A CONNECTION BETWEEN MINKOWSKI AND GALILEAN SPACETIMES IN QUANTUM MECHANICS

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Abstract. Relativistic quantum theories are equipped with a background Minkowski spacetime and non-relativistic quantum theories with a Galilean spacetime. Traditional investigations have distinguished their distinct spacetime structures and have examined ways in which relativistic theories become sufficiently like Galilean theories in a low velocity approximation or limit. A different way to look at their relationship is to see that both kinds of theories are special cases of a certain five-dimensional generalization involving no limiting procedures or approximations. When one compares them, striking features emerge that bear on philosophical questions, including the ontological status of the wave function and time reversal invariance.

1. Introduction

Let the term 'Galilean theory' apply to any theory based essentially on a Galilean (or neo-Newtonian) spacetime, a spacetime that is topologically $\mathbb{R}^4$ and comes equipped with Euclidean temporal and spatial metrics, a co-vector field picking out a time-like direction, and a constant, flat affine connection. Galilean spacetime is the background spacetime best suited to the standard interpretation of Newtonian gravitation and non-relativistic quantum mechanics. Let the term 'relativistic theory' apply to any theory based essentially on Minkowski spacetime, a spacetime that is topologically $\mathbb{R}^4$, with the Minkowski metric, a constant, flat connection, and a spacetime orientation. Minkowski spacetime is the background spacetime in relativistic electrodynamics and the Dirac theory of the electron. In this paper, I will restrict attention to theories that do not employ second quantization.

One significant philosophical issue concerns the relationship between Galilean theories and relativistic theories. The most common way to think about their relation is informed by the belief that relativistic theories are empirically more accurate, whereas Galilean theories more closely represent classical conceptions of space and time. A common task in physics textbooks is to demonstrate that in some appropriate limit, a relativistic theory behaves like a Galilean theory in physically important respects. The intricacies of these limiting processes have been often discussed, (e.g. Holland and Brown 2003), with regard to whether the limits make sense, whether there are multiple limiting processes that lead to distinct Galilean theories, etc.

This paper investigates an altogether different relationship between the relativistic and Galilean theories whose interpretational significance has been discussed very little: both theories can be embedded in a five-dimensional model such that their difference is captured by a single parameter in a simple way. It is my hope that

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this relation between Galilean and relativistic quantum mechanics will be surprising enough by itself to readers not already familiar with null cone gauge theory to generate interest in the interpretational questions raised by the five-dimensional formalism. But for the more skeptical, I will apply the formalism to the debate over the ontological status of the wave function. This serves as a demonstration that the model warrants further philosophical investigation. I do not have a specific proposal for how to interpret the five-dimensional model, but I will mention several possible approaches one could take to develop a full interpretation.

First, I will outline some preliminary positions one might take with regard to ontological status of the wave function. Then, after describing how the relativistic and Galilean theories are secretly hidden inside a five-dimensional generalization, I will focus on two applications, Galilean boosts and time reversal. In both cases the five-dimensional formalism helps to undercut arguements for anti-realism about the wave function. I will not defend the conclusion that the wave function should be treated as a physical field but just use the five-dimensional model to overcome two barriers to treating it so.

2. The Ontological Status of the Wave Function

Various attitudes taken towards the quantum mechanical wave function may for current purposes be delineated into two broad classes. In one class, the (mathematical) wave function ψ is understood to encapsulate information about fundamental physical stuff without itself representing a physical field that closely corresponds to ψ. This view has been popular historically, from early instrumentalist interpretations to modern information theoretic approaches. Call this thesis 'ψ-is-non-physical.' In the other class, the wave function is taken to be physical in some robust sense. This could involve a commitment to ψ's being just as physical as the electromagnetic field or vector potential (Albert 1996), or it could involve thinking of ψ as a holistic relation among particles (or property of particle configurations) that is physically three-dimensional (Lewis 2004) even though it is representable mathematically only using a higher-dimensional configuration space.

The distinction between the physical and non-physical interpretations of ψ is too vague to definitively categorise all existing interpretations. Any interpretation where the wave function is understood as ontologically complete is arguably in the ψ-is-physical camp because if there is nothing other than the wave function and it is not physical, then nothing is physical, which is a reductio. But, if ψ's completeness is not ontological but informational, then a range of options opens depending on how one interprets 'information' in this context. One might profess an unwillingness to speculate on the ontology underlying quantum mechanics yet remain convinced that whatever it is, our knowledge of it comes only by way of learning about ψ. In what follows, it will not matter that this distinction is somewhat vague.

Among the arguements for each side, only a few are mentioned here. In favor of the non-physicality of the wave function is that ψ in general is defined over a multi-particle configuration space instead of physical space. (Set aside Hilbert space representations for the sake of discussion.) ψ does not correspond directly to stuff in physical space but instead (in the non-relativistic theory) inhabits a configuration space of $3N$ dimensions, where N is the number of particles. The mathematical wave function cannot be reduced to a (natural) three-dimensional field and still retain all the information it encodes. This arguement for the non-physicality of the wave function is almost certainly not merely an artifact of the non-relativistic version of quantum mechanics. Although there is no known relativistic equivalent of the non-relativistic configuration space, empirical evidence demands some account of quantum entanglement, and entanglement has so far resisted being reduced to lower dimensional physics.

