The singular nature of space-time

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Abstract

We consider to what extent the fundamental question of space-time singularities is relevant for the philosophical debate about the nature of space-time. After reviewing some basic aspects of the space-time singularities within GR, we argue that the well-known difficulty to localize them in a meaningful way may challenge the received metaphysical view of space-time as a set of points possessing some intrinsic properties together with some spatio-temporal relations. Considering the algebraic formulation of GR, we argue that the space-time singularities highlight the philosophically misleading dependence on the standard geometric representation of space-time.

1 Introduction

Despite Earman's invitation to consider more carefully the question of space-time singularities (Earman 1995), only few literature in space-time philosophy has been devoted to this foundational issue.[1] This paper aims to take up this invitation and to carry out philosophical investigations about space-time singularities in the framework of the contemporary debate about the status and the nature of space-time. Indeed, there are two main positions with respect to space-time singularities and their generic character due to the famous singularity theorems: first, they can be thought of as physically meaningless, only revealing that in these cases the theory of general relativity (GR) breaks down and must be superseded by another theory (like a future theory of quantum gravity (QG) for instance).[2] Therefore, as such space-time singularities do not tell us anything physically relevant. Second, space-time singularities can be taken more 'seriously': they can well be considered as physically problematic

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but nevertheless as involving some fundamental features of space-time. In this sense, their careful study at the physical, mathematical and conceptual level may be helpful in order to understand the nature of space-time as described by GR. This paper aims to investigate this line of thought. In this framework, the question of space-time singularities is actually a fascinating one, which may be related at the same time to the question of the 'initial' state of our universe and to the question of the fundamental structure of space-time.

Roughly, the main question of this paper is the following one: in a scientific realist perspective and assuming that the space-time singularities tell us something about the nature of space-time (again, this assumption is not evident), what do they tell us? The (tricky) problem of the very definition of space-time singularities is an essential part of the question.

In Section 2, we will review the main concepts necessary to give an account of space-time singularities within GR. In particular we will see that the various attempts to define the space-time singularities in terms of local entities (like some kind of 'holes' or 'missing points' for instance) fail. We will then argue in subsection 3.1 that this may constitute a strong argument for considering space-time singularities rather as a non-local property of space-time. The central part of the paper consists in evaluating the possible consequences of space-time singularities for the metaphysical conception of space-time.

Even if taken 'seriously', space-time singularities are however not a satisfactory part of GR. In this perspective, we will briefly consider in subsection 3.2 some recent theoretical developments within the algebraic formulation of GR regarding the space-time singularities. These developments draw some possible physical (and indeed mathematical) consequences of the above mentioned aspects. This algebraic approach to space-time takes the non-local aspects of the space-time singularities as revealing that space-time is non-local and pointless at the fundamental level. The considerations about the algebraic formulation of GR underline the fact that the metaphysical conception of space-time should not be dependent on a particular formulation (like the inherently pointlike standard geometric one for instance), as it seems to be often the case.

2 Some aspects of the singular feature of space-time

2.1 Extension and incompleteness

At the present state of our knowledge, it seems to be quite commonly accepted in the relevant physics literature that there is no satisfying general definition of a space-time singularity (for instance, see Wald 1984, 212). In other terms, the notion of a space-time singularity covers various distinct aspects that cannot be all captured in one single definition. We certainly do not pretend to review all these aspects here. We rather want to focus on the first two fundamental notions that are at the heart of most of the attempts to define space-time singularities.

