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Expressivity of Imperfect Information Logics without Identity

Abstract. In this article we investigate the family of independence-friendly (IF) logics in the equality-free setting, concentrating on questions related to expressive power. Various natural equality-free fragments of logics in this family translate into existential second-order logic with prenex quantification of function symbols only and with the first-order parts of formulae equality-free. We study this fragment of existential second-order logic. Our principal technical result is that over finite models with a vocabulary consisting of unary relation symbols only, this fragment of second-order logic is weaker in expressive power than first-order logic (with equality). Results about the fragment could turn out useful for example in the study of independence-friendly modal logics. In addition to proving results of a technical nature, we address issues related to a perspective from which IF logic is regarded as a specification framework for games, and also discuss the general significance of understanding fragments of second-order logic in investigations related to non-classical logics.

Keywords: Independence-friendly logic, Existential second-order logic, Equality-free, Expressivity.

1. Introduction

We investigate the family of *independence-friendly* (IF) *logics* introduced by Hintikka and Sandu in [8]. See also [7] for an early exposition of the main ingredients leading to the idea of IF logic, and of course [5] for an even earlier discussion of ideas closely related to IF logic. Variants of IF logic have received a lot of attention recently; see [1, 4, 9, 10, 11, 14, 15, 16, 18] for example. Therefore we believe that time is beginning to be mature for technical investigations related to the expressivity of natural fragments of systems of IF logic. The focus of the current article is the expressivity of IF logic in the equality-free setting.

In [1], Caicedo, Janssen and Dechesne define a canonical version of IF logic, a version they call IF^*, and study a range of its properties. The system IF^* allows for *slashed disjunctions* and *slashed conjunctions* in addition to

Special Issue: Dependence and Independence in Logic
Edited by **Juha Kontinen, Jouko Väänänen, and Dag Westerståhl**

Studia Logica (2013) 101: 237–265
DOI: 10.1007/s11225-013-9482-7

slashed quantifiers.[1] See [1] for an exposition of the central properties of IF*. The core system studied by Mann, Sandu and Sevenster in their book [15] is similar to the system IF*. However, it does not include slashed disjunctions and conjunctions as primitive constructors. The system we study in the current article can be defined as the fragment of the system IF* without equality and without slashed *connectives.* We denote this fragment by IF$_{wo=}$.

Even though motivated by questions related to the expressive power of IF$_{wo=}$, our study concerns a wider range of logics. In fact, our study focuses on the system fESO$_{wo=}$, which is the fragment of existential second-order logic where second-order quantifiers quantify function symbols only and where the first-order parts of formulae are equality-free. We establish that each sentence of IF$_{wo=}$ can be transformed into a sentence of fESO$_{wo=}$ that defines exactly the same class of models as the original IF$_{wo=}$ sentence.

Results about fESO$_{wo=}$ automatically apply to a wider range of logics, not only IF$_{wo=}$. In general, understanding fragments of second-order logic can be very useful in the study of non-classical logics with constructors giving them the capacity to express genuinely second-order properties. In a typical case such a non-classical logic immediately translates into a fragment of second-order logic. Then, armed with theorems about fragments of second-order logic, one may immediately obtain a range of results concerning the non-classical logic under investigation. Such results can be, for example, related to decidability issues. By directing attention to *fragments* of second-order logic rather than the full system, one can often easily identify, for example, truth preserving model transformations. The very high expressive power of second-order logic seems to make it very difficult to identify directly applicable tools that enable one to produce concrete undefinability results that concern *all* of second-order logic.

On a general level, results about fESO$_{wo=}$ contribute to the study of fragments of second-order logic. On a more particular level, we believe that insights concerning the expressivity of sentences of the equality-free systems IF$_{wo=}$ and fESO$_{wo=}$ can be more or less directly useful in the study of the independence-friendly modal logics of Tulenheimo [17] and Tulenheimo and Sevenster [16] and others. This is due to the fact that formulae of such systems tend to translate to formulae of IF$_{wo=}$. This realization provides an example that should demonstrate the significance of the claim made about the study of fragments of second-order logic above.

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[1] In fact, slashed disjunctions are defined as primitive constructors; slashed conjunctions arise as abbreviations in the way analogous to the first-order case.

In this article we investigate the expressivity of *sentences* of $\mathrm{IF}_{wo=}$. A sentence of $\mathrm{IF}_{wo=}$ *defines* the class of models over which the verifying player *Eloise* has a winning strategy in the related semantic games, i.e., the class of models where the sentence is true. We begin the paper by observing that $\mathrm{IF}_{wo=}$ can define properties not definable in first-order logic FO (with equality), when the vocabulary under consideration contains at least one binary relation symbol. We then define a simple model-transformation that preserves truth of $\mathrm{fESO}_{wo=}$ sentences, but not FO sentences. The same transformation of course also preserves truth of $\mathrm{IF}_{wo=}$ sentences. Therefore we observe that $\mathrm{IF}_{wo=}$ and FO are incomparable with regard to expressive power. We discuss the significance of the preservation result in relation to a perspective from which IF logic is regarded as a specification language for games.

Finally, we ask whether $\mathrm{IF}_{wo=}$ and FO are also incomparable with regard to expressive power when attention is limited to vocabularies containing only unary relation symbols. Our principal result is that over finite models with a non-empty vocabulary containing unary relation symbols only, we have

$$\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=} \leq \mathrm{fESO}_{wo=} < \mathrm{FO}.$$

Here $\mathrm{FO}_{wo=}$ denotes first-order logic without equality. So far we have not succeeded in establishing these results without the use of somewhat involved combinatorial arguments. In addition to proving the results, we also wish to *reflect upon and promote the proof techniques* used.

2. Preliminary Considerations

Even though we have attempted to make the presentation of all results relatively self-contained and rigorous, we do assume that the reader has some degree of familiarity with IF logic. For an introduction and a tour of a wide range of central properties of IF logic, see the article [1]. The authors name the system studied in that article IF^*. The system $\mathrm{IF}_{wo=}$ we are about to define is a fragment of IF^*.

2.1. Syntax of $\mathrm{IF}_{wo=}$

We study IF logic in the equality-free setting, and for that purpose we now formally define the system $\mathrm{IF}_{wo=}$. Let $\mathcal{V}$ be a vocabulary and VAR a countably infinite set of first-order variable symbols. The set of $\mathcal{V}$-formulae of $\mathrm{IF}_{wo=}$ is defined as follows.

1. The set of $\mathcal{V}$-terms is defined as in first-order logic.
2. The set $\mathrm{ATOM}_{wo=}(\mathcal{V})$ of atomic $\mathcal{V}$-formulae is defined as in first-order logic without equality. Atomic formulae with the equality symbol are excluded from the set.
3. The set of $\mathcal{V}$-formulae of $\mathrm{IF}_{wo=}$ is generated by the following grammar.

$$\varphi ::= \alpha \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \exists v_i/W\varphi,$$

where $\alpha \in \mathrm{ATOM}_{wo=}(\mathcal{V})$, $v_i \in \mathrm{VAR}$ and W is a string that represents a finite set of variable symbols in VAR.

We also of course define, in analogy with first-order logic, the abbreviations $\forall v_i/W\varphi =_{def} \neg\exists v_i/W\neg\ \varphi$ and $(\varphi \wedge \psi) =_{def} \neg(\neg\varphi \vee \neg\psi)$. Instead of writing $Qx/\emptyset\ \varphi$, where $Q \in \{\exists, \forall\}$, we simply write $Qx\varphi$.

The set of $\mathcal{V}$-formulae of $\mathrm{IF}_{wo=}$ is exactly the set of $\mathcal{V}$-formulae of IF^* that contain neither equality nor slashed *connectives*. If we begin with the set $\mathrm{ATOM}(\mathcal{V})$ that contains all the first-order atoms of the vocabulary $\mathcal{V}$, also the atoms with an equality symbol, we obtain by the above grammar the set of $\mathcal{V}$-formulae of *slash-connective-free* IF^*.

The *set of non-logical symbols* of a formula is the set that contains exactly the relation symbols, function symbols and constant symbols that occur in the formula. (The equality symbol is not considered a non-logical symbol.) Therefore set of non-logical symbols of a $\mathcal{V}$-formula is a *subset* of $\mathcal{V}$. (The set of non-logical symbols that a $\mathcal{V}$-model gives an interpretation to is *exactly* $\mathcal{V}$.) A $\mathcal{V}$-formula may also be called a *formula of the vocabulary* $\mathcal{V}$. A $\mathcal{V}$-model may be called a *model of the vocabulary* $\mathcal{V}$.

2.2. Semantics

Let $\mathfrak{A}$ be a model with domain A. Let $X \subseteq \mathrm{VAR}$. A function $f : X \longrightarrow A$ is called a variable assignment. (The set X does not have to be finite.) If x is a variable symbol and $a \in A$, we let $f_{x:a}$ denote the variable assigment with the domain $X \cup \{x\}$ defined as follows.

1. $f_{x:a}(y) = f(y)$ if $y \neq x$,
2. $f_{x:a}(y) = a$ if $y = x$.

Let $X' \subseteq \mathrm{VAR}$ be a *finite* set of first-order variable symbols. Let V be a set of functions $f : X' \longrightarrow A$. We call such a set V a *team*. Let $h : V \longrightarrow A$ be a function mapping assignments in V to A. We define

1. $V_{x:h} = \{\ f_{x:h(f)} \mid f \in V\ \}$,

2. $V_{x:A} = \{ \ f_{x:a} \mid f \in V, \ a \in A \ \}$.

Let $Y \subseteq X'$. A function $h : V \longrightarrow A$ is called Y*-independent* if for all variable assigments $f, g \in V$ such that $f(x) = g(x)$ for all $x \in X' \setminus Y$, we have $h(f) = h(g)$. In other words, any two assignments that differ on a subset of Y, but not elsewhere, are treated similarly by h.