A separate line of reasoning points to the transformation properties of the wave function in the non-relativistic theory as evidence that ψ is non-physical. In the non-relativistic theory, the wave function transforms under boosts in a way that is wholly unmotivated if one thinks of it as a fundamental physical stuff. Its behaviour under time reversal is likewise mysterious. Neither transformation is puzzling, though, if ψ just encodes information about quantum expectation values. This issue will be discussed in greater detail in sections 4 and 5.

In favor of the physicality of the wave function, the dynamical evolution of nearly every important physical quantity takes place in the wave function. Interpretations of quantum mechanics that employ local beables, i.e., stuff in ordinary spacetime, still encode almost all physical properties—including spin, entanglement relations and (to some extent) charge—using the wave function. The interpretation that goes furthest towards segregating particle properties from the wave function is Bohmian mechanics. Bohmian particles have relative positions, relative velocities and mass. The mass appears separate from ψ in Bohm's equation and Schrödinger's equation, but the positions contribute to the determination of the particles' future motion only by way of their contribution to the full set of all particle positions together with the universal wave function. Other properties like spin states are encoded in the wave function and are not properly thought of as attributes that adhere to the particle as it moves through space. Conceivably, extra variables that in effect adhere to particles could supplant and explain the wave function's properties and evolution, but no interpretation has yet been described that replaces ψ with local beables. All current interpretations that employ extra variables do so only by supplementing ψ.

Furthermore, in the special case of a single particle, the similarities between the wave function and uncontroversially physical fields like the electromagnetic field become tighter. Loosely speaking, one can think of the single particle wave function as having a strong magnitude where the particle is located (or likely to be located). Whether such claims can be construed literally is interpretation dependent. One clear example is the Benatti, Ghirardi, Grassi mass density field interpretation (Benatti et al. 1995) where a particle is treated as fundamentally continuously distributed in space. For a single particle wave function, $|\psi(x)|^2$ corresponds literally to how much of the particle is located in an infinitesimal region around x. In most other interpretations, $|\psi(x)|^2$ merely corresponds to the probability of the particle being at x or the probability of being detected at x by some suitable experiment. Such interpretations are not especially friendly to ψ being a kind of physical stuff. In section 4 we will see that the equations governing ψ look a lot like those governing the electromagnetic potential, which appears to make ψ more friendly to an interpretation where it inhabits spacetime.

3. The Five-Dimensional Model

I will now sketch the skeletal model that I believe deserves further interpretational scrutiny. It postulates a base manifold M that is topologically $\mathbb{R}^5$ and has a constant metric of signature $(1,4)$, with a constant, flat connection. The ontology of the theory is that of a complex scalar field ψ defined on M. It is also insightful to think of $\psi = \psi_a + \psi_b i$ as a multi-vector field with ψ_a, ψ_b being real numbers and i being an appropriate multi-vector element, i.e., a volume form, in M.

The dynamics of the model is given by

$$DD\psi = 0. \tag{1}$$

The D is the vector derivative as defined in geometric algebra (see Doran and Lasenby 2003). (Readers unfamiliar with geometric algebra may interpret the dynamics in the formalism of differential forms as $\star d \star d\psi = 0$.)

For convenience, label the points of M using arbitrary rectangular coordinate axes with the variables $\{q, t, x, y, z\}$ and their corresponding unit tangent vectors $\{e_q, e_t, e_x, e_y, e_z\}$. Expressed in this coordinate system, the components of the metric are given by

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

We can also define new variables, $Q = (t+q)/\sqrt{2}$ and $T = (t-q)/\sqrt{2}$, that are rotated by an angle $\pi/4$ with respect to their lower case counterparts. The components of the metric expressed with respect to the new tangent vectors $\{e_Q, e_T, e_x, e_y, e_z\}$ is

$$G = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

The Ansatz that gives rise to the relativistic model is

$$\psi = \psi_0(t, x, y, z)e^{i(mc/\hbar)q} \tag{4}$$

whereas the Galilean model arises from

$$\psi = \psi_0(T, x, y, z)e^{i(mc/\hbar)Q}, \tag{5}$$

where ψ_0 is some complex function.

Constraining ψ with (4) is exactly what Oskar Klein proposed in his famous (1926) attempt to unify electromagnetism and gravity. In fact, he even derived a non-relativistic wave equation using a formula similar to (5). However, this remarkable feature of his model—that it entails a non-relativistic wave equation from a relativistic background theory without any technique of approximation—has not been flagged by any commentary on Klein's work, as far as I have been able to determine.

