^*\mathbf{v}\right) - \frac{\partial}{\partial t}\left(m_n^*\mathbf{v}\right) + \mathbf{v}\,\mathrm{grad}\Lambda = 0 \qquad \text{(A.4)}$$

The first term of Equation (3.4) is the electromagnetic power of the accelerated charge.

$$p_k = -\mathbf{v}\frac{\partial}{\partial t}m_e^*\mathbf{v} = \frac{1}{2}\mathbf{j}\frac{\partial \mathbf{A}}{\partial t} = -\left(\frac{\mathbf{jA}}{4}\right) \qquad \text{(A.5)}$$

Recall that r and f are not time-dependent.

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