Let $\mathfrak{A}$ denote a model and V a team. The two satisfaction relations $\models^+$ and $\models^-$ of slash-connective-free IF* are defined in the following way (cf. Definition 4.2 of [1] and Theorem 4.8 of [1]). The semantic turnstile $\models$ is reserved for ordinary first-order and second-order predicate logic.[2]

$$
\begin{array}{lcl}
\mathfrak{A}, V \models^+ t_1 = t_2 & \Leftrightarrow & \forall s \in V\big(\mathfrak{A}, s \models t_1 = t_2\big). \\
\mathfrak{A}, V \models^- t_1 = t_2 & \Leftrightarrow & \forall s \in V\big(\mathfrak{A}, s \not\models t_1 = t_2\big). \\
\mathfrak{A}, V \models^+ R(t_1, \ldots, t_m) & \Leftrightarrow & \forall s \in V\big(\mathfrak{A}, s \models R(t_1, \ldots, t_m)\big). \\
\mathfrak{A}, V \models^- R(t_1, \ldots, t_m) & \Leftrightarrow & \forall s \in V\big(\mathfrak{A}, s \not\models R(t_1, \ldots, t_m)\big). \\
\mathfrak{A}, V \models^+ \neg\varphi & \Leftrightarrow & \mathfrak{A}, V \models^- \varphi. \\
\mathfrak{A}, V \models^- \neg\varphi & \Leftrightarrow & \mathfrak{A}, V \models^+ \varphi. \\
\mathfrak{A}, V \models^+ (\varphi \vee \psi) & \Leftrightarrow & \mathfrak{A}, V_1 \models^+ \varphi \text{ and } \mathfrak{A}, V_2 \models^+ \psi \text{ for some} \\
 & & \text{teams } V_1 \text{ and } V_2 \text{ such that } V = V_1 \cup V_2. \\
\mathfrak{A}, V \models^- (\varphi \vee \psi) & \Leftrightarrow & \mathfrak{A}, V \models^- \varphi \text{ and } \mathfrak{A}, V \models^- \psi. \\
\mathfrak{A}, V \models^+ \exists x/X\varphi & \Leftrightarrow & \mathfrak{A}, V_{x:f} \models^+ \varphi \text{ for some } X\text{-independent} \\
 & & \text{function } f : V \longrightarrow A. \\
\mathfrak{A}, V \models^- \exists x/X\varphi & \Leftrightarrow & \mathfrak{A}, V_{x:A} \models^- \varphi.
\end{array}
$$

When we write $\mathfrak{A}, V \models^+ \varphi$ we always assume that the set of non-logical symbols of φ is subset of the vocabulary of $\mathfrak{A}$, and also that the free variables in φ are contained in the domain of the assignments in the team V. This convention also of course applies to the turnstile $\models^-$, and an analogous convention holds for the turnstile $\models$ of ordinary predicate logic.

The above clauses also define the semantics of $\mathrm{IF}_{wo=}$. Thus we regard $\mathrm{IF}_{wo=}$ as a fragment of slash-connective-free IF* (and IF*) both syntactically and semantically.

A variable symbol x occurs *free* in a slash-connective-free IF* formula if and only if at least one of the following conditions hold.

1. The symbol x occurs in an atomic formula that is not in the scope of any quantifier Qx/X.

[2]We frequently use the two turnstiles $\models$ and $\models^+$ in order to distinguish between predicate logic semantics and IF* semantics. A sudden change of a turnstile can be difficult to spot, so the reader is noted about this here.

2. The symbol x occurs in the slash set of some quantifier that is not in the scope of any quantifier Qx/X. A quantifier is not considered to be in the scope of itself, so for example in the formula $\exists x/\{x\}P(x)$ the variable symbol x occurs free.

A formula φ of slash-connective-free IF* is a *sentence* if no variable symbol occurs free in φ. In this article we are interested in the expressive power of IF$_{wo=}$ sentences. If φ is a slash-connective-free IF* sentence, we write $\mathfrak{A} \models^+ \varphi$ if $\mathfrak{A}, \{\emptyset\} \models^+ \varphi$. Here $\emptyset$ is the empty valuation. We say that the sentence φ is *true in* $\mathfrak{A}$ if $\mathfrak{A} \models \varphi$.

Let $\mathcal{V}$ be a vocabulary and C a class of $\mathcal{V}$-models. Let φ be a $\mathcal{V}$-sentence of IF$_{wo=}$. We say that the sentence φ *defines the class D of models with respect to the class C* if $D = \{\ \mathfrak{A} \in C \mid \mathfrak{A} \models^+ \varphi\ \}$. The obvious analogous definition of definability w.r.t. a class of models applies to predicate logic.

Let φ and ψ be sentences, possibly of different logics. We say that φ and ψ are *uniformly equivalent* if the following conditions hold.

1. The sentences φ and ψ have the same set S of non-logical symbols.
2. For all models $\mathfrak{A}$ whose vocabulary is a superset of S, the sentence φ is true in $\mathfrak{A}$ if and only if the sentence ψ is true in $\mathfrak{A}$.

Let L and L' be logics. Let C be a class of $\mathcal{V}$-models. When we assert that $L \leq L'$ *with respect to the class* C, we mean that for each $\mathcal{V}$-sentence χ of L there exists a $\mathcal{V}$-sentence χ' of L' such that the sentences χ and χ' define exactly the same class of models with respect to C. If we assert that $L \not\leq L'$ *with respect to the class* C, we mean that there exists a $\mathcal{V}$-sentence χ of L such that no $\mathcal{V}$-sentence χ' of L' defines exactly the same class of models with respect to C as χ. When we state that $L < L'$ *with respect to* C, we mean that $L \leq L'$ with respect to C and, furthermore, $L' \not\leq L$ with respect to C. Below when we indeed do assert that $L \leq L'$ (or $L < L'$) with respect to some class C, the reader may wonder whether for each $\mathcal{V}$-sentence χ of L there exists a $\mathcal{V}$-sentence χ' of L' that defines the same class of structures w.r.t. C as χ and also has the same set of non-logical symbols as χ. Regarding all the related results that we establish below, such a χ' will always be easily seen to exist by the related arguments.

The original approach to the semantics of IF logic was game theoretic. We will not discuss the details of that approach here. We will, however, very briefly and informally describe some of the ingredients of the related framework. The reader is assumed to be familiar with the semantic game of first-order logic.

The semantic game for slash-connective-free IF^* is similar to the semantic game of first-order logic. The novel feature is that on the intuitive level, when picking a witness for a quantifier Qx/X, the value of x is chosen in ignorance of the values of the variables in X. The resulting game is a game of imperfect information, whereas the semantic game for first-order logic is a game of perfect information. IF logic can be regarded as a generalization of first-order logic obtained by moving from games of perfect information to games of imperfect information. Just like the semantic game of first-order logic, the game for slash-connective-free IF^* is played by two players, *Eloise* ($\exists$) and *Abelard* ($\forall$). Unlike in the semantic game of first-order logic, it is possible that neither of the players has a winning strategy in a game corresponding to the evaluation of some sentence φ in some model $\mathfrak{A}$. In that case we say that φ is *indeterminate* in $\mathfrak{A}$.

We fully identify IF^* formulae with empty slash sets only and formulae of FO with each other – in the canonical way. In fact, FO is regarded as a fragment of IF^*. The following lemma establishes an important relationship between the standard first-order semantics and the team based semantics defined above.

Lemma 2.1 (A paraphrase of a part of Theorem 4.10 of [1]). *Let $\mathfrak{A}$ be a model, V a team and φ be a first-order formula. Then $\mathfrak{A}, V \models^+ \varphi$ if and only if for all $g \in V$, $\mathfrak{A}, g \models \varphi$.*

Our main tool in investigating $\mathrm{IF}_{wo=}$ is the logic $\mathrm{fESO}_{wo=}$, whose formulae are exactly the formulae of the type $\exists \overline{f} \varphi$, where $\overline{f}$ is a finite vector of function symbols and φ is an FO formula without equality. The function symbols are allowed to be nullary, i.e., to be interpreted as constants. We identify constant symbols and nullary function symbols. The formulae of $\mathrm{fESO}_{wo=}$ are interpreted according to the natural semantics.

Theorem 2.2. *Each sentence of* $\mathrm{IF}_{wo=}$ *translates to a uniformly equivalent sentence of* $\mathrm{fESO}_{wo=}$.

Proof. See the appendix. ■

3. Expressivity of $\mathrm{IF}_{wo=}$ and $\mathrm{fESO}_{wo=}$ over Models with a Relational Vocabulary

Let $\mathcal{V}$ be a relational vocabulary containing a binary relation symbol. We begin the current section (Section 3) by providing a rather simple proof which establishes that with respect to the class of $\mathcal{V}$-models, $\mathrm{IF}_{wo=} \not\leq \mathrm{FO}$. We then also show that over $\mathcal{U}$-models, where $\mathcal{U}$ is a relation symbol containing binary and unary relation symbols only, we have $\mathrm{FO} \not\leq \mathrm{fESO}_{wo=}$. We limit attention to relational vocabularies in this section mainly in order to streamline the exposition of the results. Note that in this section we do not, however, limit our attention to finite models only.

PROPOSITION 3.1. *Let $\mathcal{V}$ be a relational vocabulary containing at least one binary relation symbol R. Let C be the class of all $\mathcal{V}$-models. Then there is a class C' of $\mathcal{V}$-models definable w.r.t. C by a $\mathcal{V}$-sentence of* $\mathrm{IF}_{wo=}$ *and also a $\mathcal{V}$-sentence of* $\mathrm{fESO}_{wo=}$*, but not definable w.r.t. C by any $\mathcal{V}$-sentence of* FO. *In fact, C' is not definable w.r.t. C even with any $\mathcal{V}$-sentence of* MSO.[3]

PROOF. Let φ be the following sentence of IF^*.

$$\forall x \forall y \exists z/\{y\} \exists v/\{x,z\} \Big((x = y \rightarrow z = v) \ \wedge \ (z = y \rightarrow v = x) \ \wedge \ z \neq x \Big)$$

Here the symbol $\rightarrow$ is used as an abbreviation in the way familiar from first-order logic. By performing the Skolemization procedure defined in the appendix A and existentially quantifying the resulting function symbols, we obtain the following sentence φ^* of existential second-order logic.

$$\exists f \exists g \forall x \forall y \Big(\big(x = y \rightarrow g(x) = f(y)\big) \ \wedge \ \big(g(x) = y \rightarrow f(y) = x\big) \ \wedge \ g(x) \neq x \Big)$$

It is easy to see that this sentence asserts that there exists a function h that is an involution[4] and does not have a fixed point.[5] It is straighforward to show, using Lemma A.3 and Lemma 2.1, that the sentence φ^* is uniformly equivalent to φ. Therefore we conclude that the sentence φ is true in exactly those models whose domain has an even or an infinite cardinality.

Let φ' be the $\mathrm{IF}_{wo=}$ sentence obtained from φ by replacing each atom of the type $t_1 = t_2$ by the atom $R(t_1, t_2)$. Let H be the class of finite $\mathcal{V}$-models $\mathfrak{A}$ such that

$$R^{\mathfrak{A}} = \{ \ (a,a) \mid a \in Dom(\mathfrak{A}) \ \}.$$

It is clear that with respect to H, the sentence φ' defines the class H_{even} of models whose domain is even. A straightforward Ehrenfeucht-Fraïssé argument shows that the class H_{even} is not definable with respect to H by any FO sentence. In fact, a simple Ehrenfeucht-Fraïssé style argument

[3] MSO stands for *monadic second-order logic*. See [12] for an introduction to MSO.

