The singular nature of space-time

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Abstract

We consider to what extent the fundamental question of space-time singularities is relevant for the philosophical debate about the nature of space-time. After reviewing some basic aspects of the space-time singularities within GR, we argue that the well-known difficulty to localize them in a meaningful way may challenge the received metaphysical view of space-time as a set of points possessing some intrinsic properties together with some spatio-temporal relations. Considering the algebraic formulation of GR, we argue that the space-time singularities highlight the philosophically misleading dependence on the standard geometric representation of space-time.

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1 Introduction

Despite the invitation of Earman to consider more carefully the question of space-time singularities [Earman 1995], only few literature in space-time philosophy has been devoted to this foundational issue.[1] This paper aims to take up Earman's invitation and to carry out philosophical investigations about space-time singularities in the framework of the modern debate about the status of space-time. Indeed, there are two main positions with respect to space-time singularities and their generic character due to the famous singularity theorems: first, they can be thought of as physically meaningless, only revealing that in these cases the theory of general relativity (GR) breaks down and must be superseded by another theory (like a future theory of quantum gravity (QG) for instance).[2] Therefore, as such space-time singularities do not tell us anything physically relevant. Second, space-time singularities can be taken more 'seriously': they can well be considered as problematic but nevertheless as involving some fundamental features of space-time. In this sense, their careful study at the physical, mathematical and conceptual level may be helpful in order to understand the nature of space-time as described by (classical) GR. This paper aims to investigate this line of thought. In this framework, the question

[1] With the notable exceptions of [Earman 1996], [Curiel 1999] and [Mattingly 2001].

[2] They may also not occur in our universe if one of the (necessary) hypotheses of the singularity theorems were violated, see [Mattingly 2001].

of space-time singularities is actually a fascinating one, which may be related at the same time to the question of the 'initial' state of our universe and to the question of the fundamental structure of space-time.

Roughly, the main question of this paper is the following one: assuming that the space-time singularities tell us something about the nature of space-time (again, this assumption is not evident), what do they tell us? The (tricky) problem of the very definition of space-time singularities is an essential part of the question.

In section 2, we will review the main concepts necessary to give an account of space-time singularities within GR. Although these concepts reflect various aspects of the space-time singularities, there is actually no single definition that encompasses them all. In particular we will see that the various attempts to define the space-time singularities in terms of local entities (like some kind of 'holes' or 'missing points' for instance) fail.[3] We will then see in subsection 3.1 that this may constitute a strong argument for considering space-time singularities rather as a non-local property of space-time. The central part of the paper consists in evaluating the possible consequences of space-time singularities for the ontological status of space-time. We will then argue that, independently of any position in the substantivalism-relationalism debate, some aspects of the space-time singularities may urge caution when discussing ontological issues about space-time only with respect to local and pointlike considerations.

Even if taken 'seriously', space-time singularities are however not a satisfactory part of GR. In this perspective, we will briefly consider in subsection 3.2 some recent theoretical developments within the algebraic formulation of GR regarding the space-time singularities. These developments draw some possible physical (and indeed mathematical) consequences of the above mentioned aspects. This algebraic approach to space-time takes the non-local aspects of the space-time singularities as revealing that space-time is non-local and pointless at the fundamental level. The considerations about the algebraic formulation of GR underline the fact that the metaphysical conception of space-time should not be dependent on a particular formulation (like the inherently pointlike standard geometric one for instance), as it seems to be often the case.

[3]'Local' is understood in this paper in the sense of being associated to a space-time point (and its neighbourhood), see the subsection 3.1.

2 Some aspects of the singular feature of space-time

2.1 Extension and incompleteness

At the present state of our knowledge, it seems to be quite commonly accepted in the relevant physics literature that there is no satisfying general definition of a space-time singularity (for instance, see [Wald 1984, 212]). In other terms, the notion of a space-time singularity covers various distinct aspects that cannot be all captured in one single definition. We certainly do not pretend to review all these aspects here. We rather want to focus on the first two fundamental notions that are at the heart of most of the attempts to define space-time singularities.