3.1. **The Equations of Motion.**

It follows from the dynamics (1) that

$$DD\psi = -\partial_q^2\psi + \partial_t^2\psi - \partial_x^2\psi - \partial_y^2\psi - \partial_z^2\psi = 0, \tag{6}$$

and taking the q derivatives using (4) gives us

$$(\partial_t^2 - \nabla^2)\psi + \left(\frac{mc}{\hbar}\right)^2\psi = 0, \tag{7}$$

which is the Klein-Gordon equation for a spinless particle of mass m.

Similarly, it follows from the dynamics (1) that

$$DD\psi = -2\partial_T\partial_Q\psi - \nabla^2\psi = 0 \tag{8}$$

Taking the Q derivative using (5) gives us

$$2i\frac{m}{\hbar}\partial_T\psi + \nabla^2\psi = 0 \tag{9}$$

which is Schrödinger's equation for a spinless particle of mass m. (The c vanishes because in natural units there is an implicit use of cT for distances along the T axis.)

The oddity that the five-dimensional model highlights is that equations that seemingly require the symmetries of a Galilean spacetime structure can be understood in terms of the resources of a relativistic spacetime structure so long as there is a constraint that the five-dimensional ψ takes the special form of (5). The appearance of Galilean characteristics, i.e., the obtaining of Schrödinger's equation and other features to be discussed in §4 in one version of the five-dimensional model, does nothing to alter the fact that the model is built on a five-dimensional Minkowski spacetime. So, the Galilean version of the five-dimensional model is relativistic in the sense that it is a special case of a dynamics built using standard structures of special relativity, yet Galilean in the sense that its dynamics formally matches that of the four-dimensional quantum mechanics in Galilean spacetime.

3.2. **Lagrangians.** The Lagrangian density for both Schrödinger's equation and the Klein-Gordon equation take a remarkably simple form, $\mathscr{L} = |D\psi|^2$.

We can derive the usual expression for the Klein-Gordon equation,

$$\begin{aligned}\mathscr{L} &= g(D\psi^*, D\psi) \\ &= -\partial_q\psi^*\partial_q\psi + \partial_t\psi^*\partial_t\psi - \partial_x\psi^*\partial_x\psi - \partial_y\psi^*\partial_y\psi - \partial_z\psi^*\partial_z\psi \\ &= -(mc/\hbar)^2\psi^*\psi + \partial_t\psi^*\partial_t\psi - \nabla\psi^*\nabla\psi\end{aligned} \tag{10}$$

For the Galilean theory, we can verify the Lagrangian density by using G instead of g,

$$\mathscr{L} = G(D\psi^*, D\psi) = \frac{2mi}{\hbar}[\psi^*\partial_T\psi - \psi\partial_T\psi^*] - \nabla\psi^*\nabla\psi, \tag{11}$$

which generates Schrödinger's wave equation.

3.3. **Mass, Energy, Momentum Relations.** One can also see the different mass-energy-momentum relations transparently in the five-dimensional model. By substituting a generic plane wave solution $\psi = e^{i(mq-Et+\vec{p}\cdot\vec{x})}$ into (6) and $\psi = e^{i(mQ-ET+\vec{p}\cdot\vec{x})}$ into (8), we get the correct values of $-m^2 + E^2 - p^2 = 0$ for the relativistic theory and $2mE - p^2 = 0$ for the Galilean theory.

To summarise, Schrödinger's wave equation and the Klein-Gordon wave equation have been derived from the five-dimensional generalization for the case of a single, free, massive particle. The constraints (4) and (5) are only different by a rotation of the axis that appears in the exponential. This difference is all it takes to distinguish the relativistic and non-relativistic quantum theory in the special case considered. The Lagrangians for both theories appear in an especially simple form, and the mass-energy-momentum relations arise immediately. The relation between these two special cases of the five-dimensional theory is striking and provocative.

Already, one can see a formal similarity between the electromagnetic field and the wave function for a spinless particle in that the dynamical equations governing the fields are both second order differential equations using a vector derivative operating on a multi-vector field. The difference is just that the electromagnetic gauge potential (in standard formulations) is a tangent vector field (or 1-vector) in four dimensions and the spinless particle in the five-dimensional formulation is given by a scalar (or 0-vector) quantity plus five-dimensional pseudo-scalar (or 5-vector) quantity.

There exists an ongoing research programme known as light-front field theory or light-front dynamics or null cone gauge theory which to some extent captures the non-relativistic nature of relativistic spacetime. In this tradition, one takes a relativistic theory and formulates it in the so-called light-cone gauge or infinite momentum frame. This is done by picking an arbitrary direction in space, say the z-axis, and then transforming an ordinary rectangular reference frame, with axes, $\{t, x, y, z\}$, into a new reference frame, $\{t', x, y, z'\}$ where $t' = (z - t)/\sqrt{2}$ and $z' = (z + t)/\sqrt{2}$. This makes equations that are really relativistic look as if they are non-relativistic. The light-cone formulations, however, are not really Galilean theories for at least two reasons. First, the variables that look non-relativistic don't correspond to the classical quantities. For example, the m in the Schrödinger's equation of light-front field theory doesn't correspond to a particle's mass. Second, the theory only generates a two-dimensional Euclidean physical space. To get the three-dimensional physical space, one must use a fifth dimension that has no ordinary interpretation in terms of space and time. In standard light-front field theory, the appearance of a Galilean equation is merely formal in the sense that one can cook up variables that obey Schrödinger's wave equation but they do not correspond to any natural physical quantities. In the five-dimensional model above, the m really does play the role of a particle mass; the T really does play the role of time.

