Notre Dame Journal of Formal Logic
Volume 40, Number 4, Fall 1999

Nonconstructive Properties of Well-Ordered T_2 Topological Spaces

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Abstract We show that none of the following statements is provable in Zermelo-Fraenkel set theory (ZF) answering the corresponding open questions from Brunner in "The axiom of choice in topology":

(i) For every T_2 topological space (X, T) if X is well-ordered, then X has a well-ordered base,

(ii) For every T_2 topological space (X, T), if X is well-ordered, then each open cover of X has a well-ordered open refinement,

(iii) For every T_2 topological space (X, T), if X has a well-ordered dense subset, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $\{x\} = \cap f(\{x\} \times W)$ for each $x \in X$.

1 Introduction Let (X, T) be a T_2 topological space and let $\mathcal{B}$ be a base for X. Clearly,

$$|T| \leq |2^X| \tag{1}$$

and

$$|X| \leq |2^{\mathcal{B}}|. \tag{2}$$

(The map $f : X \to \mathcal{P}(\mathcal{B})(=$ the powerset of $\mathcal{B})$, $f(x) = \{B \in \mathcal{B} : x \in B\}$ is obviously $1 : 1$). We then have the following proposition.

Proposition 1.1 *In Fraenkel-Mostowski permutation models, a T_2 topological space (X, T) is well-ordered if and only if X has a well-ordered base.*

Proof: From (1) and the fact that in every permutation model Form 91 in Howard and Rubin [4], PW : *The powerset of a well-ordered set can be well-ordered* holds, we have that if X is well-ordered, then T is well-ordered. Similarly from (2) it follows that if X has a well-ordered base, then X is well-ordered. $\square$

Received February 1, 2000; revised November 1, 2000

In Cohen models however Proposition 1.1 may fail. Indeed, in the basic Cohen model, model $\mathcal{M}1$ of [4], the real line $\mathbb{R}$ with the standard topology has a countable base, but $\mathbb{R}$ is not well-ordered. There remains the question:

> *If (X, T) is a well-ordered T_2 topological space, then does X have a well-ordered base?*

Motivated by this question, Brunner [1] defined the following statements:

(A1) Form 148 in [4]: For every T_2 topological space (X, T), if X is well-ordered, then X has a well-ordered base.

(A2) For every T_2 topological space (X, T), if X is well-ordered, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $f(\{x\} \times W)$ is a neighborhood base at x for each $x \in X$.

(A3) For every T_2 topological space (X, T), if X is well-ordered, then each open cover of X has a well-ordered open refinement.

(A4) For every T_2 topological space (X, T), if X is well-ordered,then X satisfies $(*)$: if $O \subseteq T$ covers X, there is a mapping $f : X \to T$ such that $x \in f(x)$ and $f[X]$ refines O.

(A5) For every T_2 topological space (X, T), if X is well-ordered, then $(*)$ is a hereditary property of X.

(A6) For every T_2 topological space (X, T), if X has a well-ordered dense subset, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $\{x\} = \cap f(\{x\} \times W)$ for each $x \in X$.

Clearly, each of the above statements is a theorem of ZFC (ZF with the axiom of choice AC, Form 1 in [4]). Brunner [1] asks whether these statements are provable in ZF minus the axiom of regularity (ZF^0) and Howard and Rubin [4] ask whether 148 implies AC. The aim of this paper is to show that none of (Ai), i = 1, 2, 3, 4, 5, 6, is a theorem of ZF and that 148 does not imply AC in ZF^0. In particular, we show that

1. (A1), (A2), and (A6) are equivalent to AC in ZF,
2. (A3), (A4), and (A5) imply Form 13 in [4]: *Every Dedekind finite subset (i.e., it has no countably infinite subset) of $\mathbb{R}$ is finite*.

Before setting out with proofs let us make a straightforward remark on the interrelation between the statements (A1) up to (A5).

(i) (A1) $\Longleftrightarrow$ (A2).