The first is the notion of *extension* of a space-time Lorentz manifold (together with the interrelated notion of continuity and differentiability conditions).[4] The idea is to insure that what we count as singularities are not merely (regular) 'holes' or 'missing points' in our space-time Lorentz manifold that could be covered ('filled') by a 'bigger' but regular space-time Lorentz manifold with respect to certain continuity and differentiability conditions (or C^k-conditions). These latter conditions (together with the notion of extension) are therefore essential to the most consensual definitions of space-time singularities. But, at this level, there are two major ambiguities that are part of the difficulties to define space-time singularities. First, extensions are not unique and all possible extensions must be carefully considered in order to discard (regular) singularities that can be removed by a mere regular extension. Given certain C^k-conditions, we will therefore always consider maximal space-time Lorentz manifolds.[5] A space-time singularity will therefore be defined with respect to certain C^k-conditions (and indeed should be called a C^k-singularity; these conditions are often implicit and not always mentioned). This fact leads to the second difficulty: it is not clear what are exactly the necessary and sufficient continuity and differentiability conditions for a space-time Lorentz manifold to

[4] An extension of a space-time Lorentz manifold (M, g) is any space-time Lorentz manifold (M', g') (of same dimension) where (M', φ) is an envelopment of M and such that $\varphi^*(g) = g'|_{\varphi(M)}$ holds.

[5] A space-time Lorentz manifold is maximal with respect to the differentiability condition C^k if there is no C^k-regular extension, which is an extension where the metric g' is C^k at the boundary $\partial' M$ of $\varphi(M)$ in M'.

be physically meaningful.[6]

Strongly related with the idea of extension, the second essential notion in order to give an account of space-time singularities is the notion of *curve incompleteness*, which is the feature that is widely recognized as the most general characterization so far of space-time singlarities (see for instance [Wald 1984, §9.1]). Moreover, it is actually curve incompleteness that is predicted by the singularity theorems as the generic singular behaviour for a wide class of solutions.[7] The broad idea is that we should look at the behavior of physically relevant curves (namely geodesics and curves with a bounded acceleration) in the space-time Lorentz manifold for 'detecting' space-time singularities (which actually do not belong to the space-time Lorentz manifold): in particular, the idea is that an (inextendible) half-curve of finite length (with respect to a certain generalized affine parameter) may indicate the existence of a space-time singularity. The obvious intuition behind this idea is that, roughly, the (inextendible) curve has finite length because it 'meets' the singularity (it must be clear that this way of speaking is actually misleading in the sense that the 'meeting' does not happen in the space-time Lorentz manifold). Pictorially, anything moving along such an incomplete (non-space-like) curve (like an incomplete geodesic or an incomplete curve with a bounded acceleration) would literally 'disappear' after a finite amount of proper time or after a finite amount of a generalized affine parameter (again, we must be very careful when using such pictures; for instance, the event of the 'disappearance' itself is not part of the space-time Lorentz manifold). In more formal terms, a (maximal) space-time Lorentz manifold is said to be b-complete if all inextendible C^1-half curves have infinite length as measured by the generalized affine parameter (it is b-incomplete otherwise).[8] The link with the initial intuition comes from the fact that it can be shown that b-completeness entails the completeness of geodesics and of curves with a bounded acceleration (but not vice versa).

[6] A possible guideline would be to require that these conditions secure that the fundamental laws of GR, that is, the Einstein field equations and the Bianchi identity, are well defined, see [Earman 1995, §2.7].

[7] However, the notion of curve incompleteness does not encompass all aspects of space-time singularities (like for instance certain aspects linked with the violation of the cosmic censorship).

[8] The generalized affine parameter u for a C^1-half-curve $\gamma(t)$ is defined by $u := \int_0^t (\sum_{i=0}^3 (V^\alpha(t))^2)^{\frac{1}{2}} dt$, where $V(t) = V^\alpha(t) e_\alpha(p)$ is the tangent vector expressed in the parallel propagated orthonormal basis e_α.