The first is the notion of *extension* of a space-time Lorentz manifold (together with the interrelated notion of continuity and differentiability conditions).[3] The idea is to insure that what we count as singularities are not merely (regular) 'holes' or 'missing points' in our space-time Lorentz manifold that could be covered ('filled') by a 'bigger' but regular space-time Lorentz manifold with respect to some continuity and differentiability conditions (or C^k-conditions). These latter conditions (together with the notion of extension) are therefore essential for any characterization of space-time singularities. But, at this level, there are two major ambiguities that are part of the difficulties to define space-time singularities. First, extensions are not unique and all possible extensions must be carefully considered in order to discard (regular) singularities that can be removed by a mere regular extension. Given some C^k-condition, we will always consider maximal space-time Lorentz manifolds.[4] A space-time singularity will therefore be defined with respect to certain C^k-conditions (and indeed should be called a C^k-singularity; these conditions are often implicit and not always mentioned). This fact leads to the second difficulty: it is not clear what are exactly the necessary and sufficient continuity and differentiability conditions for a space-time Lorentz manifold to be physically meaningful.[5]

Strongly related with the idea of extension, the second essential notion in order to give an account of space-time singularities is the notion of *curve incompleteness*, which is the feature that is widely recognized as the most consensual characterization so far of space-time singlarities (see for instance Wald 1984, §9.1). Moreover, it is actually curve incompleteness that is predicted by the singularity theorems as the generic singular behaviour for a wide class of solutions.[6] The broad idea is that we should look at the behavior of physically relevant curves (namely geodesics and curves with a bounded acceleration) in the space-time Lorentz manifold for 'detecting' space-time singularities (which actually do not belong to the space-time Lorentz manifold): in particular, the idea is that an (inextendible) half-curve of finite length (with respect to a certain generalized affine parameter) may indicate the existence of a space-time singularity. The obvious intuition behind this idea is that, roughly, the (inextendible) curve has finite length because it 'meets' the singularity (it must be clear that this way of speaking is actually misleading in the sense that the 'meeting' does not happen in the space-time Lorentz manifold). Pictorially, anything moving along such an incomplete (non-spacelike) curve (like an incomplete geodesic or an incomplete curve with a bounded acceleration) would literally 'disappear' after a finite amount of proper time or after a finite amount of a generalized affine parameter (again, we must be very careful when using such pictures; for instance, the event of the 'disappearance' itself is not part of the space-time Lorentz manifold). In more formal terms, a (maximal) space-time Lorentz manifold is said to be b-complete if all inextendible C^1-half curves have infinite length as measured by the generalized affine parameter (it is b-incomplete otherwise).[7] The link with the initial intuition comes from the fact that it can be shown that b-completeness entails the completeness of geodesics and of curves with a bounded acceleration (but not vice versa).

2.2 Boundary

The most widely accepted standard definition of a singular space-time is the following one: a (maximal) space-time (Lorentz manifold) is said to be singular if and only if it is b-incomplete. However, b-incompleteness only indirectly refers (if at all) to space-time singularities in the sense of localized singular parts of space-time (like space-time points where something 'goes wrong'). Space-time singularities are actually not part of the space-time Lorentz manifold (M, g) representing space-time (within GR) in the sense that they cannot be merely represented by points $p \in (M, g)$ (or regions) where some physical quantity related to the space-time structure (like the scalar curvature for instance) goes to infinity.[8]

Boundary constructions can be understood as attempts to describe space-time singularities directly in terms of local properties that can be ascribed to certain boundary points 'attached' to the space-time Lorentz manifold. The physical motivation is to do local physics, that is, to study the space-time structure 'near' (or even 'at') the singularities. It will suffice for our purpose here to only briefly consider some aspects of the so-called b- and a-boundary constructions.