(ii) (A1) $\Longrightarrow$ (A3).

(iii) (A3) $\Longleftrightarrow$ (A4) $\Longleftrightarrow$ (A5).

For any undefined topological notion the reader is referred to Willard [9].

2 Results We begin by observing the following.

Theorem 2.1 *(A1) does not imply AC in ZF^0.*

Proof: Let $\mathcal{N}$ be the basic Fraenkel model (model $\mathcal{N}$ 1 in [4]). By Proposition 1.1 we have that (A1) holds in $\mathcal{N}$. On the other hand, AC fails in $\mathcal{N}$ (see [4]) meaning that (A1) does not imply AC in ZF^0 as required. □

However in ZF, (A1) is equivalent to AC as Theorem 2.3 clarifies. In particular, we show that both (A1) and (A6) are equivalent to the set-theoretic principle PW (see the introduction) which in ZF is known to be equivalent to AC (see Felgner and Jech [3]). We recall first the notion of an independent family of sets.

Definition 2.2 Let $\theta \geq \omega$ be an ordinal number. A family $\mathcal{A} \subseteq \mathcal{P}(\theta)$ is said to be *independent* if and only if for any finite collection $A_1, \ldots, A_m, B_1, \ldots, B_n$ of distinct elements of $\mathcal{A}$, $|A_1 \cap \cdots \cap A_m \cap (E \backslash B_1) \cap \cdots \cap (E \backslash B_n)| = |\theta|$.

Theorem 2.3 *In ZF the following statements are equivalent:*

(i) *PW,*
(ii) *(A1),*
(iii) *(A6).*

Proof: (i) $\rightarrow$ (ii) This is straightforward.

(ii) $\rightarrow$ (i) Fix an ordinal number $\kappa \geq \omega$ and let $\mathcal{A} = \{a_i : i \in 2^\kappa\} \subseteq \mathcal{P}(\kappa)$ be an independent family (see Kunen [6], Exercise (A6), p. 288). The existence of such a family can be proved in ZF^0. We show that 2^κ is well-ordered.

For each $i \in 2^\kappa$, let $G_i = \{x \in \mathcal{P}(\kappa) : |x \triangle a_i| < \omega\}$ where $\triangle$ denotes the operation of symmetric difference. Since for all $i, j \in 2^\kappa$, $i \neq j$, $a_i \triangle a_j$ is infinite, we have that $G_i \cap G_j = \varnothing$. Put $G = \cup\{G_i : i \in 2^\kappa\}$. For each $x \in [\kappa]^{<\omega} (= \{x \in \mathcal{P}(\kappa) : |x| < \omega\})$, $i \in 2^\kappa$ and $g \in G_i$, put

$$B(x, i, g) = \{y \in [\kappa]^{<\omega} : x \subseteq y \text{ and } y \cap g = \varnothing\}. \tag{3}$$

Claim 2.4 *The family $\{B(x, i, g) : x \in [\kappa]^{<\omega}, i \in 2^\kappa, g \in G_i\}$ is a cover of $[\kappa]^{<\omega}$.*

Proof of Claim 2.4: Fix $x \in [\kappa]^{<\omega}$ and let $i \in 2^\kappa$. Then $a_i \backslash x \in G_i$ and $x \in B(x, i, a_i \backslash x)$ finishing the proof of the Claim 2.4. □

Let $\mathcal{B} = \{B(x, g) : x \in [\kappa]^{<\omega}, g = \cup Q, Q \in [G]^{<\omega}\}$ where $B(x, g) = \{y \in [\kappa]^{<\omega} : x \subseteq y \text{ and } y \cap g = \varnothing\}$.