2.2 Boundary

The most widely accepted standard definition of a singular space-time is the following one: a (maximal) space-time (Lorentz manifold) is said to be singular if and only if it is b-incomplete. However, b-incompleteness only indirectly refers (if at all) to space-time singularities in the sense of localized singular parts of space-time (like space-time points where something 'goes wrong'). Space-time singularities are actually not part of the space-time Lorentz manifold (M, g) representing space-time (within GR) in the sense that they cannot be merely represented by points $p \in M$ (or regions of M) where some physical quantity related to the space-time structure (like the curvature for instance) goes to infinity.[9]

Boundary constructions can be understood as attempts to describe space-time singularities directly in terms of certain local properties that can be ascribed to certain boundary (ideal) points 'attached' to the space-time Lorentz manifold. It will suffice for our purpose here to only briefly consider some aspects of the so-called b- and a-boundary constructions.

The main idea of the b-boundary construction is to consider the b-incomplete curves to define (singular) boundary points (as their endpoints) that can be 'attached' to the space-time Lorentz manifold. Schmidt's procedure [Schmidt 1971] provides a way to construct such a (singular) boundary ∂M (called b-boundary) using the equivalence between the b-completeness of the space-time Lorentz manifold (M, g) and the Cauchy completeness of the total space OM of the orthonormal frame bundle $\pi : OM \to M$. In order to establish the possible intuition of localization of space-time singularities with the help of these boundary points, it is necessary to endow the singular boundary with some differential or at least some topological structure. But it has been shown that the b-boundary of the closed Friedman-Robertson-Walker (FRW) solution, which constitute part of the so-called 'standard model' of contemporary cosmology, consists of a single point that is *not Hausdorff separated* from points of the space-time Lorentz manifold ([Bosshard 1976] and [Johnson 1977]). Being not Hausdorff separated from points of M, this unique boundary point, which should repre-

[9] Space-time is represented within GR by a pair (M, g), where M is in general assumed to be a 'nice' (paracompact, connected, oriented, Hausdorff) 4-dimensional differentiable manifold and g is a C^k ($k \geq 2$ in general) Lorentz metric, solution of the Einstein field equations and defined everywhere on M.

sent the *two* singularities of the closed FRW model, is actually 'arbitrarily close' to the points of M. It is then very difficult to give physical meaning to such a behavior in terms of local entities or properties since any (regular) points $p \in M$ has the singular boundary point in his (arbitrarily small) neighborhood:[10] at least any (usual) sense of localization of the singularities seems then to be lost - indeed one of the main motivations for attaching boundary points to the space-time Lorentz manifold is lost (see [Earman 1995, 36-37]). Moreover, such bad topological behavior has been shown to be a feature of all boundary constructions that share with the b-boundary construction certain natural (and rather weak) conditions [Geroch, Liang & Wald 1982].

With the help of the central notion of extension or envelopment, the a-boundary construction aims to truly capture the idea of 'missing points', according to which space-time singularities have to be considered as points in a 'bigger' manifold. More precisely, the motivation of the a-boundary construction is that singularities in a space-time Lorentz manifold have to be considered as points (or subsets) of the topological boundary of the (image of the) manifold with respect to an envelopment (such subsets are called boundary sets). In order to overcome the already mentioned difficulty of the non-uniqueness of the possible envelopments of a given manifold (section 2.1), the a-boundary is defined as a set of equivalence classes of boundary sets (with respect to different envelopments) under a relevant equivalence relation (called the mutual covering relation, see [Scott & Szekeres 1994]). The a-boundary points representing (essential) space-time singularities are further defined with respect to incomplete curves. Avoiding the technical details, it is sufficient for our purpose here to emphasize that a space-time singularity is then represented by an *equivalence class* of boundary *sets*, most of which are in general not singletons (and not even necessarily connected). In this framework, any interpretation of a space-time singualrity as a local or pointlike space-time entity to which local properties could be ascribed seems problematic too (see [Curiel 1999, 133-36]).