The main idea of the b-boundary construction is to consider the b-incomplete curves to define (singular) boundary points (as their endpoints) that can be 'attached' to the space-time Lorentz manifold. Schmidt's procedure provides a way to construct such a (singular) boundary ∂M (called b-boundary) using the equivalence between the b-completeness of the space-time Lorentz manifold (M, g) and the Cauchy completeness of the total space OM of the orthonormal frame bundle $\pi : OM \rightarrow M$ (Schmidt 1971). In order to make physical sense of the idea of localizing the space-time singularities with the help of these boundary points, it is necessary to endow the singular boundary with some differential or at least some topological structure. But it has been shown that the b-boundary of the closed Friedman-Lemaître-Robertson-Walker (FLRW) solution, which is part of the 'standard model' of contemporary cosmology, consists of a single point that is not Hausdorff separated from points of the space-time Lorentz manifold (Bosshard 1976; Johnson 1977). Being not Hausdorff separated from points of M, this unique boundary point, which should represent the two singularities of the closed FLRW model, is actually 'arbitrarily close' to the points of M.[9] It is then very difficult to give physical meaning to such a behavior in terms of local entities or properties since any (regular) points $p \in (M, g)$ has the singular boundary point in his (arbitrarily small) neighborhood:[10] at least any (usual) sense of localization of the singularities, which is indeed one of the main motivations for boundary points, seems then to be lost (Earman 1995, 36-37). Moreover, such bad topological behavior has been shown to be a feature of all boundary constructions that share with the b-boundary construction certain natural (and rather weak) conditions (Geroch Can-bin and Wald 1982).

With the help of the central notion of extension or envelopment, the a-boundary construction aims to truly capture the idea of 'missing points', according to which space-time singularities have to be considered as points in a 'bigger' manifold. More precisely, the motivation of the a-boundary construction is that singularities in a space-time Lorentz manifold have to be considered as points (or subsets) of the topological boundary of the (image of the) manifold with respect to an envelopment (such subsets are called boundary sets). In order to overcome the already mentioned difficulty of the non-uniqueness of the possible envelopments of a given manifold (subsection 2.1), the a-boundary is defined as a set of equivalence classes of boundary sets (with respect to different envelopments) under a relevant equivalence relation (called the mutual covering relation, see Scott and Szekeres 1994). The a-boundary points representing (essential) space-time singularities are further defined with respect to incomplete curves. Avoiding the technical details, it is sufficient for our purpose here to emphasize that a space-time singularity is then represented by an equivalence class of boundary sets, most of which are in general not singletons (and not even necessarily connected). In this framework, any interpretation of a space-time singualrity as a pointlike or local space-time entity to which local properties could be ascribed seems problematic too (Curiel 1999, 133-36).

3 Singular feature and space-time metaphysics

3.1 A non-local feature of space-time

We have seen that space-time singularities cannot be merely described by space-time Lorentz manifold points (or regions) where something 'goes wrong' (where the scalar curvature 'blows up' for instance). This is actually intimately related to the dynamical nature of the space-time structure as described by GR: space-time singularities are indeed singularities of the space-time structure itself and there is no a priori fixed (space-time) structure or entity with respect to which the space-time singularities could be defined. The point is that the very characterizations of space-time singularities within GR, in terms of curve (b-)incompleteness or with the help of boundary constructions, do not enable us to conceive space-time singularities as meaningful *local* entities or properties, which are defined here to be those that can be associated with (and determined at) a space-time point and its (arbitrarily small) neighborhood. In order to be physically meaningful, such definition of local entities and propeties requires that a (topological) separation assumption holds among the space-time points. For instance, it seems meaningful to require that distinct local properties can be associated with (determined at) distinct space-time points together with their disjoint (arbitrarily small) neighbourhood, no matter how close to one another they are (Hausdorff condition). Such (topological) separation assumption lies then at the heart of the definition of local entities and properties and is in general part of the standard differential geometric representation of space-time, in which, therefore, "a significant amount of locality is being presupposed" (Earman 1987, 453). As we have seen above, this aspect of locality can be violated by space-time singularities in some physically important cases. In general, there is no clear and necessary link between the singular behaviour of space-time and the existence of any particular local entities like space-time (boundary) points or local properties instantiated at particular space-time (boundary) points (this is clearly underlined in Curiel 1999; see also Dorato 1998, 340). In this sense, this singular behaviour seems to constitute an irreducible *non-local* feature of space-time.[11] More precisely and as a consequence of this non-local aspect, the singular behaviour of space-time cannot be reduced to (is not supervenient on) some intrinsic properties[12] instantiated at some particular space-time (boundary) points. In this perspective, it bears some analogy with some other non-local aspects of the space-time structure, like the gravitational energy and, in a certain sense, some (irreducible) global topological properties. But it does not merely amount to the widely recognized non-supervenience of space-time relations on intrinsic properties of the space-time points[13] (Cleland 1984). Whereas a particular space-time relation needs to be instantiated between particular space-time points, what we want to stress here is that space-time may possess some fundamental features that are actually independent of the existence of any particular space-time points (and of any intrinsic properties instantiated at particular space-time points). Such non-local features of space-time may challenge therefore the received atomisitic (local) view of space-time (and of the world) as a set of points possessing some intrinsic properties together with some space-time relations (like within Lewis' thesis of Humean supervenience).[14] So, it seems that not only quantum physics, but also classical general relativistic physics may threaten this traditional metaphysical conception of the world.[15]