Claim 2.5 *$\mathcal{B}$ is a base for a T_2 topology $T_{\mathcal{B}}$ on $[\kappa]^{<\omega}$.*

Proof of Claim 2.5: By Claim 2.4 we have that $\mathcal{B}$ is a cover of $[\kappa]^{<\omega}$. On the other hand, if $x \in B(x_1, g_1) \cap B(x_2, g_2)$, then since $x \cap g_1 = x \cap g_2 = \varnothing$, we have that $B(x, g_1 \cup g_2) \in \mathcal{B}$ and clearly, $x \in B(x, g_1 \cup g_2) \subseteq B(x_1, g_1) \cap B(x_2, g_2)$. Therefore, $\mathcal{B}$ is a base. We show now that $\mathcal{B}$ generates a T_2 topology on $[\kappa]^{<\omega}$. Fix $x, y \in [\kappa]^{<\omega}$ with $x \neq y$ and let $g \in G$ be such that $(x \cup y) \cap g = \varnothing$ (take, for example, any $i \in 2^\kappa$ and put $g = a_i \backslash (x \cup y)$). Then $V_x = B(x, g \cup (y \backslash x))$ and $V_y = B(y, g \cup (x \backslash y))$ are disjoint neighborhoods of x and y, respectively. Assume otherwise and let $z \in V_x \cap V_y$. Then $x \subseteq z$, $z \cap (g \cup (y \backslash x)) = \varnothing$, and $y \subseteq z$, $z \cap (g \cup (x \backslash y)) = \varnothing$. Thus,

$$y \subseteq z \text{ and } y \cap (y \backslash x) = \varnothing \tag{4}$$

and

$$x \subseteq z \text{ and } x \cap (x \backslash y) = \varnothing \tag{5}$$

By (4) we have that $y \subseteq x$ and by (5), $x \subseteq y$. Therefore, $x = y$, a contradiction. This completes the proof of Claim 2.5. □

Since $([\kappa]^{<\omega}, T_B)$ is a well-ordered T_2 space, let by (A1) $\mathcal{W} = \{W_j : j \in \aleph\}$ be a well-ordered base. Consider now the open cover $\mathcal{U} = \{B(\varnothing, i, g) : i \in 2^\kappa, g \in G_i\}$ where $B(x, i, g)$ is given by (3). Then $\mathcal{V} = \{V \in \mathcal{W} : V \subseteq U$ for some $U \in \mathcal{U}\}$ is clearly a well-ordered open refinement of $\mathcal{U}$. For every $V \in \mathcal{V}$, let

$$H_V = \{i \in 2^\kappa : \exists g \in G_i, V \subseteq B(\varnothing, i, g)\}. \tag{6}$$

Claim 2.6 *For each $V \in \mathcal{V}$, H_V is finite.*

Proof of Claim 2.6: Assume the contrary and let $V_0 \in \mathcal{V}$ be such that H_{V_0} is infinite. As each G_i can be well-ordered uniformly ($\{a_i \triangle x : x \in [\kappa]^{<\omega}\}$ is a uniform well-ordering of G_i) we may define an infinite set $\{g_i \in G_i : i \in H_{V_0}\}$ such that $V_0 \subseteq B(\varnothing, i, g_i)$ for all $i \in H_{V_0}$. Fix $B(x_0, g)$ a basic open set contained in V_0. Then $g = g_{i_1} \cup g_{i_2} \cup \cdots \cup g_{i_n}$ for some $n \in \omega$ and $g_{i_j} \in G_{i_j}$, $j \leq n$. Since $B(x_0, g) \subseteq \cap\{B(\varnothing, i, g_i) : i \in H_{V_0}\}$, we have that $(\cup\{g_i : i \in H_{V_0}\})\backslash g = \varnothing$ (otherwise fix $i \in H_{V_0}$ and $y \in g_i \backslash g$, then $x_0 \cup \{y\} \in B(x_0, g) \backslash B(\varnothing, i, g_i)$, a contradiction). Since $|g_i \triangle a_i| < \omega$, it follows immediately that for all $i \in H_{V_0}$, $F_i = a_i \backslash (a_{i_1} \cup a_{i_2} \cup \cdots \cup a_{i_n})$ is finite. This contradicts the fact that $\mathcal{A}$ is an independent family and completes the proof of Claim 2.6. □

Since $\mathcal{A}$ is an independent family, $\mathcal{U}$ has no finite subcover. Furthermore, as $\mathcal{W}$ is a base it is clear that $2^\kappa = \cup\{H_V : V \in \mathcal{V}\}$ and since κ is well-ordered, 2^κ is linearly ordered (e.g., lexicographically). Thus, each H_V is well-ordered and consequently 2^κ is well-ordered finishing the proof of (ii) → (i).