The five-dimensional model contains a significant lacuna: an explanation of what accounts for the Ansätze (4) and (5). Without some story about the physical structures that justify discarding most of the solutions for the general equation (1), the model is incomplete. I think ultimately, one should not get too hung up on what the best interpretation is for the partial model above because it presumably will be interesting physically only insofar as it can be incorporated as a special case of a more comprehensive theory. So, the ultimate answer as to how to interpret it, will hinge on how it can be extended to accommodate a wider range of physical theories. So, while I advocate no specific proposal for a completion here, there are several routes one might take. First, one could refashion my model into one based on a four-dimensional base manifold, which would allow one to maintain a standard interpretation of spacetime. This would involve maintaining the five-dimensional character of the model only in the tangent space, i.e., by treating the model as an extension of a fiber bundle interpretation of special relativity with an extension of the tangent bundle into a fifth dimension. This would have the benefit of not being revisionary about spacetime itself but would make the five-dimensional tangent structure and the associated transformation properties a bit more ad hoc. Second, one could somehow compactify one of the dimensions of the base manifold, presumably as $\mathbb{R}^4 \times S$. The compactification by itself would not be sufficient to explain (4) and (5), but could supplement a more thorough explanation by at least accounting for why we do not perceive the fifth dimension as an extra degree of spatial freedom. Third, one could maintain the $\mathbb{R}^5$ topology of the base manifold and attempt to subsume the Ansätze, 4 and 5, under a more comprehensive theory as the result of the imposition of a special boundary condition. This strategy would be much more revisionary than the other two, but also possibly more informative and explanatory. There are undoubtedly other strategies one could attempt, but so far as I can tell, there is no reason to think completing such a model is impossible. I will continue the discussion assuming that some unobjectionable completion exists that does not force a significant reappraisal of the interpretation.

4. Galilean Spacetime

It is no great surprise that the Klein-Gordon equation can be formulated as a wave equation on the five-dimensional M. The more interesting result is that the non-relativistic quantum mechanics is also naturally formulated using the same manifold, metric, and connection. This is especially surprising because the Galilean spacetime structure that underlies non-relativistic quantum mechanics is not a metric space. There is no Galilean spacetime metric, but instead a Euclidean metric for space, another for time, and an affine connection to bind space and time together. The key advance the five-dimensional formalism offers is a natural way to understand classical spacetime structure *within a metric space*. This perhaps offers some hope for a version of classical mechanics that eschews the traditional complicated spacetime structure, though the fifth dimension may introduce more interpretational difficulties than is desirable.

One traditionally strange feature of the wave function in non-relativistic quantum mechanics is that while it is complex-valued, it is not a complex field. Its boost transformation involves a more complicated space and time dependent phase shift. Starting with one inertial frame where spacetime points are labeled $(t, \vec{x})$, there is another inertial frame $(t', \vec{x}')$ moving away from it with constant velocity $\vec{v}$. In the traditional treatment, (e.g. Ballentine 1998), under the combined transformations

$$\begin{aligned} \vec{x}' &= \vec{x} - \vec{v}t \\ t' &= t \\ \psi'(t', \vec{x}') &= e^{i\frac{m}{\hbar}\left(-\frac{\vec{v}^2 t}{2} + \vec{v}\cdot\vec{x}\right)}\psi(t, \vec{x}), \end{aligned}$$

Schrödinger's equation holds for $\psi'(t', \vec{x}')$ whenever it holds for $\psi(t, \vec{x})$. One can also verify that $|\psi'|^2 = |\psi|^2$, so that the probability distribution in the position representation is preserved under a Galilean boost. Strictly speaking, the standard spacetime transformations of the wave function do not constitute a Galilean group but only a group up to an overall global phase shift. This is universally thought to be harmless because overall phases are physically undetectable, so when we say non-relativistic quantum mechanics is a Galilean theory, this phase irrelevance is taken for granted.

If one interprets quantum mechanics such that ψ merely codifies statistical properties about measurement outcomes, the boost transformation is entirely unmysterious. In order to ensure that the standard connection between ψ and probabilities of outcomes is invariant under boosts, one must insist that $|\psi|^2$ be invariant under boosts. The needed transformation rule for ψ can then be derived using this assumption and leaving open how the phase varies with regard to $\vec{x}$ and t, as described in (Ballentine 1998).

But the traditional boost transformation should strike one as wholly artificial if ψ is to be interpreted as a kind of fundamental physical stuff. The phase shift is just stuck in by hand and attributed to the intrinsic transformation properties of the wave function stuff even though there is no natural field structure that it represents. That the boost transformation is mass dependent makes this intrinsic transformation property of ψ appear even more artificial. The seeming artificiality of the boost equation counts in favor of interpreting ψ as non-physical.