This should prevent us from limiting our ontological considerations about space-time only to local properties and entities. In particular, we should not put too much ontological weight on local and intrinsic properties of space-time points (as well as on space-time points themselves). Indeed, this sceptical attitude towards space-time points and their possible intrinsic properties may well receive support from the GR-principle of active general covariance (or of invariance under active diffeomorphisms) and the related hole argument. Due to this fundamental physical principle, a wide range of philosophers of physics and physicists agree on the fact that, within GR, space-time points cannot be physically individuated (and therefore 'localized'), possessing intrinsic properties for instance, independently of the space-time relations (structure) as represented by the metric (see for instance Dorato 2000, Rovelli 2004, ch.2).

As regards the ontological status of space-time, taking into consideration ('seriously') the singular feature of space-time (and more generally non-local - or global - aspects of space-time) seems to favour a non-atomistic space-time metaphysics, be it substantivalist or relationalist. Such conception is understood in the broad sense of an ontology that does not give priority to local entities, like space-time points or pointlike bits of matter, with or without intrinsic properties, over the global structure in which they are embedded, like the space-time or world structure with its non-local aspects. Of course, this broad metaphysical framework can be refined, according to the 'ontological space' one leaves to local entities.[16] We argue that both substantivalism and relationalism need to accommodate these non-atomistic (or structural) and non-local (or global) aspects.[17] Indeed, within a scientific realist perspective, this focus on structures has strong flavours of a structural realist metaphysics.

According to a rather radical approach to the question of the singular behaviour of space-time, it may be the case that the moral of the 'space-time singularities problem' is that the very concept of a space-time point (or pointlike bits of matter) is challenged at the fundamental level.[18] The singular feature would then reveal the fundamental non-local and non-pointlike (or pointless) nature of space-time, which would need to be described in other mathematical (non-pointlike or pointless) terms. These could be algebraic.

3.2 Algebraic approaches

If the philosophical analysis of the singular feature of space-time is able to shed some new light on the possible nature of space-time (as we have tried to show), one should not lose sight of the fact that, although connected to fundamental issues in cosmology, like the 'initial' state of our universe, space-time singularities involve unphysical behaviour (like, for instance, the very geodesic incompleteness implied by the singularity theorems or some possible infinite value for physical quantities) and constitute therefore a physical problem that should be overcome.[19] We now want to consider some recent theoretical developments that directly address this problem by drawing some possible physical (and mathematical) consequences of the above considerations.