[4] An involution is a function h such that $h(h(x)) = x$ for all inputs x.

[5] A fixed point of a unary function h is a point x such that $h(x) = x$.

establishes that H_{even} is not definable w.r.t. H even by any sentence of MSO.[6]

Recall that C denotes the class of all $\mathcal{V}$-models. Since there is no FO (or MSO) sentence that defines w.r.t. H the same class of models as φ', there is no FO (or MSO) sentence that defines exactly the same class of models as φ' w.r.t. C.

Since by Theorem 2.2 we see that φ' can be transformed to a uniformly equivalent $\text{fESO}_{wo=}$ sentence, it follows that $\text{fESO}_{wo=} \not\leq$ FO (and also that $\text{fESO}_{wo=} \not\leq$ MSO) over $\mathcal{V}$-models. ■

By the proof it is immediate that Proposition 3.1 holds even if we restrict attention to finite $\mathcal{V}$-models.

3.1. Bloating Models

We now define a model-transformation under which truth of $\text{fESO}_{wo=}$ sentences is preserved.

Definition 3.2. Let $\mathcal{U}$ be a relational vocabulary containing unary and binary relation symbols only.[7] (We restrict our attention to at most binary relation symbols for the sake of simplicity.) Let $\mathfrak{A}$ be a $\mathcal{U}$-model with the domain A, and let $a \in A$. Let S be some set such that $S \cap A = \emptyset$. Define the $\mathcal{U}$-model $\mathfrak{B}$ as follows.

1. The domain of $\mathfrak{B}$ is the set $A \cup S$.
2. Let $P \in \mathcal{U}$ be a unary relation symbol. We define $P^{\mathfrak{B}}$ as follows.
   (a) For all $v \in A$, $v \in P^{\mathfrak{B}}$ iff $v \in P^{\mathfrak{A}}$.
   (b) For all $s \in S$, $s \in P^{\mathfrak{B}}$ iff $a \in P^{\mathfrak{A}}$.
3. Let $R \in \mathcal{U}$ be a binary relation symbol. We define $R^{\mathfrak{B}}$ as follows.
   (a) For all $\overline{v} \in A \times A$, $\overline{v} \in R^{\mathfrak{B}}$ iff $\overline{v} \in R^{\mathfrak{A}}$.
   (b) For all $s \in S$ and all $v \in A$, $(v, s) \in R^{\mathfrak{B}}$ iff $(v, a) \in R^{\mathfrak{A}}$.
   (c) For all $s \in S$ and all $v \in A$, $(s, v) \in R^{\mathfrak{B}}$ iff $(a, v) \in R^{\mathfrak{A}}$.
   (d) For all $s, s' \in S$, $(s, s') \in R^{\mathfrak{B}}$ iff $(a, a) \in R^{\mathfrak{A}}$.

We call the model $\mathfrak{B}$ a *bloating* of $\mathfrak{A}$. Figure 1 illustrates how this model transformation affects models.

We note that bloatings are closely related to for example *surjective strict homomorphisms* (see Definition 2.1 of [2]).

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[6]See [12] for an introduction to the Ehrenfeucht-Fraïssé games for MSO.

[7]Let us agree that this mode of speaking allows for the case where $\mathcal{U} = \emptyset$.

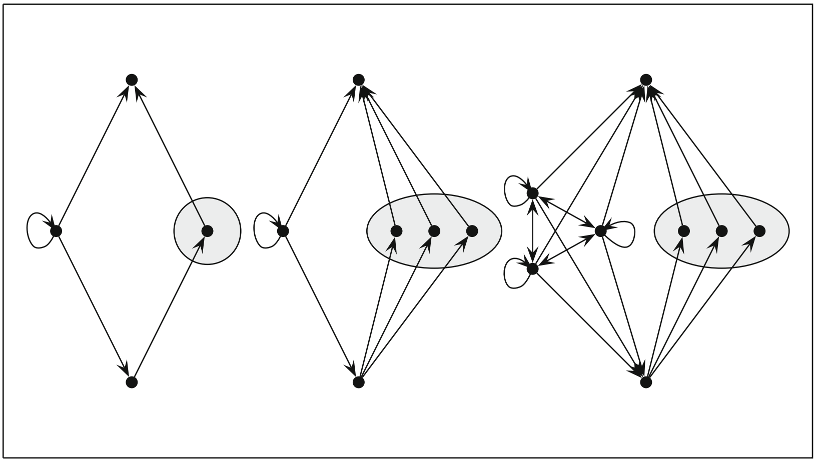

Figure 1. The figure shows three connected structures of a vocabulary consisting of one binary and one unary relation symbol. The shaded areas correspond to the extensions of the unary relation symbol. The structure in the middle is a bloating of the structure on the left. The structure in the middle is obtained from the one on the left by adding two new copies of the middle right element. The structure on the right is a bloating of the structure in the middle obtained by adding two copies of the middle left element.

THEOREM 3.3. *Let $\mathcal{U}$ be a vocabulary containing unary and binary relation symbols only. Truth of* $\mathrm{fESO}_{wo=}$ *sentences is preserved from $\mathcal{U}$-models to their bloatings.*

PROOF. Let $\mathfrak{A}$ be a $\mathcal{U}$-model and φ a sentence of $\mathrm{fESO}_{wo=}$. The formula φ can be transformed into a uniformly equivalent formula $\exists\overline{f}\psi$, where $\exists\overline{f}$ is a vector of existentially quantified function symbols (some of them perhaps nullary) and ψ is a first-order sentence such that the following conditions hold.

1. The formula ψ is of the type $\forall\overline{x}\,\psi'$, where $\forall\overline{x}$ is a string of universal first-order quantifiers and ψ' is a quantifier-free formula.
2. The quantifier free part ψ' of ψ is in negation normal form, i.e., negations occur only in front of atomic formulae.

This normal form is obtained by first transferring the first-order part of φ into prenex normal form without nested quantification of the same variable and then Skolemizing the first-order part of the resulting sentence. The quantifier-free part of the resulting sentence is then put into negation normal form. The freshly introduced Skolem functions are prenex quantified existentially, so the set of non-logical symbols of $\exists\overline{f}\psi$ is the same as that of φ. The process of transferring φ into the described normal form does not introduce equality, so $\exists\overline{f}\psi$ is a sentence of $\mathrm{fESO}_{wo=}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in Definition 3.2. The models there had the domains A and $A \cup S$, respectively, and the element $a \in A$ was used in order to define $\mathfrak{B}$. We assume that $\mathfrak{A} \models \exists\overline{f}\psi$ and expand $\mathfrak{A}$ to a model $\mathfrak{A}' = (\mathfrak{A}, \overline{f^{\mathfrak{A}'}})$ such that $\mathfrak{A}' \models \psi$. We then expand $\mathfrak{B}$ to a model $\mathfrak{B}' = (\mathfrak{B}, \overline{f^{\mathfrak{B}'}})$ as follows.

1. For each k-ary symbol f, we let $f^{\mathfrak{B}'} \upharpoonright A^k = f^{\mathfrak{A}'}$. Note that when $k = 0$, i.e., when f is a constant symbol, then $f^{\mathfrak{B}'} = f^{\mathfrak{A}'}$.
2. For each k-tuple $\overline{w} \in (A \cup S)^k$ containing points from the set S, we define the k-tuple $\overline{w}'$, where each co-ordinate value $s \in S$ of $\overline{w}$ is replaced by the element a. We then set $f^{\mathfrak{B}'}(\overline{w}) = f^{\mathfrak{A}'}(\overline{w}')$.

We then establish that $\mathfrak{B}' \models \psi$. The proof is a simple induction on the structure of ψ. For each variable assignment h with codomain A, let $g(h)$ denote the set of all variable assignments with codomain $A \cup S$ that can be obtained from h by allowing some subset of the variables mapping to the element a to map to elements in S. We prove that for every variable assignment h with codomain A and every subformula χ of ψ,

$$\mathfrak{A}', h \models \chi \;\Rightarrow\; \forall h' \in g(h)(\mathfrak{B}', h' \models \chi).$$

The cases for atomic and negated atomic formulae form the basis of the induction. The claim for these formulae follows immediately with the help of the observation that $h(t) = h'(t)$ for all h and $h' \in g(h)$ and terms t that *contain function symbols*, i.e., terms that are not variable symbols. We will next establish this claim by induction on the nesting depth of function symbols.

The basis of the induction deals with the terms of nesting depth one, i.e., terms of the type $f(x_1, \ldots, x_k)$ and c, where the symbols $x_1, \ldots, x_k$ are variable symbols and the symbol c is a constant symbol. It is immediate that $h(t) = h'(t)$ for all h and $h' \in g(h)$ and all such terms t of nesting depth one.

Now let $f(t_1, \ldots, t_k)$ be a term of nesting depth $n + 1$. By the induction hypothesis, for each one of the terms t_i that is not a variable symbol, we have $h(t_i) = h'(t_i)$. For the terms t_i that are variable symbols and for which $h(t_i) \neq a$, we have $h(t_i) = h'(t_i)$. For the terms t_i that are variable symbols and for which $h(t_i) = a$, we have either $h'(t_i) = a$ or $h'(t_i) \in S$. We therefore notice that we obtain the tuple $(h(t_1), \ldots, h(t_k))$ from the tuple $(h'(t_1), \ldots, h'(t_k))$ by replacing the elements $u \in S$ of the tuple $((h'(t_1), \ldots, h'(t_k))$ by the element a. Therefore we conclude, by the definition of the function $f^{\mathfrak{B}'}$, that

$$f^{\mathfrak{B}'}(h'(t_1),\ldots,h'(t_k)) = f^{\mathfrak{A}'}(h(t_1),\ldots,h(t_k)).$$

This concludes the induction on terms and therefore the basis of the original induction on the structure of ψ has now been established. We return to the original induction.

The connective cases are trivial and the quantifier case relatively straightforward. We discuss the details of the quantifier case here.

Assume $\mathfrak{A}', h \models \forall x\alpha(x)$. We need to show that for all $h' \in g(h)$, $\mathfrak{B}', h' \models \forall x\alpha(x)$. Assume, for contradiction, that for some $h'' \in g(h)$ we have $\mathfrak{B}', h'' \not\models \forall x\alpha(x)$. Therefore, for some $u \in A \cup S$, we have $\mathfrak{B}', h''\frac{u}{x} \not\models \alpha(x)$. It suffices to show that $h''\frac{u}{x} \in g(h\frac{v}{x})$ for some $v \in A$. This suffices, as the assumption $\mathfrak{A}', h \models \forall x\alpha(x)$ first implies that $\mathfrak{A}', h\frac{v}{x} \models \alpha(x)$, which in turn then implies, by the induction hypothesis, that $\mathfrak{B}', h''\frac{u}{x} \models \alpha(x)$.