However, the situation changes if we consider the five-dimensional model. We can achieve the same Galilean covariance in the five-dimensions with

$$\begin{aligned}
\vec{x}' &= \vec{x} - \vec{v}T \\
T' &= T \\
Q' &= Q - \frac{1}{2}\left|\frac{\vec{v}}{c}\right|^2 cT + \frac{\vec{v}}{c}\cdot\vec{x} \\
\psi'(Q', T', \vec{x}') &= \psi(Q, T, \vec{x}).
\end{aligned}$$

Notice that while c does appear in the boost transformation, it does so in a way that rightly draws no important distinction between slower-than-c and faster-than-c Galilean boosts.

An advantage of embedding non-relativistic quantum mechanics in the five-dimensional formulation is that it allows us to see how the wave packet can be just an ordinary scalar field: ψ itself is invariant under boosts. In the single particle case, this makes ψ seem much more like a physical field. If one sets aside the fact that the fifth dimension has not yet been given a clear physical interpretation, ψ is even less mysterious, vis-à-vis boost transformations, than the electromagnetic field. This helps to undercut the arguement that boost transformation properties motivate denying ψ an interpretation as a physical stuff.

Of course, nothing I have said here overcomes the fact that in the multiple particle case ψ is defined over a configuration space. Until one solves the much harder problem of finding a theory with fundamental ψ-like fields defined over a lower dimensional space like space or spacetime from which one derives all the important aspects of the full ψ in configuration space, the arguement here can only count as a small first step towards demonstrating the physicality of ψ.

5. Time Reversal Invariance

There has been much recent philosophical discussion of time-reversal invariance, (e.g. Albert 2000, Earman 2002, Malament 2004, Leeds 2006, North 2008), especially a debate concerning whether classical relativistic electromagnetism is time-reversal invariant, correctly understood. Although valuable lessons have been thereby gained about how to think of time-reversal invariance, I believe the overall framework of the debate is suboptimally structured in two ways. First, the debate is focused on electromagnetism in isolation from other theories. Although, for historical and interpretative purposes, it is useful to examine the features of non-quantum relativistic electromagnetism, it is even more useful to examine its time-reversal characteristics in light of other phenomena that we believe to be true, specifically the **CPT** symmetry of quantum field theory. Second, the debate primarily concerns whether there is sufficient justification for the traditional way of defining time-reversal invariance instead of seeking constraints on our interpretations of the physical structures that illuminates why the physics is time-reversal invariant under the traditional definitions.

Concerning the first issue, consider the interesting fact that in quantum field theory, we know that electromagnetic interactions and strong interactions obey the **C**, **P**, and **T** symmetries individually and the combined operation **CPT**. Weak interactions however violate the individual **C**, **P**, and **T** symmetries yet still obey the combined **CPT**. This is a striking phenomenon because is prima facie puzzling that violations of **PT**, the flipping of an extrinsic spatio-temporal relation among material fields, is exactly counterbalanced by violations of **C**, the reversal of an apparently intrinsic property of material fields. How is it that there are never any violations of the combined spatiotemporal flipping and reversal of the internal charge properties of particles? Why is it that these violations only occur in weak interactions? Is it a coincidence that particles have mass if and only if they interact via the weak force, or is that a clue that the existence of mass itself is closely related to the fact that the weak force violates the individual **C**, **P**, and **T** symmetries? There are no doubt many possible schemes for explaining the relation between **C** and **PT**. One particularly appealing potential answer is that spacetime and matter are not metaphysically distinct—that there is some fundamental structure that is somehow a fusion of matter and spacetime, with the existence of matter being somehow related to additional geometrical structure, so that the **CPT** symmetry is really geometric through and through, not a coincidental counterbalancing of internal material properties with external spatiotemporal symmetries. Classical gauge theory, string theory and super-symmetry are examples of programmes consistent with such an exploration.

How exactly a programme to geometrise matter will play out is very uncertain, but I think it is useful to examine simple classical systems like relativistic electromagnetism as a step on the way to understanding **CPT** symmetry. Arntzenius and Greaves (2008) adopt one approach, arguing that the time-reversal operation is really **CT**, so that the **CPT** transformation is really just a spatiotemporal flip. Another possibility is that there are extra dimensions and performing a charge-conjugation operation really just amounts to performing the same kind of geometric flipping on the extra dimensions that one ordinarily does for spatial and temporal reversals. There are presumably many other possible avenues for explaining how charge conjugation and spatiotemporal reversals are intimately related so that **CPT** symmetry can hold fast while weak interactions violate each individual symmetry.