Indeed, according to the algebraic approaches to space-time, the singular feature of space-time is an indicator for the fundamental non-local character of space-time: it is conceived actually as a very important part of GR that reveals the fundamental pointless structure of space-time. This latter cannot be described by the usual mathematical tools like standard differential geometry, since, as we have seen above, it presupposes some "amount of locality" and is inherently pointlike. The mathematical roots of such considerations are to be found in the full equivalence of, on the one hand, the usual (geometric) definition of a differentiable manifold M in terms of a set of points with a topology and a differential structure (compatible atlases) with, on the other hand, the definition using only the algebraic structure of the (commutative) ring $C^\infty(M)$ of the smooth real functions on M (under pointwise addition and multiplication; indeed $C^\infty(M)$ is a (concrete) algebra). For instance, the existence of points of M is equivalent to the existence of maximal ideals of $C^\infty(M)$.[20] Indeed, all the differential geometric properties of the space-time Lorentz manifold (M, g) are encoded in the (concrete) algebra $C^\infty(M)$. Moreover, the Einstein field equations and their solutions (which represent the various space-times) can be constructed only in terms of the algebra $C^\infty(M)$.[21] Now, the algebraic structure of $C^\infty(M)$ can be considered as primary (in exactly the same way in which space-time points or regions, represented by manifold points or sets of manifold points, may be considered as primary) and the manifold M as derived from this algebraic structure. Indeed, one can define the Einstein field equations from the very beginning in abstract algebraic terms without any reference to the manifold M as well as the abstract algebras, called the 'Einstein algebras', satisfying these equations. The standard geometric description of space-time in terms of a Lorentz manifold (M, g) can then be considered as inducing a mathematical (Gelfand) representation of an Einstein algebra. Without entering into too many technical details, the important point for our discussion is that Einstein algebras and sheaf-theoretic generalizations thereof reveal the above discussed non-local feature of (essential) space-time singularities from a different point of view.[22] In the framework of the b-boundary construction $\overline{M} = M \cup \partial M$ (see subsection 2.2), the (generalized) algebraic structure C corresponding to M can be prolonged to the (generalized) algebraic structure $\overline{C}$ corresponding to the b-completed $\overline{M}$ such that $\overline{C}_M = C$, where $\overline{C}_M$ is the restriction of $\overline{C}$ to M; then in the singular cases (like the closed FLRW solution), only constant functions (and therefore only zero vector fields)[23] can be prolonged. This underlines the non-local feature of the singular behaviour of space-time, since constant functions are non-local in the sense that they do not distinguish points. This fundamental non-local feature suggests non-commutative generalizations of the Einstein algebras formulation of GR, since non-commutative spaces are highly non-local (see for instance Demaret Heller and Lambert 1997 and references therein). We will not discuss this matter here. It is sufficient for us to stress that, in general, non-commutative algebras have no maximal ideals, so that the very concept of a point has no counterpart within this non-commutative framework. Therefore, according to this line of thought, space-time, at the fundamental level, is completely non-local (pointless indeed). Then it seems that the very distinction between singular and non-singular is not meaningful anymore at the fundamental level; within this framework, space-time singularities are 'produced' at a less fundamental level together with standard physics and its standard differential (commutative) geometric representation of space-time (see Heller 2001 and references therein).

Although these theoretical developments are rather speculative, it must be emphasized that the algebraic representation of space-time itself is "by no means esoteric" (Butterfield & Isham 2001, §2.2.2). Starting from an algebraic formulation of the theory, which is completely equivalent to the standard geometric one, it provides another point of view on space-time and its singular behaviour that should not be dismissed too quickly. At least it underlines the fact that our interpretative framework for space-time should not be dependent on the traditional atomistic and local (pointlike) conception of space-time (induced by the standard differential geometric formulation). Indeed, this misleading dependence on the standard differential geometric formulation seems to be at work in some reference arguments in contemporary philosophy of space-time, like in the hole argument[24] or in the field argument (Field 1980, 35). According to the latter argument, field properties occur at space-time points or regions, which must therefore be presupposed. Such an argument seems to fall prey to the standard differential geometric representation of space-time and fields, since within the algebraic formalism of GR, (scalar) fields - elements of the algebra C^∞ - can be interpreted as primary and the manifold (points) as a secondary derived notion.[25]