If $u \in A$, let $v = u$. Then, as $h'' \in g(h)$, we have $h''\frac{u}{x} = h''\frac{v}{x} \in g(h\frac{v}{x})$. If $u \in S$, we let $v = a$. Then, as $h'' \in g(h)$, we have $h''\frac{u}{x} \in g(h\frac{a}{x}) = g(h\frac{v}{x})$. ■

An immediate consequence of Theorem 3.3 is that FO $\not\leq$ fESO$_{wo=}$ with respect to $\mathcal{U}$-models because there exist $\emptyset$-sentences of first-order logic whose truth is not preserved under bloating. In fact, we clearly have FO $\not\leq$ fESO$_{wo=}$ even with respect to the class of $\mathcal{U}$-models with a finite domain.

Theorem 3.3 is interesting when regarding IF logic as a kind of a specification language for games. Let ψ be a sentence of IF*, and let S be the set of non-logical symbols occurring in ψ. Let C be the class of all models whose vocabulary is a superset of S. We may regard the sentence ψ as a collection of *rules* that *specifies*, for each $\mathfrak{M} \in C$, the semantic game for establishing whether $\mathfrak{M} \models^+ \psi$. The model $\mathfrak{M}$ may be regarded as a *board* on which games with various different kinds of rules can be played – one collection of rules for each sentence whose set of non-logical symbols is a subset of the vocabulary of $\mathfrak{M}$.

Let $\mathcal{U}$ be a vocabulary of the type defined in Theorem 3.3. Let the $\mathcal{U}$-sentence φ of IF$_{wo=}$ specify some class of games and assume we know some board (i.e., a model) on which Eloise has a winning strategy in the related game (i.e., φ is true in that model). Theorem 3.3 then gives us a whole range of new, larger boards where she has a winning strategy in the game specified by φ. So, winning is preserved under bloatings. On the other hand, non-winning and in fact even indeterminacy are clearly preserved under reverse bloatings, as can be directly seen by the following dualization argument. Assume $\mathfrak{B}$ is a bloating of $\mathfrak{A}$ and assume that φ is indeterminate in $\mathfrak{B}$. If $\mathfrak{A} \models^+ \varphi$, then $\mathfrak{B} \models^+ \varphi$, which is a contradiction, so necessarily $\mathfrak{A} \not\models^+ \varphi$.

To conclude that φ is indeterminate in $\mathfrak{A}$, it now suffices to establish that $\mathfrak{A} \not\models^{-} \varphi$. Assume for the sake of contradiction that $\mathfrak{A} \models^{-} \varphi$. Therefore $\mathfrak{A} \models^{+} \neg\varphi$, whence $\mathfrak{B} \models^{+} \neg\varphi$. Hence $\mathfrak{B} \models^{-} \varphi$, which is a contradiction.

4. Expressivity of $\mathrm{fESO}_{wo=}$ and $\mathrm{IF}_{wo=}$ over Finite Models with a Unary Relational Vocabulary

We now turn our attention to finite models with a unary relational vocabulary. Over such finite models, the picture is quite different from the case where there is a binary relation symbol in the vocabulary. We will show that over the class of finite $\mathcal{V}$-models, where $\mathcal{V}$ is an arbitrary non-empty vocabulary containing unary relation symbols only, we have

$$\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=} \leq \mathrm{fESO}_{wo=} < \mathrm{FO}.$$

We first establish that $\mathrm{fESO}_{wo=} < \mathrm{FO}$, and then that $\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}$. We already know that $\mathrm{IF}_{wo=} \leq \mathrm{fESO}_{wo=}$ (Lemma 2.2).

4.1. $\mathrm{fESO}_{wo=} < \mathrm{FO}$ over Finite Models with a Unary Relational Vocabulary

For the duration of the current subsection (subsection 4.1), fix $\mathcal{V}$ to be a relational vocabulary containing unary relation symbols only. The vocabulary $\mathcal{V}$ may be empty or infinite. In this subsection we establish that $\mathrm{fESO}_{wo=} < \mathrm{FO}$ with respect to the class of finite $\mathcal{V}$-models. Therefore also $\mathrm{IF}_{wo=} < \mathrm{FO}$ over that class. We begin by making a number of auxiliary definitions.

Let $U \subseteq \mathcal{V}$ be a finite unary vocabulary. A *unary U-type* (with the free variable x) is a conjunction τ with $|U|$ conjuncts such that for each $P \in U$, exactly one of the formulae $P(x)$ and $\neg P(x)$ is a conjunct of τ; in the case $U = \emptyset$, τ is the formula $x = x$. Let $T = \{\tau_1, \ldots, \tau_{|T|}\}$ be the set of unary U-types.[8] The domain of each U-model $\mathfrak{A}$ is partitioned into some number $n \leq |T|$ of sets S_i such that the elements of S_i *realize*, i.e., satisfy, the type $\tau_i \in T$. An element $a \in Dom(\mathfrak{A})$ realizes (satisfies) the type τ_i if and only if $\mathfrak{A} \models \tau_i(a)$ in the usual sense of first-order logic.

Let $n \in \mathbb{N}$, and let $k = 2^n$. Any relation

$$R \;\subseteq\; \mathbb{N}^k \setminus \{0\}^k$$

[8] We assume that types have some standard ordering of conjuncts and bracketing, so that there are exactly $2^{|U|}$ different unary U-types; for each subset S of U, there exists exactly one unary U-type τ such that for each $P \in U$, $P(x)$ is a conjunct of τ if and only if $P \in S$.

is called a *spectrum*. We associate sentences of FO and $\mathrm{fESO}_{wo=}$ with spectra in a way specified in the following definition.

Definition 4.1. Consider the set S containing exactly all the FO and $\mathrm{fESO}_{wo=}$ sentences of the vocabulary $\mathcal{V}$. Let $\varphi \in S$. Let $U \subseteq \mathcal{V}$ be the finite set of unary relation symbols occurring in φ. Let $T = \{\tau_1, \ldots, \tau_{|T|}\}$ be the finite set of unary U-types, and let $\leq^T$ denote a linear ordering of the types in T defined such that $\tau_i \leq^T \tau_j$ iff $i \leq j$. Define the relation $R_\varphi \subseteq \mathbb{N}^{|T|}$ such that $(n_1, \ldots, n_{|T|}) \in R_\varphi$ iff there exists a finite U-model $\mathfrak{A}$ of φ such that for all $i \in \{1, \ldots, |T|\}$, the number of points in the domain of $\mathfrak{A}$ that satisfy τ_i is n_i. We call such a relation R_φ the *spectrum of* φ (with respect to the ordering $\leq^T$).

Notice that the class of finite $\mathcal{V}$-models defined by φ is completely characterized by the spectrum $R_\varphi \subseteq \mathbb{N}^{|T|}$ in the sense that there is a canonical one-to-one correspondence between the isomorphism classes of finite U-models that satisfy φ and the tuples $\overline{r} \in R_\varphi$. See Figure 2 for an illustration of a spectrum of a sentence of FO with a unary relational vocabulary.

We then define a special family of spectra and then establish that this family exactly characterizes the expressive power of FO over the class of (finite) $\mathcal{V}$-models.

Definition 4.2. Let $l = 2^{l'}$ for some $l' \in \mathbb{N}$. Let $R \subseteq \mathbb{N}^l$ be a spectrum for which there exists a number $n \in \mathbb{N}_{\geq 1}$ such that for all co-ordinate positions $i \in \{1, \ldots, l\}$, all integers $k, k' > n$ and all $m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_l \in \mathbb{N}$, we have

$$(m_1, \ldots, m_{i-1}, k, m_{i+1}, \ldots, m_l) \in R$$
$$\Leftrightarrow$$
$$(m_1, \ldots, m_{i-1}, k', m_{i+1}, \ldots, m_l) \in R.$$

We call such a number n a *stabilizer* of the spectrum R. A spectrum with a stabilizer is called a *stabilizing spectrum.*

Proposition 4.3. *A spectrum R is a stabilizing spectrum if and only if R is the spectrum of some* FO *sentence.*

Proof. Let $R \subseteq \mathbb{N}^k$ be a stabilizing spectrum. Let $n \in \mathbb{N}_{\geq 1}$ be a stabilizer of R. Define the set $S = \{0, 1, \ldots, n\} \cup \{\infty\}$, where ∞ is simply a symbol. Define the function $c : \mathbb{N} \longrightarrow S$ as follows.

$$c(x) = \begin{cases} x & \text{if } x \leq n, \\ \infty & \text{if } x > n. \end{cases}$$

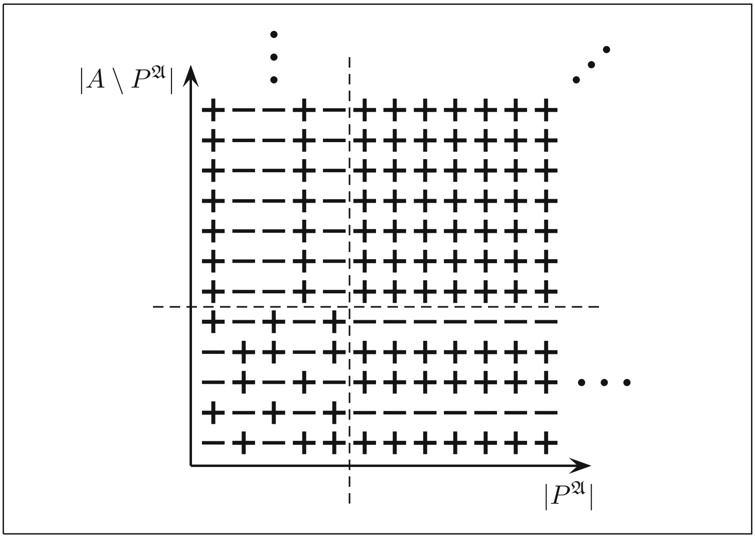

Figure 2. The figure illustrates a stabilizing spectrum that corresponds to some FO sentence φ with the set $\{P\}$ of non-logical symbols. P is a unary relation symbol. A plus symbol occurs at the position (i, j) iff there exists a $\{P\}$-model $\mathfrak{A}$ satisfying φ such that $|P^{\mathfrak{A}}| = i$ and $|A \setminus P^{\mathfrak{A}}| = j$, where $A = Dom(\mathfrak{A})$. In other words, the number of points in the domain of $\mathfrak{A}$ satisfying the type $P(x)$ is i and the number of points satisfying the type $\neg P(x)$ is j. The spectra for FO sentences divide the xy-plane into four distinct regions. The upper right region always contains either only plus symbols or only minus symbols. In the bottom left region, any distribution is possible. (The point $(0, 0)$ always contains a minus symbol though since we do not allow for models to have an empty domain.)