[10]The same problem arises in the case of the Schwarzschild solution.

3 Singular feature and the ontological status of space-time

3.1 A non-local feature of space-time

Neither curve (b-)incompleteness nor boundary constructions enable us to conceive space-time singularities as local entities or local properties. We have seen that they cannot be naively described by space-time Lorentz manifold points (or regions) where something 'goes wrong' (where the curvature 'blows up' for instance). This is actually intimately related to the dynamical nature of the space-time structure as described by GR: space-time singularities are indeed singularities of the space-time structure itself and there is no a priori fixed (space-time) structure or entity with respect to which the space-time singularities could be defined. Therefore, it seems indeed more accurate to speak of 'singular feature of space-time' in a sense that is not committed to any notion of localized entity or property as the (actually misleading) talk of 'space-time singularities' is (see [Earman 1995, 28]). Of course, this is not a mere semantic move, and, in a scientific realist perspective, we want now to consider some possible ontological implications of this singular feature for the nature of space-time.

This singular feature seems to be an irreducible *non-local* feature of space-time in the sense that it is based neither on the existence of any particular local entities like space-time points nor on local properties instantiated at particular space-time points (this is underlined in [Curiel 1999]). These notions of local properties and local entities are understood here as being closely linked to the concept of a point (and its neighbourhood) and to the notion of intrinsicality (which is understood in the sense of being independent of accompaniment or loneliness, see [Langton & Lewis 1998]). In this sense, it implies in particular that the non-local character of the singular feature of space-time cannot be tied to some intrinsic properties instantiated at some particular space-time points (or in some particular ('local') space-time region). In this perspective, it bears some analogy with some non-local aspects of the gravitational energy (and, in a certain sense, with some global topological features of space-time). But this does not merely amount to the widely recognized non-supervenience of space-time relations on intrinsic properties of the space-time points (or of their relata, see [Cleland 1984]). Whereas a particular space-time relation needs to be instantiated between particular space-time points, what we want to stress here is that space-time may possess some fundamental features that are actually independent of the existence of any particular space-time points (or of any particular ('local') space-time region indeed). Such non-local (non-pointlike) features of space-time may challenge therefore the received view of space-time as a set of points possessing some intrinsic properties together with some space-time relations (like in Lewis' thesis of Humean supervenience).[11] In particular, this should at least prevent us from putting too much ontological weight on local (and intrinsic) properties and local entities (like space-time points for instance). Moreover, this sceptical attitude towards space-time points and their possible intrinsic properties may well receive support from the GR-principle of active general covariance (or of invariance under active diffeomorphisms) and the related hole argument. Indeed, due to this fundamental physical principle, a wide range of philosophers of physics and physicists agree on the fact that, within GR, space-time points cannot be physically individuated (and therefore 'localized'), possessing intrinsic properties for instance, independently of the space-time relations as represented by the metric (see for instance [Dorato 2000] and [Rovelli 2004, ch.2]). As mentioned already above, from the physical point of view, all this is linked with the fundamental dynamical nature of space-time as described by GR: there are no a priori fixed space-time points with respect to which other spatio-temporal entities or features (like the space-time singularities) can be ('pointlikely') described and localized.

Therefore, as regards the ontological status of space-time, taking into consideration ('seriously') the singular feature of space-time (and similarly the gravitational energy) would rather favour some 'non-pointlike and non-intrinsic conception' of space-time, be it substantivalist or relationalist (relationalism is here understood in the strong reductive sense, that is, the position according to which space-time is reduced to non-spatio-temporal relations among matter or to non-spatio-temporal properties of matter). Indeed, it seems difficult for the relationalist to reduce these non-local (non-pointlike) space-time features to

[11] To include merely the non-local features of space-time in the supervenience basis would be a rather ad hoc solution. So, we see that not only quantum physics, but also classical general relativistic physics may threaten Humean supervenience.

pointlike bits of matter. In an analoguous way, it is quite difficult to maintain a pointlike substantivalist position (with respect to space-time points bearing intrinsic properties) that does not fall victim to the hole argument and that can account for these non-local features. Indeed, according to a rather radical approach to the question of the singular behaviour of space-time, it may be the case that the moral of the 'space-time singularities problem' is that the very concept of a space-time point[12] is challenged at the fundamental level. From a substantivalist point of view, the singular feature would then reveal the fundamental non-local (non-pointlike) nature of space-time, which would need to be described in other mathematical (non-pointlike) terms. These could be algebraic.