4 Conclusion

Taking up Earman's invitation to consider space-time singularities 'seriously' has led us to deal with fundamental issues about the nature of space-time. Indeed, we have seen that space-time may possess some fundamental non-local features, like the singular feature, that challenge the traditional atomistic view about space-time (as in Lewis' Humean supervenience thesis). According to this received view, space-time is conceived as a set of points, at which intrinsic properties are instantiated, together with the space-time relations. Indeed, the very concept of a space-time point seems to lie at the heart of the challenge. It cannot be merely postulated anymore (as in the field argument), since it is indeed a secondary derived notion within the algebraic formulation of GR. This latter formualtion may with reason be considered as deserving to play a role in the interpretative issues about space-time - at least to the same extent as the standard differential geometric formulation does. Actually, the alleged interpretational problems with respect to space-time singularities may find part of their roots in the misleading dependence on the atomisitc and local conception of space-time, which is actually induced by this standard differential geometric representation of space-time. And this gives a structuralist flavour to space-time as described by GR and independently of the formulation. But this is a story for another time.

References

Bain, J. (2003), “Einstein Algebras and the Hole Argument”, *Philosophy of Science* 70: 1073-1085.

Bosshard, B. (1976), “On the *b*-boundary of the closed Friedmann model”, *Communications in Mathematical Physics* 46: 263-268.

Butterfield, J. (2006), “Against *Pointillisme* about Mechanics”, *British Journal for the Philosophy of Science* 57: 709-753.

—— and C. Isham (2001), “Spacetime and the philosophical challenge of quantum gravity”, in C. Callender and N. Huggett (eds.), *Physics meets philosophy at the Planck scale*, Cambridge: Cambridge University Press, 33-89.

Cleland, C. (1984), “Space: An Abstract System of Non-Supervenient Relations”, *Philosophical Studies* 46:19-40.

Curiel, E. (1999), “The Analysis of Singular Spacetimes”, *Philosophy of Science* 66 (Proceedings): S119-145.

Demaret, J., M. Heller, and D. Lambert (1997), “Local and Global Properties of the World”, *Foundations of Science* 2: 137-176.

Dorato, M. (1998), review of Earman 1995, *British Journal for the Philosophy of Science* 49: 338-347.

—— (2000), “Substantivalism, relationism, and structural spacetime realism”, *Foundations of Physics* 30: 1605-1628.

Earman, J. (1987), “Locality, Nonlocality and Action at a Distance: A Skeptical Review of Some Philosophical Dogmas”, in R. Kargon and P. Achinstein (eds.), *Kelvins Baltimore Lectures and Modern Theoretical Physics*, Cambridge: Massachusetts Institute of Technology Press.

—— (1995), *Bangs, Crunches, Whimpers, and Shrieks: Singularities and Acausalities in Relativistic Spacetimes*, New York: Oxford University Press.

—— (1996), “Tolerance for Spacetime Singularities”, *Foundations of Physics* 26: 623-640.

Esfeld, M. and V. Lam (2007), “Moderate structural realism about spacetime”, fortcoming in *Synthese*.

Field, H. (1980), *Science without numbers. A defence of nominalism.*, Oxford: Blackwell.

Geroch, R. (1972), "Einstein Algebras", *Communications of Mathematical Physics* 26: 271-275.

——, L. Can-bin, and R. Wald (1982), "Singular boundaries of space-times", *Journal of Mathematical Physics* 23: 432-435.

Heller, M. (2001), "The Classical Singularity Problem - History and Current Research", in V. Martinez, V. Trimble, and M. Pons-Borderia (eds.), "Historical Development of Modern Cosmology", *ASP Conferences Series* 252: 121-145.

Johnson, R. (1977), "The Bundle Boundary in Some Special Cases", *Journal of Mathematical Physics* 18: 898-902.

Langton, R., and D. Lewis (1998), "Defining 'intrinsic' ", *Philosophy and Phenomenological Research* 58: 333-345.

Mallios, A. and I. Raptis (2003), "Finitary, Causal and Quantal Vacuum Einstein Gravity", *International Journal of Theoretical Physics* 42: 1479-1619.

Mattingly, J. (2001), "Singularities and Scalar Fields: Matter Theory and General Relativity", *Philosophy of Science* 68 (Proceedings): S395-406.