Define

$$R_0 = \{ \ \big(c(r_1), \ldots, c(r_k)\big) \mid (r_1, \ldots, r_k) \in R \ \}.$$

Let $(s_1, \ldots, s_k) \in R_0$. For each $i \leq k$ define a first-order sentence φ_i such that the following conditions hold.

1. If $s_i \leq n$, then φ_i asserts that there are exactly s_i elements that satisfy the type τ_i.
2. If $s_i = \infty$, then φ_i asserts that there are at least $n + 1$ elements that satisfy the type τ_i.

Let $\psi_{(s_1,\ldots,s_k)}$ be the conjunction of the sentences φ_i. Let χ_R be the disjunction of the sentences $\psi_{(s_1,\ldots,s_k)}$, where $(s_1, \ldots, s_k) \in R_0$. The set R_0 is finite, so the disjunction is a first-order sentence. Since the spectrum R is a stabilizing spectrum and n a stabilizer of R, we see that the disjunction χ_R defines the spectrum R.

The fact that each spectrum of an FO sentence is stabilizing follows by a straightforward Ehrenfeucht-Fraïssé argument. ■

Next we define some order theoretic concepts and then prove a number of related results that are used in the proof of the main theorem (Theorem 4.7) of the current section.

A structure $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ is a *partial order* if $\leq^{\mathfrak{A}} \subseteq A \times A$ is a reflexive, transitive and antisymmetric binary relation. Given a partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$, we let $<^{\mathfrak{A}}$ denote the irreflexive version of the order $\leq^{\mathfrak{A}}$. A partial order is *well-founded* if no strictly decreasing infinite sequence occurs in it. That is, a partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ is well-founded if for each each sequence $s : \mathbb{N} \longrightarrow A$ there exist numbers $i, j \in \mathbb{N}$ such that $i < j$ and $s(j) \not<^{\mathfrak{A}} s(i)$. An *antichain* $S \subseteq A$ of a partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ is a set such that for all distinct elements $s, s' \in S$, we have $s \not\leq^{\mathfrak{A}} s'$ and $s' \not\leq^{\mathfrak{A}} s$. In other words, the distinct elements s and s' are incomparable. A well-founded partial order that does not contain an infinite antichain is called a *partial well order*, or a *pwo*.

Let $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ and $\mathfrak{B} = (B, \leq^{\mathfrak{B}})$ be partial orders. The *Cartesian product* $\mathfrak{A} \times \mathfrak{B}$ of the structures is the partial order defined in the following way.

1. The domain of $\mathfrak{A} \times \mathfrak{B}$ is the Cartesian product $A \times B$.
2. The binary relation $\leq^{\mathfrak{A}\times\mathfrak{B}} \subseteq (A \times B) \times (A \times B)$ is defined in a pointwise fashion as follows.

$$(a, b) \leq^{\mathfrak{A}\times\mathfrak{B}} (a', b') \Leftrightarrow \Big(a \leq^{\mathfrak{A}} a' \text{ and } b \leq^{\mathfrak{B}} b'\Big)$$

For each $k \in \mathbb{N}_{\geq 1}$ and each partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$, we let $\mathfrak{A}^k = (A^k, \leq^{\mathfrak{A}^k})$ denote the partial order where the relation $\leq^{\mathfrak{A}^k} \subseteq A^k \times A^k$ is again defined in the pointwise fashion as follows.

$$(a_1, \ldots, a_k) \leq^{\mathfrak{A}^k} (a'_1, \ldots, a'_k) \quad \Leftrightarrow \quad \forall i \in \{1, \ldots, k\} : a_i \leq^{\mathfrak{A}} a'_i$$

We call the structure $\mathfrak{A}^k$ the *k-th Cartesian power* of $\mathfrak{A}$. We let $(\mathbb{N}^k, \leq)$ denote the k-th Cartesian power of the linear order $(\mathbb{N}, \leq)$. When $S \subseteq \mathbb{N}^k$, we let $(S, \leq)$ denote the partial order with the domain S and with the ordering relation inherited from the structure $(\mathbb{N}^k, \leq)$. In other words, for all $\overline{s}, \overline{s}' \in S$, we have $\overline{s} \leq^{(S,\leq)} \overline{s}'$ if and only if $\overline{s} \leq^{(\mathbb{N}^k,\leq)} \overline{s}'$. We simply write $\overline{u} \leq \overline{v}$ in order to assert that $\overline{u} \leq^{(\mathbb{N}^k,\leq)} \overline{v}$, when $\overline{u}, \overline{v} \in \mathbb{N}^k$.

The following lemma is a paraphrase of Lemma 5 of [13], where the lemma is credited to Higman [6].

LEMMA 4.4. *The Cartesian product of any two partial well orders is a partial well order.*

Variants of the next lemma are often attributed to Dickson [3]. The lemma follows immediately from Lemma 4.4.

LEMMA 4.5 (Dickson's Lemma variant). *Let $k \in \mathbb{N}_{\geq 1}$. The structure $(\mathbb{N}^k, \leq)$ does not contain an infinite antichain.*

PROOF. The structure $(\mathbb{N}, \leq)$ is a pwo, and the property of being a pwo is preserved under taking finite Cartesian products by Lemma 4.4. Therefore the structure $(\mathbb{N}^k, \leq)$ is a pwo. By definition, a pwo does not contain an infinite antichain. ■

Let $l \in \mathbb{N}_{\geq 1}$ and let $R \subseteq \mathbb{N}^l$ be a relation such that for all $\overline{u}, \overline{v} \in \mathbb{N}^l$, if $\overline{u} \in R$ and $\overline{u} \leq \overline{v}$, then $\overline{v} \in R$. We call the relation R *upwards closed* with respect to $(\mathbb{N}^l, \leq)$. When the exponent l is irrelevant or known from the context, we simply say that the relation R is upwards closed.

THEOREM 4.6. *Let $l' \in \mathbb{N}$ and $l = 2^{l'}$. Let $R \subseteq \mathbb{N}^l$ be a spectrum that is upwards closed with respect to $(\mathbb{N}^l, \leq)$. Then R is a stabilizing spectrum.*

PROOF. As $\emptyset$ is a stabilizing spectrum, we assume without loss of generality that $R \neq \emptyset$. We begin the proof by defining a function f that maps each non-empty subset of the set $\{1, \ldots, l\}$ to a natural number. Let $C \subseteq \{1, \ldots, l\}$ be a non-empty set. Let $R(C)$ denote the set consisting of exactly those tuples $\overline{w} \in R$ that have a non-zero co-ordinate value at each co-ordinate position $i \in C$ and a zero co-ordinate value at each co-ordinate position $j \in \{1, \ldots, l\} \setminus C$. Define the value $f(C) \in \mathbb{N}$ as follows.

1. If $R(C) = \emptyset$, let $f(C) = 0$.
2. If $R(C) \neq \emptyset$, choose some $\overline{w} \in R(C)$. Let $W \subseteq R(C)$ be a maximal antichain of $(R(C), \leq)$ with $\overline{w} \in W$, i.e., let W be an antichain of $(R(C), \leq)$ such that for all $\overline{u} \in R(C) \setminus W$, there exists some $\overline{v} \in W$ such that $\overline{u} < \overline{v}$ or $\overline{v} < \overline{u}$. By Lemma 4.5, we see that the set W is finite. Thus there exists a maximum co-ordinate value occurring in the tuples in W. Let $f(C)$ be equal to this value.

(Notice that we have some freedom of *choice* when defining the function f, so there need not be a unique way of defining f.)

With the function f defined, call

$$n = max(\{\ f(C) \mid C \subseteq \{1, \ldots, l\},\ C \neq \emptyset\ \}).$$

We establish that n is a stabilizer for the relation R. We assume, for the sake of deriving a contradiction, that there exist integers $k, k' > n$ and $m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_l \in \mathbb{N}$ such that the equivalence

$$\begin{array}{cc} & (m_1, \dots, m_{i-1}, k, m_{i+1}, \dots, m_l) \in R \\ \Leftrightarrow & \\ & (m_1, \dots, m_{i-1}, k', m_{i+1}, \dots, m_l) \in R. \end{array}$$

does not hold. Let $k < k'$. As by assumption the relation R is upwards closed, it must be the case that

$$\begin{array}{cc} & (m_1, \dots, m_{i-1}, k, m_{i+1}, \dots, m_l) \notin R \\ \text{and} & \\ & (m_1, \dots, m_{i-1}, k', m_{i+1}, \dots, m_l) \in R. \end{array}$$

Otherwise we would immediately reach a contradiction. Call

$$\begin{array}{cc} & \overline{w}_k = (m_1, \dots, m_{i-1}, k, m_{i+1}, \dots, m_l) \\ \text{and} & \\ & \overline{w}_{k'} = (m_1, \dots, m_{i-1}, k', m_{i+1}, \dots, m_l). \end{array}$$

Let $C^* \subseteq \{1, \dots, l\}$ be the set of co-ordinate positions where the tuple $\overline{w}_{k'}$ (and therefore also the tuple $\overline{w}_k$) has a non-zero co-ordinate value. Let $W(C^*)$ denote the maximal antichain of $(R(C^*), \leq)$ chosen when defining the value of the function f on the input C^*. The tuple $\overline{w}_{k'}$ cannot belong to the set $W(C^*)$, since the co-ordinate value k' is greater than n, and therefore greater than any of the co-ordinate values of the tuples in $W(C^*)$. Hence, as $W(C^*)$ is a maximal antichain of $(R(C^*), \leq)$ and $\overline{w}_{k'} \in R(C^*)$, we conclude that there exists a tuple $\overline{u} \in W(C^*)$ such that $\overline{w}_{k'} < \overline{u}$ or $\overline{u} < \overline{w}_{k'}$. Since $k' > f(C^*)$, we must have $\overline{u} < \overline{w}_{k'}$. Therefore, as also $k > f(C^*)$, we conclude that $\overline{u} < \overline{w}_k$. Since R is upwards closed and $\overline{u} \in R$, we have $\overline{w}_k \in R$. This is a contradiction, as desired. ■

The following theorem is the main result of the current section.