3.2 Algebraic approaches

If the philosophical analysis of the singular feature of space-time is able to shed some new light on the possible nature of space-time (as we have tried to show), one should not lose sight of the fact that, although connected to fundamental issues in cosmology, like the 'initial' state of our universe, space-time singularities involve unphysical behaviour (like, for instance, the very geodesic incompleteness implied by the singularity theorems or some possible infinite value for physical quantities like the curvature) and constitute therefore a physical problem that should be overcome.[13] We now want to consider some recent theoretical developments that directly address this problem by drawing some possible physical (and mathematical) consequences of the above considerations.

Indeed, according to the algebraic approaches to space-time, the singular feature of space-time is an indicator for the fundamental global character of space-time: it is conceived actually as a very important part of GR that reveals the fundamental pointless structure of space-time, which therefore cannot be described by the usual mathematical tools like standard differential geometry - which is inherently pointlike. The mathematical roots of such considerations are to be found in the full equivalence of, on the one hand, the usual (geometric) definition of a differentiable manifold M in terms of a set of points with

[12] Or the concept of (non-spatio-temporal) pointlike and intrinsic property in a relationalist perspective.

[13] However this does not entail that GR is either false or incomplete, see [Earman 1996].

a topology and a differential structure (compatible atlases) with, on the other hand, the definition using only the algebraic structure of the (commutative) ring $C^{\infty}(M)$ of the smooth real functions on M (under pointwise addition and multiplication; indeed $C^{\infty}(M)$ is a (concrete) algebra). For instance, the existence of points of M is equivalent to the existence of maximal ideals of $C^{\infty}(M)$.[14] Indeed, all the differential geometric properties of the space-time Lorentz manifold (M, g) are encoded in the (concrete) algebra $C^{\infty}(M)$. Moreover, the Einstein field equations and theirs solutions (which represent the various space-times) can be constructed only in terms of the algebra $C^{\infty}(M)$.[15] Now, the algebraic structure of $C^{\infty}(M)$ can be considered as primary (in exactly the same way in which space-time points or regions, represented by manifold points or sets of manifold points, may be considered as primary) and the manifold M as derived from this algebraic structure. Indeed, one can define the Einstein field equations from the very beginning in abstract algebraic terms without any reference to the manifold M as well as the abstract algebras, called the 'Einstein algebras', satisfying these equations. The standard geometric description of space-time in terms of a Lorentz manifold (M, g) can then be considered as inducing a mathematical (Gelfand) representation of an Einstein algebra. Without entering into too many technical details, the important point for our discussion is that Einstein algebras and sheaf-theoretic generalizations thereof reveal the above discussed non-local feature of (essential) space-time singularities from a different point of view.[16] In the framework of the b-boundary construction $\overline{M} = M \cup \partial M$ (see subsection 2.2), the (generalized) algebraic structure C corresponding to M can be prolonged to the (generalized) algebraic structure $\overline{C}$ corresponding to the b-completed $\overline{M}$ such that $\overline{C}_M = C$, where $\overline{C}_M$ is the restriction of $\overline{C}$ to M; then in the singular cases (like the closed FRW solution), only constant functions (and therefore only zero vector fields)[17] can be

[14]A maximal ideal of a commutative algebra $\mathcal{A}$ is the largest proper subset of - indeed a subgroup of the additive group of - $\mathcal{A}$ closed under multiplication by any element of $\mathcal{A}$. The corresponding maximal ideal of $C^{\infty}(M)$ to a point $p \in M$ is the set of all vanishing functions at p.