(i) → (iii) Since in ZF, AC $\Longleftrightarrow$ PW, this is straightforward.

(iii) → (i) Fix an ordinal number κ. Since $|\kappa| < |2^\kappa|$ we may assume without loss of generality that $\kappa \subseteq 2^\kappa$. Let $\mathcal{W} = \{W_f : f \in 2^\kappa \backslash \kappa\}$ be an independent family of subsets of κ. Define a topology T on $X = 2^\kappa$ by requiring: All points in κ to be isolated whereas neighborhoods of $f \in 2^\kappa \backslash \kappa$ are all sets of the form

$$V_f = \{f\} \cup (W_f \backslash (\cup Q \cup A)), \; Q \in [\mathcal{W} \backslash \{W_f\}]^{<\omega}, \; A \in [W_f]^{<\omega}.$$

(X, T) is a T_2 space. Indeed, let $x, y \in X$, $x \neq y$. We consider the following cases.

Case 1: $x, y \in \kappa$. Then $\{x\}, \{y\}$ are the required disjoint neighborhoods of x and y, respectively.

Case 2: $x \in \kappa, y \in 2^\kappa \backslash \kappa$. Then $\{x\}, \{y\} \cup (W_y \backslash \{x\})$ are the required disjoint neighborhoods of x and y, respectively.

Case 3: $x, y \in 2^\kappa \backslash \kappa$. Then $\{x\} \cup (W_x \backslash W_y), \{y\} \cup (W_y \backslash W_x)$ are the required disjoint neighborhoods of x and y, respectively.

Thus, (X, T) is a T_2 space having the well-ordered set κ as a dense subset. Adjoin an extra point ∞ to X and extend the topology T by declaring neighborhoods of ∞ to be all supersets of $\{\infty\}$ missing finitely many sets $\{f\} \cup W_f$, $f \in 2^\kappa \backslash \kappa$. Thus, each neighborhood of ∞ misses only finitely many elements of $2^\kappa \backslash \kappa$. Clearly $Y = X \cup \{\infty\}$ with the extended topology T^∞ is a T_2 space having κ as a dense subset.

Let, by (A6), $\{Z_i : i \in \aleph\}$ be a well-ordered family of neighborhoods of $\{\infty\}$ such that $\{\infty\} = \cap\{Z_i : i \in \aleph\}$. Then $2^\kappa \backslash \kappa = \cup\{(2^\kappa \backslash \kappa) \backslash Z_i : i \in \aleph\}$ and by the above each set $(2^\kappa \backslash \kappa) \backslash Z_i$ is finite. As 2^κ is linearly ordered, $(2^\kappa \backslash \kappa) \backslash Z_i$ is well-ordered. Thus, $2^\kappa \backslash \kappa$ is well-ordered finishing the proof of (iii) $\to$ (i) and of the theorem. □