The second issue was that the debate about time-reversal invariance was initiated as a dispute over whether the traditional way of defining time reversal invariance is the proper way to define it. Standardly, a theory is said to be time reversal invariance iff $\phi(i) \rightarrow \phi(f)$ is a lawful physical evolution whenever $\mathbf{T}\phi(f) \rightarrow \mathbf{T}\phi(i)$ is also a lawful physical evolution of some physical state $\phi(t)$. **T** here represents a suitably defined time reversal transformation on an instantaneous state, and the debate concerns what counts as a suitable transformation. It is standard practice in physics to allow at least some non-trivial time reversal transformations, typically flipping the signs of some mathematical terms corresponding to physical magnitudes. The purpose of non-trivial time reversal transformations is to take into account aspects of the instantaneous state that are intrinsically temporal like velocities. Specifically, in electromagnetism, T flips the sign of the magnetic field components but not those of the electric field.

Albert (2000) contends in effect that we should impose constraints on $\mathbf{T}$ that arise from our not-too-theoretically-biased ontological commitments. Because velocities are a priori a first derivative of position with respect to time, their sign should flip under a $\mathbf{T}$ transformation, but because electromagnetic fields are not a priori the derivatives of anything, no electromagnetic field terms should flip signs. Arguing for the other side, Earman (2002) contends that the changes of sign are appropriate given the role that the physical quantity plays. Malament (2004) also contends the traditional transformations for the electromagnetic field are justified. Concision prevents me from engaging in this debate adequately, so I will just submit that the mere fact that there is some $\mathbf{T}$ function that is a symmetry of the physics and preserves the magnitudes of macroscopic parameters of any instantaneous state on which it operates, e.g., volume, pressure, and energy, is enough to warrant interest in such a $\mathbf{T}$. The task should be to explore what interpretational consequences such a $\mathbf{T}$ has for the various physical fields. The task is not to assume we antecedently know what the electromagnetic field is really like and then derive the "correct" time-reversal transformation from it, but to find specifications of $\mathbf{T}$ such that (1) the physics is symmetric under them and (2) $\mathbf{T}$ bears a close enough relation to time-reversal, e.g., by serving adequately in a thermodynamic reversibility arguement. Then, one should seek geometric interpretations for the stuff of electromagnetism such that $\mathbf{T}$ and related symmetries $\mathbf{P}$ and $\mathbf{C}$ hold. Furthermore, one can examine how these interpretations might extended to help us understand the significance of weak interactions.

How the physical quantities transform under time-reversal depends greatly on which quantities are treated as fundamental and which are treated as derivative. It is standard practice in all theories with point-like particles to think of their world lines in spacetime as fundamental and to treat their tangents (4-velocities) at any point p and three dimensional velocities as derivative, so that in virtue of what it means to be a point-particle velocity, velocities flip sign under a time reversal operation.

Consider, for example, that the accepted formula for the time reversal of a spinless quantum particle (in the normal four-dimensional treatment) is given (up to a unitary transformation) by complex conjugation, $\mathbf{T}\psi(t) = \psi^*(-t)$. It is trivial that Schrödinger's equation (and Bohm's equation) are time reversal invariant using this definition of $\mathbf{T}$ and not time reversal invariant using $\mathbf{T}\psi(t) = \psi(-t)$ or $\mathbf{T}\psi(t) = -\psi(-t)$. This result by itself vindicates interest in time reversal using the complex conjugation definition, but one still needs to consider whether the $\mathbf{T}$ transformation plays the right role with regard to various elements in the theory to deserve the label 'time reversal.'

How one fills in this story depends on the ontological status of the wave function. In the case where ψ is considered non-physical, complex conjugation can be vindicated as time reversal by demonstrating that it is kinematically permissible in the sense of being compatible with all commutation relations, and that it corresponds to classical time reversal in a classical limit. Such demonstrations are easily available in standard texts, (e.g. Sachs 1987, Earman 2002). The basic idea is just that complex conjugation converts a wave packet into a new wave packet whose

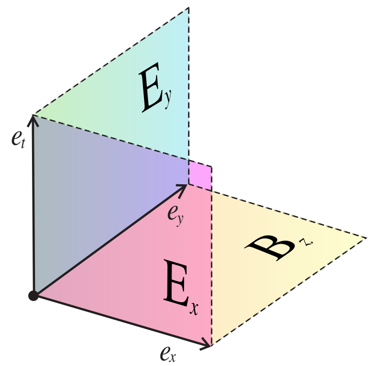

FIGURE 1. Three of the Six Electromagnetic Field Components

expectation values concerning position are the same and whose expectation values concerning momentum measurements are of the same magnitude but oppositely directed, just as needed for capturing the idea that time reversal involves flipping the directions of velocities.

In the case where ψ is considered physical, more is required. One needs an interpretation of the kind of stuff ψ is that justifies the complex conjugation definition of $\mathbf{T}$, an interpretation of ψ's intrinsic geometrical structure. As before, the focus is on the single particle case, with hope that the time reversal transformation will unproblematically extend to the multi-particle wave function once that problem is solved.