THEOREM 4.7. *Let $\mathcal{V}$ be an arbitrary vocabulary containing unary relation symbols only. We have* $\mathrm{fESO}_{wo=} < \mathrm{FO}$ *with respect to the class of finite $\mathcal{V}$-models.*

PROOF. It is immediate by Theorem 3.3 that we have $\mathrm{FO} \not\leq \mathrm{fESO}_{wo=}$ over finite $\mathcal{V}$-models. It therefore suffices to show that $\mathrm{fESO}_{wo=} \leq \mathrm{FO}$ over finite $\mathcal{V}$-models. To show this, Let φ be an arbitrary $\mathrm{fESO}_{wo=}$ sentence of the vocabulary $\mathcal{V}$. By Proposition 4.3 it suffices to establish that the spectrum R_φ of φ is stabilizing. By Theorem 3.3, the spectrum R_φ is upwards closed. Therefore, by Theorem 4.6, R_φ is a stabilizing spectrum. ■

Corollary 4.8. *Let $\mathcal{V}$ be an arbitrary vocabulary containing unary relation symbols only. We have* $\mathrm{IF}_{wo=} < \mathrm{FO}$ *with respect to the class of finite* $\mathcal{V}$-*models.*

It is easy to see that the argument leading to Theorem 4.7 applies in a more general context. Let $\mathcal{U}$ be a *finite* unary relational vocabulary, and restrict attention to definability with respect to the class C of finite $\mathcal{U}$-models. Now Theorem 4.7 applies not only to $\mathrm{fESO}_{wo=}$ but to any logic L such that the classes definable in L w.r.t. C are closed under bloating. Here the restriction to finite models is crucial. For let L' be a logic whose language consists of exactly one formula, φ. Let the semantics of L' dictate that φ is true in a model $\mathfrak{A}$ iff the domain of the model $\mathfrak{A}$ is infinite. Then truth of L' sentences is closed under bloating, but FO and L' are incomparable with regard to expressive power. Note also that our proof is nonconstructive in the sense that without further information, the current formulation of the argument leaves open the question whether there is an *effective translation* from the system L considered into FO.

4.2. $\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}$ over Finite Models with a Unary Relational Vocabulary

In this subsection (subsection 4.2), let $\mathcal{V}$ denote a fixed *non-empty* vocabulary containing unary relation symbols only. In the current subsection we establish that over the class of finite $\mathcal{V}$-models, we have

$$\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}.$$

Let $P \in \mathcal{V}$. Let $\mathfrak{M}$ be a $\mathcal{V}$-model with exactly three points, two of which satisfy P. For other symbols $Q \in \mathcal{V}$, let $Q^{\mathfrak{M}} = \emptyset$. Let $\mathfrak{N}$ be a $\mathcal{V}$-model whose domain contains exactly two points, one satisfying P and the other one not. For other symbols $Q \in \mathcal{V}$, let $Q^{\mathfrak{N}} = \emptyset$. Consider the following $\mathrm{IF}_{wo=}$ sentence, where $\leftrightarrow$ is the usual familiar abbreviation.

$$\forall x \exists y \exists z/\{x\}\big(P(y) \wedge (P(x) \leftrightarrow P(z))\big)$$

The sentence is true in the model $\mathfrak{M}$ but not true in $\mathfrak{N}$. However, we will establish that there exists no $\mathrm{FO}_{wo=}$ sentence φ such that exactly one of the models $\mathfrak{M}$ and $\mathfrak{N}$ satisfies φ. We show this by applying a very simple back and forth argument. In the article [2], a characterization of the expressivity of $\mathrm{FO}_{wo=}$ is formulated in terms of a class of back and forth systems. We show that $\mathfrak{M}$ and $\mathfrak{N}$ satisfy exactly the same $\mathrm{FO}_{wo=}$ sentences by employing the tools defined in [2].

Definition 4.9 (cf. Definition 4.1 in [2].). Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{W}$-models, where $\mathcal{W}$ is a vocabulary containing relation symbols only. A relation

$$p \ \subseteq \ Dom(\mathfrak{A}) \times Dom(\mathfrak{B})$$

is said to be a *partial relativeness correspondence* if for any n-ary relation symbol $R \in \mathcal{W}$ and any $(a_1, b_1), \ldots, (a_n, b_n) \in p$,

$$(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \ \Leftrightarrow \ (b_1, \ldots, b_n) \in R^{\mathfrak{B}}.$$

Definition 4.10 (cf. Definition 4.2 in [2].). Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{W}$-models, where $\mathcal{W}$ contains relation symbols only. Let $A = Dom(\mathfrak{A})$ and $B = Dom(\mathfrak{B})$. We write $\mathfrak{A} \sim_n \mathfrak{B}$, where $n \in \mathbb{N}_{\geq 1}$, if there exists a sequence $(I_k)_{k \in \{0,1,..,n\}}$ of sets I_k of partial relativeness correspondences $p \subseteq A \times B$ such that the following conditions hold.

1. Every I_k is a non-empty set of partial relativeness correspondences.
2. For any $i \in \{1, 2, \ldots, n\}$, any $p \in I_i$ and any $a \in A$, there exists a $q \in I_{i-1}$ such that $p \subseteq q$ and $a \in Dom(q)$.
3. For any $i \in \{1, 2, \ldots, n\}$, any $p \in I_i$ and any $b \in B$, there exists a $q \in I_{i-1}$ such that $p \subseteq q$ and $b \in Ran(q)$.

Proposition 4.11 (A weakened version of Proposition 4.5 in [2]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{W}$-models, where $\mathcal{W}$ is a finite vocabulary containing relation symbols only. Then $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same* $\mathrm{FO}_{wo=}$ *$\mathcal{W}$-sentences of the quantifier rank $n \in \mathbb{N}_{\geq 1}$ if and only if $\mathfrak{A} \sim_n \mathfrak{B}$.*

We then prove the main result of the subsection.

Theorem 4.12. *Let $\mathcal{V}$ be an arbitrary non-empty vocabulary containing unary relation symbols only. We have* $\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}$ *with respect to the class of finite $\mathcal{V}$-models.*

Proof. Let C be the class of finite $\mathcal{V}$-models. By Lemma 2.1, we have $\mathrm{FO}_{wo=} \leq \mathrm{IF}_{wo=}$ over C.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be as defined in the beginning of the current subsection. To conclude the proof, it suffices to establish that for each finite set $U \subseteq \mathcal{V}$, the U-reducts $\mathfrak{M}' = \mathfrak{M} \upharpoonright U$ and $\mathfrak{N}' = \mathfrak{N} \upharpoonright U$ satisfy $\mathfrak{M}' \sim_n \mathfrak{N}'$ for all $n \in \mathbb{N}_{\geq 1}$.

Let $U \subseteq \mathcal{V}$ and assume without loss of generality that $P \in U$. Call $\mathfrak{M}' = \mathfrak{M} \upharpoonright U$ and $\mathfrak{N}' = \mathfrak{N} \upharpoonright U$. Let $n \in \mathbb{N}_{\geq 1}$ and define the sets I_k as follows.

1. $I_n \ = \ \{\emptyset\}$.

2. $I_{k-1} \;=\; \{\ (a,b)\ \mid\ \mathfrak{M}' \models P(a)\ \Leftrightarrow\ \mathfrak{N}' \models P(b)\ \}.$

We immediately observe that the back and forth system $(I_k)_{k\in\{0,1,\ldots,n\}}$ satisfies the required properties. Therefore $\mathfrak{M}' \sim_n \mathfrak{N}'$ for all $n \in \mathbb{N}_{\geq 1}$, whence $\mathfrak{M}'$ and $\mathfrak{N}'$ satisfy exactly the same U-sentences of $\mathrm{FO}_{wo=}$.

We conclude that the models $\mathfrak{M}$ and $\mathfrak{N}$ satisfy exactly the same $\mathcal{V}$-sentences of $\mathrm{FO}_{wo=}$. ■

5. Concluding Remarks

We have investigated the expressive power of IF logic in the equality-free setting. The results obtained have been established through a study of the logic $\mathrm{fESO}_{wo=}$. We have established that over $\mathcal{V}$-models, where $\mathcal{V}$ is a relational vocabulary containing a binary relation symbol, both logics $\mathrm{IF}_{wo=}$ and $\mathrm{fESO}_{wo=}$ are incomparable with FO with regard to expressive power. However, we have also established that when limiting attention to finite models with a non-empty unary relational vocabulary, we have

$$\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=} \leq \mathrm{fESO}_{wo=} < \mathrm{FO}.$$

We have also identified a model-transformation that preserves truth of $\mathrm{IF}_{wo=}$ sentences, and discussed a perspective from which IF logic is regarded as a specification language for games. Finally, perhaps the main contribution of the paper is the method of proof applying the notions of a spectrum and a stabilizer and leading to Theorem 4.7. The proof makes use of Dickson's Lemma.

A natural continuation to the investigations in this paper could involve dealing with some loose ends that were left undiscussed here. This could include, for example, a look at infinite models with unary vocabularies.

In this article we have concentrated on the role of equality in IF logic. It would be interesting to compare different logics in the IF family by identifying differences in the roles that different logical constructors – such as negation and identity – play in different logics. For example, the full systems of dependence logic [18] and IF^* coincide in expressive power on the level of sentences, both being able to exactly capture existential second-order logic, but the logics differ in expressivity when a suitable subset of the available logical constructors is uniformly removed from both systems. Trivially, if we remove identity from both logics and consider the class of models with the empty vocabulary, IF^* will have no formulae at all (unless we allow for primitive atoms such as $\top$ or $\bot$), but dependence logic will. Further investigations along such lines should lead to a deeper understanding of the strengths and weaknesses different systems have in relation to different applications.

In addition to technical investigations related to the familiar logics in the IF family, it would be interesting to have a look at team semantics (and game semantics) from a rather general point of view. For example, one could study *generalized atoms* defined atop team semantics. Such atoms would make assertions about teams in the spirit of dependence logic. To list one possibility, one could assert for example that the variable x obtains at least as many values in a team as y. Also, one could generalize the notion of a team and consider, for example, ordered teams. From the game theoretic point of view, investigations related to systems with further players in addition to Eloise and Abelard could be interesting and intriguing. The possibilities are endless indeed.

A. Appendix – Translating from $\mathrm{IF}_{wo=}$ into $\mathrm{fESO}_{wo=}$

In this appendix we show that sentences of $\mathrm{IF}_{wo=}$ translate into uniformly equivalent sentences of $\mathrm{fESO}_{wo=}$. Basic properties of IF logic can be rather different from what one might expect based on experience in ordinary predicate logic, and the related issues have proved tricky. (For more information on this matter, see [1] for example.) Therefore we feel that it makes sense to give a relatively detailed account of the translation from $\mathrm{IF}_{wo=}$ into $\mathrm{fESO}_{wo=}$.

Let φ be a first-order formula and let t be a first-order term. By $\varphi(x/t)$ we denote the formula obtained from φ by replacing exactly all the free occurrences (if any) of the variable symbol x by the term t. The following lemma describes a simple substitution property of first-order logic.