[15]The original idea is due to [Geroch 1972].

[16]There are indeed several algebraic approaches to GR. For instance, according to the Abstract Differential Geometry program of A.Mallios and I.Raptis, space-time singularities are artifacts of our mathematical (C^{∞}-)representation of space-time: indeed, they simply disappear once GR is written in purely algebraic (sheaf-theoretic) terms, see [Mallios & Raptis 2003]. In the following, we rather brielfy consider the (less radical) approach of M.Heller et al., which emphasizes some interesting points for our discussion.

[17]In the algebraic formalism, vector fields are abstract 'derivations'.

prolonged [Heller & Sasin 1994]. This underlines the non-local feature of the singular behaviour of space-time, since constant functions are non-local (non-pointlike) in the sense that they do not distinguish points. This fundamental non-local feature suggests non-commutative generalizations of the Einstein algebras formulation of GR (see [Heller, Pysiak & Sasin 2004] for instance), since non-commutative spaces are highly non-local. We will not discuss this matter here. It is sufficient for us to stress that, in general, non-commutative algebras have no maximal ideals, so that the very concept of a point has no counterpart within this non-commutative framework. Therefore, according to this line of thought, space-time, at the fundamental level, is completely non-local (pointless indeed). Then, at this fundamental level, it seems that the very distinction between singular and non-singular is not meaningful anymore; within this framework, space-time singularities are then 'produced' together with the standard (commutative) space-time geometry through a kind of 'transition' (in the history of the universe).[18]

Although these theoretical developments are rather speculative, it must be emphasized that the algebraic representation of space-time itself is "by no means esoteric" [Butterfield & Isham 2001, §2.2.2]. Starting from an algebraic formulation of the theory, which is completely equivalent to the standard geometric one, it provides another point of view on space-time and its singular behaviour that should not be dismissed too quickly. At least it underlines the fact that our interpretative framework for space-time should not be dependent on the standard local (pointlike) conception of space-time (which is induced by the standard geometric formulation). Indeed, this misleading dependence on the formalism seems to be at work in some reference arguments in modern philosophy of space-time, like in the hole argument[19] or in the field argument by Hartry Field. According to the latter argument, field properties occur at space-time points or regions, which must therefore be presupposed [Field 1980, 35]. Such an argument seems to fall prey to the local (pointlike) representation of space-time and fields, since within the algebraic formalism of GR, (scalar) fields - elements of the algebra C^∞ - can be interpreted as primary and the manifold

[18] See [Heller 2001] and references therein.

[19] The hole argument has been recently discussed in [Bain 2003] within the framework of the algebraic formulation of GR.

(points) as a secondary derived notion.[20] However, it must be stressed that, as such, this position does not speak for reductive relationalism, since a particular field, namely the metric or gravitational field, encodes all space-time features and can therefore still be considered as the representative of space-time.

4 Conclusion

Taking up Earman's invitation to consider space-time singularities 'seriously' has led us to deal with fundamental issues about the nature of space-time. Indeed, we have seen that space-time may possess some fundamental non-local (and non-pointlike) features, like the singular feature, that challenge the traditional metaphysical view about space-time. According to this received view, space-time is conceived as a set of points, at which (intrinsic) properties can be instantiated, together with the space-time relations. Indeed, the very concept of a space-time point seems to lie at the heart of the challenge. It cannot be merely postulated anymore (as in Field's argument), since it is indeed a secondary derived notion within the algebraic formulation of GR, which may with reason be considered as deserving to play a role in the interpretative issues about space-time - at least to the same extent as the standard geometric formulation does. Actually, it seems that the alleged interpretational problems with respect to space-time singularities may find part of theirs roots in the misleading dependence on the local and pointlike conception of space-time, which is actually induced by this standard geometric representation of space-time.

[20] And this does not even take into account the fact that, within sheaf-theoretic or non-commutative generalizations, the very concept of a point may be challenged at the fundamental level.

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