Consider as an exemplar of geometric interpretation, the case of the electromagnetic field in ordinary four-dimensional Minkowski spacetime. The electromagnetic field on a space-like hypersurface is standardly time-reversed by changing the signs of either the electric field components or magnetic field components, depending on one's conventions, but not both together. One way of expressing the electromagnetic field components in Minkowski spacetime, (e.g. Baez and Munaian 1994), is

$$F = E_x e_{tx} + E_y e_{ty} + E_z e_{tz} + B_x e_{yz} + B_y e_{zx} + B_z e_{xy}.$$

where e_{ij} is shorthand for a basis 2-vector $e_i e_j$ (or in the formalism of differential forms, the 2-form $dx_i \wedge dx_j$). The electromagnetic field is interpreted geometrically as 6 magnitudes corresponding to infinitessimal rectangular patches pictured in Fig. Fig. 1.

The relevant time reversal operation is a transformation $\mathbf{T}$ that in effect reverses the order of time slices in a foliation of spacetime. For all properties inhabiting a tangent space, the complete transformation is equivalent to transforming the basis tangent vector e_t with $-e_t$, which makes the original charge-current transform as

$$\mathbf{T}(\rho e_t + J_x e_x + J_y e_y + J_z e_z) = \rho e_t - J_x e_x - J_y e_y - J_z e_z$$

F transforms according to

$$\mathbf{T}F = E_x e_{tx} + E_y e_{ty} + E_z e_{tz} - B_x e_{yz} - B_y e_{zx} - B_z e_{xy}.$$

This formula mismatches the textbook treatment of time-reversal where one flips the sign of the magnetic but not the electric field because the standard treatment inserts an additional overall sign change for F under time reversal, which is justified in Malament (2004) and Leeds (2006) by noting the representation of the electromagnetic field as a tensor makes it implicitly relative to a choice of temporal orientation. Either way, the transformation can be subsumed under an overall geometrical interpretation of electromagnetism that maintains the time-reversal invariance of Maxwell's equations. This geometrical formalism also correctly handles the behaviour of F under all the proper Lorentz transformations as naturally following from how the basis tangent vectors transform under boosts and rotations.

Turning now to quantum mechanics, it is mysterious why ψ, if it is to be a physical field represented as complex-valued with no geometrical structure, should undergo complex conjugation under the action of time reversal. That suggests that we should enrich the standard interpretation of complex numbers with a more geometrical interpretation. There is a long tradition of thinking of the imaginary unit i in terms of geometry, going back to its inception. Space considerations forbid me from providing a thorough exploration of the myriad ways one might try to represent the geometric role of i in quantum mechanics, so I will just focus on two ideas that come from the geometric algebra formulation of physics, as elaborated by Hestenes (2003, 1999) and Doran and Lasenby (2003). In the geometric algebra formulation, a complex value can be split up into its real and imaginary parts, $\psi = \psi_a + \psi_b i$, with the real part being interpreted as a real scalar field and the imaginary part interpreted as a field whose unit magnitude is a volume element with a spatiotemporal orientation, as depicted in Fig. 2. In either a four-dimensional Minkowski spacetime or the five-dimensional version, the unit volume element (either a 4-vector or 5-vector) multiplied with itself does indeed equal -1, which is a requirement for it to represent the imaginary unit i adequately.

The benefit of the geometric interpretation is that it is automatically compatible with the algebraic properties of complex numbers in quantum mechanics, yet it provides a way of seeing quite directly why it is that complex conjugation is the right way to time-reverse a quantum field. First, consider the ordinary four-dimensional relativistic theory with a spinless massive particle and no external potential. Time reversal can be interpreted geometrically as a reflection of the physical fields about a plane perpendicular to any time-like axis t. When one maps the points in a flat manifold (with rectilinear axes) according to $(t, x, y, z) \to (-t, x, y, z)$, that induces a transformation on the tangent space so that e_t goes to $-e_t$. Thus, $\psi_a + \psi_b e_t e_x e_y e_z \to \psi_a - \psi_b e_t e_x e_y e_z$, as desired.

Because Galilean spacetime is not a metric space, it is impossible to define a volume form for it, at least in the normal way, so this interpretation of complex numbers cannot apply to non-relativistic quantum mechanics in its four-dimensional form. However, the situation changes if we use the five-dimensional model.

In the five-dimensional non-relativistic model, it is T that corresponds to Newtonian time, so one should consider how ψ is affected under the transformation $e_T \to -e_T$. Because $e_T = (e_t - e_q)/\sqrt{2}$, reversing e_T is accomplished by $e_t \to -e_t, e_q \to -e_q$. So, the mapping of the five-dimensional base manifold that effects time reversal is $(q, t, x, y, z) \to (-q, -t, x, y, z)$. Insofar as we are concerned merely with time reversal, it is possible to represent i in multiple ways, e.g., as e_{txyz} or e_{Txyz}, and get the desired transformation properties for ψ, complex conjugation, in the Galilean special case (5).