Lemma A.1. *Let φ be a first-order formula. Let $y_1, \ldots, y_m$ be a collection of $m \in \mathbb{N}$ distinct variables such that no quantifier Qy_i occurs in φ. Let x be a variable distinct from $y_1, \ldots, y_m$. Let $\mathfrak{A}$ be a model with domain A. Let X be a set of variables containing at least all the free variables in φ, and also the variables $y_1, \ldots, y_m$ and x. (Note that any of the variables $x, y_1, \ldots, y_m$ may occur free in φ, but does not have to.) Let $g : X \longrightarrow A$ be a variable assignment function. Let f be a function symbol with the arity m that occurs neither in φ nor in the vocabulary of $\mathfrak{A}$. Let $\mathfrak{A}^*$ be an expansion of $\mathfrak{A}$ with an m-ary function $f^{\mathfrak{A}^*}$ such that $f^{\mathfrak{A}^*}\big(g(y_1), \ldots, g(y_m)\big) = g(x)$. Then we have $\mathfrak{A}, g \models \varphi$ if and only if $\mathfrak{A}^*, g \models \varphi\big(x/f(y_1, \ldots, y_m)\big)$.*

Proof. Straightforward. ■

Note that in the above lemma, if $m = 0$, then f is a constant symbol and $f^{\mathfrak{A}^*}$ is the element $g(x) \in A$.

A slash-connective-free IF* sentence is *regular* iff there are no nested occurrences of quantifiers quantifying the same variable symbol. In other words, a quantifier Qx/X never occurs in the scope of another quantifier $Q'x/Y$. Let φ be a slash-connective-free IF* sentence. We say that φ is in *processed form* iff the following conditions are satisfied.

1. The sentence φ is in prenex normal form, i.e., it is of the type $\overline{Q}\psi$, where $\overline{Q}$ is a vector of quantifiers and ψ is a quantifier-free formula.
2. The sentence φ is regular.
3. Each universal quantifier of φ has an empty slash set. In other words, if a quantifier of the type $\forall x/X$ occurs in φ, then X denotes the empty set.

LEMMA A.2. *Let φ be a sentence of slash-connective-free* IF* *in prenex normal form and φ' the sentence obtained from φ by making all the slash sets of the universal quantifiers of φ empty. Then for all models $\mathfrak{A}$ we have $\mathfrak{A} \models^+ \varphi$ if and only if $\mathfrak{A} \models^+ \varphi'$.*

PROOF. The satisfaction clause for $\models^+$ for universal quantification is as follows (see Definition 4.2 of [1]).

$$\mathfrak{A}, V \models^+ \forall x/X\,\psi \quad \Leftrightarrow \quad \mathfrak{A}, V_{x:A} \models^+ \psi$$

The satisfaction clause for $\models^+$ does not depend on X. Therefore it is immediate for the prenex normal form sentences φ and φ' that for all models $\mathfrak{A}$, we have $\mathfrak{A} \models^+ \varphi$ if and only if $\mathfrak{A} \models^+ \varphi'$. ■

Let φ be a slash-connective-free IF* sentence in processed form. Let $\mathfrak{A}$ be a model with domain A. Let $\varphi = \overline{Q}\psi$, where $\overline{Q}$ is a vector of at least $m \in \mathbb{N}$ quantifiers and ψ is a quantifier-free formula. Let X be a set of variable symbols, and let $\mathcal{T}$ be the set of all teams that consist of variable assignments mapping a subset of X to A. Assume the function

$$S : \{0, 1, 2, \ldots, m\} \longrightarrow \mathcal{T}$$

is a sequence of teams such that the following conditions are satisfied.

1. $S(0) = \{\emptyset\}$.
2. If $i < m$ and $S(i) = V$, then the following conditions hold.

(a) If the $(i+1)$-th quantifier[9] in $\overline{Q}$ is $\exists x/Y$, then $S(i+1) = V_{x:f}$, where $f : V \longrightarrow A$ is a Y-independent function.

(b) If the $(i+1)$-th quantifier of $\overline{Q}$ is $\forall x/Y$, then $S(i+1) = V_{x:A}$.

We call such a sequence S a *sequence of teams for* $(\mathfrak{A}, \varphi)$. If $\overline{Q}'$ is the suffix of $\overline{Q}$ obtained by removing the first m quantifiers of $\overline{Q}$ and if $\mathfrak{A}, S(m) \models^+ \overline{Q}'\psi$, then we call S a *verifying sequence for* $(\mathfrak{A}, \varphi)$. We have $\mathfrak{A} \models^+ \varphi$ if and only if there exists a verifying sequence for $(\mathfrak{A}, \varphi)$.

Let $g : X \longrightarrow A$ be a finite variable assignment function mapping some set X of first-order variable symbols to some set A. Let $x \in X$. We let g_{-x} denote the variable assignment function $g \setminus \{\ (x, g(x))\ \}$. Let V be a team of assignments with the domain X. We let $V_{-x} = \{\ g_{-x} \mid g \in V\ \}$.

We then define a certain *Skolemization operation* for sentences of slash-connective-free IF* logic in processed form. Let φ be such a sentence of the form $\overline{Q}\ \exists x/X\ \overline{Q}'\ \psi$, where

1. $\overline{Q}$ is a (possibly empty) string $Q_1 y_1/Y_1\ \ldots\ Q_k y_k/Y_k$ consisting of $k \in \mathbb{N}$ quantifiers,
2. $\overline{Q}'$ is a possibly empty string consisting of $l \in \mathbb{N}$ *universal* quantifiers,
3. ψ is a quantifier-free formula.

Let $y_{i_1}, \ldots, y_{i_m}$ enumerate the variable symbols in $\{y_1, \ldots, y_k\} \setminus X$ in the order they occur in $\overline{Q}$.[10] Let $\overline{y}$ denote the tuple $(y_{i_1}, \ldots, y_{i_m})$. Let f be a function symbol that does not occur in φ. We define

$$Sk_f(\varphi)\ =\ \overline{Q}\ \overline{Q}'\psi(x/f(\overline{y})).$$

The Skolemization operation Sk_f removes the innermost existential quantifier $\exists x/X$ of the sentence φ and replaces each occurrence of the symbol x in ψ by the term $f(\overline{y})$. For example

$$Sk_f\big(\ \forall x \exists y \exists z/\{x\} R(x,z)\ \big)\ =\ \forall x \exists y R(x, f(y))$$

and

$$Sk_g\big(\ \forall x \exists y R(x, f(y))\ \big)\ =\ \forall x R(x, f(g(x))).$$

[9]The j-th quantifier in $\overline{Q}$ means the j-th quantifier from the left in $\overline{Q}$. The leftmost quantifier in $\overline{Q}$ is the *first* (as opposed to zeroeth) quantifier in $\overline{Q}$.

[10]A variable in $\{y_1, \ldots, y_k\} \setminus X$ may have occurrences in slash sets too. Only the first occurrence of a variable counts, and the first occurrence cannot be an occurrence in a slash set.

Let φ be a slash-connective-free IF* sentence in processed form. We let $Sk(\varphi)$ denote the sentence obtained from φ by eliminating all the existential quantifiers of φ by successively applying operations of the type Sk_f, where f is always a fresh function symbol. The expression $Sk(\varphi)$ specifies neither which function symbols were used in the elimination process nor the order in which they were used, but such details will be clear from the context.

Let $\mathfrak{A}$ be a model with domain A and $f^* : A^k \longrightarrow A$ a function. By $(\mathfrak{A}, f^*)$ we denote the expansion of the model $\mathfrak{A}$ by the function f^*. We always assume that the corresponding function symbol f does not occur in the vocabulary of $\mathfrak{A}$. We may also expand $\mathfrak{A}$ by multiple functions $f_1^*, \ldots, f_m^*$. The resulting model is then denoted by $(\mathfrak{A}, f_1^*, \ldots, f_m^*)$.

LEMMA A.3. *Let $\mathfrak{A}$ be a model and let φ a sentence of slash-connective-free* IF* *in processed form. Assume the quantifier prefix of φ contains an existential quantifier. Let f be a function symbol that occurs neither in φ nor in the vocabulary of $\mathfrak{A}$. We have $\mathfrak{A} \models^+ \varphi$ if and only if there exists a function f^* such that $(\mathfrak{A}, f^*) \models^+ Sk_f(\varphi)$.*

PROOF. As above, let φ be of the form $\overline{Q}\ \exists x/X\ \overline{Q}'\ \psi$, where

1. $\overline{Q}$ is a possibly empty string $Q_1 y_1/Y_1\ \ldots\ Q_k y_k/Y_k$ consisting of $k \in \mathbb{N}$ quantifiers,
2. $\overline{Q}'$ is a possibly empty string consisting of $l \in \mathbb{N}$ *universal* quantifiers,
3. ψ is a quantifier-free formula.

Let $y_{i_1}, \ldots, y_{i_m}$ enumerate the variable symbols in $\{y_1, \ldots, y_k\} \setminus X$ in the order they occur in $\overline{Q}$. Let $\overline{y}$ denote the tuple $(y_{i_1}, \ldots, y_{i_m})$.

Assume first that $\mathfrak{A} \models^+ \varphi$. Let S be a verifying sequence for $(\mathfrak{A}, \varphi)$ such that $\{0, 1, \ldots, k+1\}$ is the domain of S. Thus we have $\mathfrak{A}, S(k) \models^+ \exists x/X \overline{Q}' \psi$ and $\mathfrak{A}, S(k+1) \models^+ \overline{Q}' \psi$. Furthermore, there exists an X-independent function $h : S(k) \longrightarrow A$ such that $S(k+1) = S(k)_{x:h}$. We define a function $f^* : A^m \longrightarrow A$ as follows.

1. In the case $(a_1, \ldots, a_m) = (s(y_{i_1}), \ldots, s(y_{i_m}))$ for some $s \in S(k)$, we let $f^*(a_1, \ldots, a_m) = h(s)$.
2. If there is no $s \in S(k)$ such that $(a_1, \ldots, a_m) = (s(y_{i_1}), \ldots, s(y_{i_m}))$, then we define the value of $f^*(a_1, \ldots, a_m)$ arbitrarily.

Notice that the function f^* is well defined, as h is an X-independent function. Note also that if $m = 0$, then f is a constant symbol and f^* an element of A.