Remark 2.7 The statement "If (X, T) is a T_2 space with a well-ordered dense subset, then each open cover of X has a well-ordered open refinement" has also been considered in [1] where it is shown not to be a theorem of ZF; in the basic Cohen model, the Moore plane (see Steen and Seebach [8], Example 82) is a separable T_2 space having an open cover with no well-ordered open refinement. Via the latter proof, Brunner implicitly suggests that the above statement implies a well-known weak choice principle, namely, the axiom of choice for families of nonempty subsets of $\mathbb{R}$, AC($\mathbb{R}$), and Form [79 A] in [4]. However, following the proof of Theorem 2.3 we deduce that the above statement is equivalent to AC in ZF. Indeed, let (X, T) be the T_2 space of Theorem 2.3 and let $O = \{\{f\} \cup W_f : f \in 2^\kappa \backslash \kappa\} \cup \{\{x\} : x \in \kappa\}$. Clearly, O is an open cover of X. Let $V = \{V_i : i \in \aleph\}$ be a well-ordered open refinement of O. For each $f \in 2^\kappa \backslash \kappa$, let i_f be the least $i \in \aleph$ such that $f \in V_i$. Then $V_{i_f} \subseteq \{g\} \cup W_g$ for some $g \in 2^\kappa \backslash \kappa$. Necessarily, $g = f$ and consequently the function $f \mapsto V_{i_f}$ is $1:1$ meaning that 2^κ is well-ordered.

Brunner [1] $(*)$ of a T_2 space (see the Introduction) as topologically the most interesting (among the other fifteen properties of T_2 spaces he considers in [1]) because as he points out it is both a weakening of metacompact and well-ordered local weight. In Theorem 2.8 we show that the statement (A3) implies Form 13. Since (13) fails in the basic Cohen model for the set of the countably many added Cohen reals (see Cohen [2]) we have that (A3) is not provable in ZF. Thus, neither of the statements (A4) or (A5) is provable in ZF.

Theorem 2.8 *(A3) implies the statement: For every ordinal number $\kappa \geq \omega$, every infinite subset of 2^κ has an infinite well orderable subset. In particular, for $\kappa = \omega$, (A3) implies Form 13.*

Proof: Fix an ordinal number $\kappa \geq \omega$ and let A be an infinite subset of 2^κ. Assume that A has no infinite well-orderable subset and let $\mathcal{A}$, G_i, G and $([\kappa]^{<\omega}, T_B)$ be defined as in the proof of Theorem 2.3. By (A3) let $\mathcal{W} = \{W_j : j \in \aleph\}$ be a well-ordered open refinement of the open cover $\mathcal{U} = \{B(\varnothing, i, g) : i \in A, g \in G_i\}$ where $B(x, i, g)$ is given by (3) in the proof of Theorem 2.3. For each $j \in \aleph$, let

$$H_j = \{i \in A : \exists g \in G_i, W_j \subseteq B(\varnothing, i, g)\}.$$

As in Claim 2.6 of the proof of Theorem 2.3 it can be shown that H_j is finite for all $j \in \aleph$ and since U has no finite subcover, it follows that there is an infinite set $M \subseteq \aleph$ such that $H_j \neq H_{j'}$ for all $j, j' \in M$, $j \neq j'$. Consequently, $H = \cup\{H_j : j \in M\}$ is infinite and as 2^κ is linearly ordered it follows that H is an infinite well-ordered subset of A. This contradicts our assumption and completes the proof of the theorem. □

Remark 2.9 In view of Theorem 2.8 one expects that (13) does not imply back (A3). Indeed, this is the case. In particular, Monro ([7], p. 37) constructs a symmetric extension N of a countable transitive model of ZF $+V = L$ such that N satisfies AC($\mathbb{R}$) (hence, it satisfies 13), but there is a cardinal $\kappa \in N$ and an infinite subset of 2^κ (the set G^* of p. 37) having no countably infinite subsets in N. Thus, the conclusion of Theorem 2.8 fails for the ordinal number κ and consequently (A3) fails in N as well.

3 Summary

The following diagram summarizes the results of the paper.

$$\boxed{\boxed{\mathrm{AC} \equiv (\mathrm{A1}) \equiv (\mathrm{A2}) \equiv (\mathrm{A6})}}$$
$$\Downarrow$$
$$\boxed{\boxed{(\mathrm{A3}) \equiv (\mathrm{A4}) \equiv (\mathrm{A5})}}$$
$$\Downarrow$$
$$\boxed{\boxed{13}}$$

Acknowledgments We would like to thank Professor Dr. Norbert Brunner for bringing to our attention the interrelations between the statements (Ai) defined on pages 2 and 3.

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