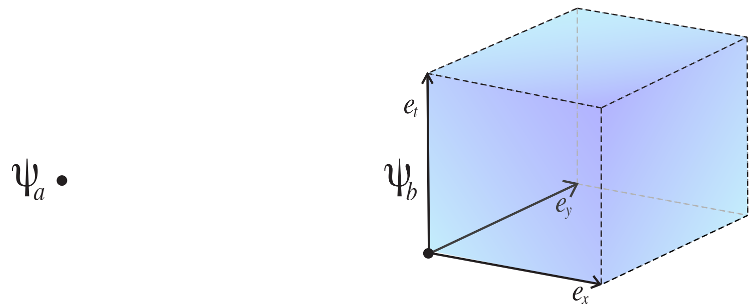

FIGURE 2. Geometric algebra representation of $\psi = \psi_a + \psi_b e_{txyz}$ with e_z suppressed.

What this shows is that it is possible to represent the non-relativistic ψ in a geometrically explicit manner that realises time-reversal as naturally as standard treatments of electromagnetism realise the transformation properties of the electromagnetic field. Nothing stands in the way of retaining the more traditional framework and interpreting ψ with regard to the geometrical structure of four-dimensional Galilean spacetime, having the imaginary part but not the real part linked to the vector structure that picks out the difference between time-like and space-like vectors and possibly linked to the affine connection. But because there are no structures in Galilean spacetime corresponding to spatiotemporal volume elements, a geometric interpretation of i in the four-dimensional Galilean theory would be rather ad hoc.

It is important to recognise that the mere fact we have found some structure to represent i is not terribly surprising because there is a range of choices for how to represent ψ in the five-dimensional model. A more detailed investigation would involve postulating geometric interpretations of multiple fundamental fields all embedded within the forty-two degrees of freedom provided by the tangent space for a multi-vector field in a way that respects what we know of the fields empirically and illuminates geometrical relations among them, especially getting all the fields to transform under **C**, **P**, and **T** appropriately in unison. That is far too large a task to take up here, but it is important that the five-dimensional model does provide a tool whereby geometric relationships between variables whose transformation properties we think we understand can be translated back and forth between a relativistic and non-relativistic version.

6. Conclusion

The most interesting feature of the five-dimensional model is simply the fact that there exists this non-trivial relationship between the Galilean and relativistic theories—a relationship not captured by thinking of the non-relativistic theory as a limiting case of the relativistic. I know of no other similar cases where two major competing scientific theories with significantly different interpretations of their common subject matter have been embedded into a single model in such a non-trivial way. I suggest this case is worth studying further not only for its application to physics but for its potential impact on conceptual translation and incommensurability between competing paradigms.

References

[1] HOLLAND, P. and BROWN, H. (2003) "The Non-Relativistic Limits of the Maxwell and Dirac Equations: The Role of Galilean and Gauge Invariance, **Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics** 34 (2), pp. 161–87.

[2] ALBERT, D. (1996) Elementary Quantum Metaphysics in A. FINE and S. GOLDSTEIN (eds.) **Bohmian Mechanics and Quantum Theory: An Appraisal** (Dordrecht, Kluwer), pp. 277–284.

[3] LEWIS, P. (2004) Life in Configuration Space, **British Journal for the Philosophy of Science**, 55 (4), pp. 713–729.

[4] BENATTI, F., GHIRARDI, G. C., GRASSI, R. (1995) Describing the Macroscopic World: Closing the Circle within the Dynamical Reduction Program. **Foundations of Physics**, 25, pp. 5–38.

[5] KLEIN, O., (1926) Quantum Theory and Five-Dimensional Relativity, **Zeitschrift für Physik**, 37, pp. 895.

[6] DORAN, C. and LASENBY, A. (2003) **Geometric Algebra for Physicists** (Cambridge, Cambridge University Press).

[7] HESTENES, D. (2003) Spacetime Physics with Geometric Algebra **American Journal of Physics**, 71 (7), pp. 691–714.

[8] HESTENES, D. (1999) **New Foundations for Classical Mechanics** (Dordrecht, Kluwer).

[9] BALLENTINE, L. (1998) **Quantum Mechanics: A Modern Development** (Singapore, World Scientific).

[10] SACHS, R. (1987) **The Physics of Time Reversal** (Chicago, University of Chicago Press), pp. 33–41.

[11] EARMAN, J. (2002) "What Time Reversal Invariance is and Why it Matters," **International Studies in the Philosophy of Science**, 16 (3), pp. 245–264.

[12] LEEDS, S. (2006) "Discussion: Malament on Time Reversal," **Philosophy of Science**, 73, pp. 448–458.

[13] ALBERT, D. (2000) **Time and Chance**. Cambridge, MA: Harvard University Press.

[14] MALAMENT, D. (2004) "On the Time Reversal Invariance of Classical Electromagnetic Theory," **Studies in History and Philosophy of Modern Physics**, 35 pp. 295–315.

[15] BAEZ, J., MUNIAIN, J. P. (1994) **Gauge Fields, Knots, and Gravity**. World Scientific Publishing.

[16] NORTH, J. (2008) "Two Views on Time Reversal," **Philosophy of Science**, 75, pp. 201–223.

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