Call $S^* = S \restriction \{0, 1, \ldots, k\}$. We have $S^*(i) = S(i)$ for all $i \leq k$, and therefore S^* is a sequence for $\big((\mathfrak{A}, f^*), Sk_f(\varphi)\big)$. To prove that S^* is a *verifying* sequence for $\big((\mathfrak{A}, f^*), Sk_f(\varphi)\big)$, it suffices to show that

$$(\mathfrak{A}, f^*), S^*(k) \models^+ \overline{Q}'\psi(x/f(\overline{y})).$$

We know that $\mathfrak{A}, S(k+1) \models^+ \overline{Q}'\psi$. The formula $\overline{Q}'\psi$ is a first-order formula. Thus, by Lemma 2.1, we have $\mathfrak{A}, g \models \overline{Q}'\psi$ for all $g \in S(k+1)$. Notice that for all $g \in S(k+1)$, we have

$$f^*(g(y_{i_1}), \ldots, g(y_{i_m})) = f^*(g_{-x}(y_{i_1}), \ldots, g_{-x}(y_{i_m})) = h(g_{-x}) = g(x).$$

Hence, as $\mathfrak{A}, g \models \overline{Q}'\psi$ for all $g \in S(k+1)$, we conclude by Lemma A.1 that

$$(\mathfrak{A}, f^*), g \models \overline{Q}'\psi(x/f(\overline{y}))$$

for all $g \in S(k+1)$. Notice that the variable symbol x does not occur in the formula $\overline{Q}'\psi(x/f(\overline{y}))$. Thus we have

$$(\mathfrak{A}, f^*), g' \models \overline{Q}'\psi(x/f(\overline{y}))$$

for all $g' \in S(k+1)_{-x} = S^*(k)$. Therefore, by Lemma 2.1, we have

$$(\mathfrak{A}, f^*), S^*(k) \models^+ \overline{Q}'\psi(x/f(\overline{y})).$$

Thus S^* is a verifying sequence for $\big((\mathfrak{A}, f^*), Sk_f(\varphi)\big)$, and hence $(\mathfrak{A}, f^*) \models^+ Sk_f(\varphi)$.

Assume then that there exists a function $f^* : A^m \longrightarrow A$ such that $(\mathfrak{A}, f^*) \models^+ Sk_f(\varphi)$. Therefore there of course exists a verifying sequence T for $\big((\mathfrak{A}, f^*), Sk_f(\varphi)\big)$ such that $\{0, 1, \ldots, k\}$ is the domain of T. We have

$$(\mathfrak{A}, f^*), T(k) \models^+ \overline{Q}'\psi(x/f(\overline{y})).$$

Define the function $h : T(k) \longrightarrow A$ such that $h(s) = f^*(s(y_{i_1}), \ldots, s(y_{i_m}))$ for all assignments $s \in T(k)$. As none of the variables $y_{i_1}, \ldots, y_{i_m}$ occur in X, the function h is an X-independent function. Define a sequence T^* with the domain $\{0, 1, \ldots, k+1\}$ as follows.

1. For $i \leq k$, let $T^*(i) = T(i)$.
2. Let $T^*(k+1) = T(k)_{x:h}$.

Notice that as h is an X-independent function, the sequence T^* is a sequence for $(\mathfrak{A}, \varphi)$. We will prove that T^* is a verifying sequence for $(\mathfrak{A}, \varphi)$ by showing that $\mathfrak{A}, T^*(k+1) \models^+ \overline{Q}'\psi$.

We have $(\mathfrak{A}, f^*), T(k) \models^+ \overline{Q}'\psi(x/f(\overline{y}))$. As the formula $\overline{Q}'\psi(x/f(\overline{y}))$ is a first-order formula, we have

$$(\mathfrak{A}, f^*), g \models \overline{Q}'\psi(x/f(\overline{y}))$$

for all $g \in T(k)$ by Lemma 2.1. The variable symbol x does not occur in the formula $\overline{Q}'\psi(x/f(\overline{y}))$, so we have

$$(\mathfrak{A}, f^*), g' \models \overline{Q}'\psi(x/f(\overline{y}))$$

for all $g' \in T(k)_{x:h} = T^*(k+1)$. Therefore, as

$$g'(x) = h(g'_{-x}) = f^*(g'_{-x}(y_{i_1}), \ldots, g'_{-x}(y_{i_m})) = f^*(g'(y_{i_1}), \ldots, g'(y_{i_m}))$$

for all $g' \in T^*(k+1)$, we conclude by Lemma A.1 that

$$\mathfrak{A}, g' \models \overline{Q}'\psi$$

for all $g' \in T^*(k+1)$. Thus $\mathfrak{A}, T^*(k+1) \models^+ \overline{Q}'\psi$ by Lemma 2.1. Hence T^* is a verifying sequence for $(\mathfrak{A}, \varphi)$, and therefore $\mathfrak{A} \models^+ \varphi$, as required. ■

THEOREM A.4. *Every sentence of* $\mathrm{IF}_{wo=}$ *translates into a uniformly equivalent sentence of* $\mathrm{fESO}_{wo=}$.

PROOF. Let χ be a sentence of $\mathrm{IF}_{wo=}$. By Theorems 10.1 and 10.2 of [1], any sentence of IF^* can be transformed into a uniformly equivalent sentence in prenex normal form without introducing equality or slashed connectives. By Theorem 9.3 of [1], any sentence of IF^* in can be turned into a uniformly equivalent regular sentence, again without introducing equality or slashed connectives, and furthermore, if the original sentence is in prenex normal form, then so is the result of the transformation. By Lemma A.2, every $\mathrm{IF}_{wo=}$ sentence in prenex normal form is uniformly equivalent to the sentence obtained from the original sentence by making all the slash sets of universal quantifiers empty. Thus there exists an $\mathrm{IF}_{wo=}$ sentence φ that is uniformly equivalent to χ and in processed form. If φ does not contain existential quantifiers, we are done with the proof. Therefore we assume that φ contains $m \in \mathbb{N}_{\geq 1}$ existential quantifiers.

By Lemma A.3, we have the following chain of equivalences.

$$\begin{aligned}\mathfrak{A} \models^+ \varphi &\Leftrightarrow \exists f_1^* \big((\mathfrak{A}, f_1^*) \models^+ Sk_{f_1}(\varphi)\big)\\ &\Leftrightarrow \exists f_1^* \exists f_2^* \big((\mathfrak{A}, f_1^*, f_2^*) \models^+ Sk_{f_2}(Sk_{f_1}(\varphi))\big)\\ &\ \vdots\\ &\Leftrightarrow \exists f_1^* \ldots \exists f_m^* \big((\mathfrak{A}, f_1^*, \ldots, f_m^*) \models^+ Sk(\varphi)\big)\end{aligned}$$

The formula $Sk(\varphi)$ is a first-order formula. Thus we have

$$(\mathfrak{A}, f_1^*, \ldots, f_m^*) \models^+ Sk(\varphi) \ \Leftrightarrow\ (\mathfrak{A}, f_1^*, \ldots, f_m^*) \models Sk(\varphi)$$

by Lemma 2.1. (Notice the use two different turnstiles $\models^+$ and $\models$.) Therefore we conclude that

$$\mathfrak{A} \models^+ \varphi \ \Leftrightarrow\ \exists f_1^* \ldots \exists f_m^* \big((\mathfrak{A}, f_1^*, \ldots, f_m^*) \models Sk(\varphi)\big).$$

Hence

$$\mathfrak{A} \models^+ \varphi \ \Leftrightarrow\ \mathfrak{A} \models \exists f_1 \ldots \exists f_m Sk(\varphi).$$

The sentence $\exists f_1 \ldots \exists f_m Sk(\varphi)$ is a sentence of fESO$_{wo=}$, and it is uniformly equivalent to the original sentence χ. ∎

Acknowledgements. The author participates in the European Science Foundation project Logic for Interaction (LINT). The research leading to the current article was supported by MALJA Graduate School in Mathematical Logic (Academy of Finland grant 118815) and Tampere Graduate School in Information Science and Engineering TISE (Academy of Finland grant 129892). The author wishes to thank Lauri Hella, Kerkko Luosto, Allen Mann, Jonni Virtema and the referees for valuable feedback.

References

[1] Caicedo, X., F. Dechesne, and T. M. V. Janssen, Equivalence and Quantifier Rules for Logic with Imperfect Information, *Logic Journal of the IGPL* 17(1):91–129, 2009.

[2] Casanovas, E., P. Dellunde, and R. Jansana, On Elementary Equivalence for Equality-free Logic, *Notre Dame Journal of Formal Logic* 37(3):506–522, 1996.

[3] Dickson, L. E., Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors, *American Journal of Mathematics* 35(4):413–422, 1913.

[4] Galliani, P., *Game Values and Equilibria for Undetermined Sentences of Dependence Logic*, M.Sc. thesis, ILLC, 2008.

[5] Henkin, L., Some Remarks on Infinitely Long Formulas, in *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 29 September 1959*, Pergamon Press, 1961, pp. 167–183.

[6] Higman, G., Ordering by Divisibility in Abstract Algebras, in *Proceedings of the London Mathematical Society 1952*, 2:326–336, 1952.

[7] Hintikka, J., Quantifiers vs. Quantification Theory, *Dialectica* 27:329–358, 1973.

[8] Hintikka, J., and G. Sandu, Informational Independence as a Semantical Phenomenon. In J. E. Fenstad et al., (eds.), *Logic, Methodology and Philosophy of Science VIII*, volume 126 of Studies in Logic and the Foundations of Mathematics, North-Holland, 1989, pp. 571–589.

[9] Kontinen, Jarmo, *Coherence and Complexity in Fragments of Dependence Logic*. Ph.D. thesis, ILLC, 2010.

[10] Kontinen, Juha, and V. Nurmi Team Logic and Second-Order Logic, In *Proceedings of WoLLIC 2009*, Springer Lecture Notes in Computer Science 5514, 2009.

[11] Kontinen, Juha, and J. Väänänen, On Definability in Dependence Logic, *Journal of Logic, Language and Information* 18(3):317–332, 2009.

[12] Libkin, L., *Elements of Finite Model Theory*, Springer, 2004.

[13] Malicki, M., and A. Rutkowski, On Operations and Linear Extensions of Well Partially Ordered Sets, *Order* 21(1):7–17, 2004.

[14] Mann, A. L., Independence-Friendly Cylindric Set Algebras, *Logic Journal of the IGPL* 17(6):719–754, 2009.

[15] Mann, A. L., G. Sandu, and M. Sevenster, *Independence-Friendly Logic: A Game-Theoretic Approach*, London Mathematical Society Lecture Note Series (No. 386), Cambridge University Press, 2011.

[16] Tulenheimo, T., and M. Sevenster, Approaches to Independence Friendly Modal Logic, in J. van Benthem, D. Gabbay, and B. Löwe, (eds.), *Interactive Logic, Selected Papers from the 7th Augustus de Morgan Workshop*, London. Texts in Logic and Games 1, Amsterdam: Amsterdam University Press, 2007, pp. 247–280.

[17] Tulenheimo, T., *Independence Friendly Modal Logic*. Ph.D. thesis, Department of Philosophy, University of Helsinki, 2004.

[18] Väänänen, J., *Dependence Logic*, Vol. 70 of *London Mathematical Society Student Texts*, Cambridge University Press, 2007.

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