Rickles, D. and S. French (2007), "Quantum Gravity Meets Structuralism: Interweaving Relations in the Foundations of Physics", in D. Rickles, S. French and J. Staasi (eds.), *The Structural Foundations of Quantum Gravity*, Oxford: Oxford University Press.

Rovelli, C. (2004), *Quantum gravity*, Cambridge: Cambridge University Press.

Schmidt, B. (1971), "A New Definition of Singular Points in General Relativity", *General Relativity and Gravitation* 1: 269-280.

Scott, S. and P. Szekeres (1994), "The abstract boundary - a new approach to singularities of manifolds", *Journal of Geometry and Physics* 13: 223-253.

Senovilla, J. (1997), "Singularity Theorems and Their Consequences", *General Relativity and Gravitation* 29: 701-848.

Wald, R. (1984), *General Relativity*, Chicago: The University of Chicago Press.

Notes

[1]Some notable exceptions are Earman 1996, Curiel 1999 and Mattingly 2001 for instance.

[2]They may merely not occur in our universe if one of the (necessary) hypotheses of the singularity theorems were violated, see Mattingly 2001. For a detailed physical discussion of these hypotheses, see Senovilla 1997.

[3]An extension of a space-time Lorentz manifold (M, g) is any space-time Lorentz manifold (M', g') of same dimension where (M', φ) is an envelopment of M and such that $\varphi^*(g) = g'|_{\varphi(M)}$ holds.

[4]A space-time Lorentz manifold (M, g) is maximal with respect to some C^k-condition if there is no extension (M', g') where the metric g' is C^k at the boundary $\partial' M$ of $\varphi(M)$ in M'.

[5]A possible guideline would be to require that these conditions secure that the fundamental laws of GR, that is, the Einstein field equations and the Bianchi identity, are well defined, see Earman 1995, §2.7.

[6]However, the notion of curve incompleteness does not encompass all aspects of space-time singularities (like for instance certain aspects linked with the violation of the cosmic censorship).

[7]The generalized affine parameter u for a C^1-half-curve $\gamma(t)$ is defined by $u := \int_0^t (\sum_{\alpha=0}^3 (V^\alpha(t))^2)^{\frac{1}{2}} dt$, where $V(t') = V^\alpha(t') e_\alpha(p)$, $p = \gamma(t')$, is the tangent vector expressed in the parallel propagated orthonormal basis e_α.

[8]Space-time is represented within GR by a pair (M, g), where M is in general assumed to be a 'nice' (paracompact, connected, Hausdorff, oriented) 4-dimensional differentiable manifold and g is a C^k ($k \geq 2$ in general) Lorentz metric, solution of the Einstein field equations and defined everywhere on M.

[9]A topological space $\overline{M}$ is Hausdorff if $\forall p, q \in \overline{M}, p \neq q, \exists$ open sets $U, V \subset \overline{M}$ such that $p \in U$ and $q \in V$ and $U \cap V = \emptyset$; p and q are said to be Hausdorff separated. So, if two points of a topological space (like $\overline{M}$) are not Hausdorff separated, it is not possible for them to find two ('arbitrarily small') disjoint neighborhoods (open sets): it is in this topological sense that they can be considered as 'arbitrarily close'.

[10]The same problem arises in the case of the Schwarzschild solution.

[11]From the semantical point of view, this favours then the "adjective conceptions of space-time singularities" (Earman 1995, 28).

[12]In the standard view, intrinsic properties are those whose instantiation is independent of accompaniment or loneliness, see Langton and Lewis 1998.

[13]Or pointlike bits of matter; the main claim here does not side with any position in the substantivalism-relationalism debate, but for simplicity we mainly use the space-time points talk.

[14]To include merely the non-local features of space-time in the supervenience basis would be a rather ad hoc solution.

[15]Butterfield has recently argued that also classical mechanics excludes this atomistic conception about space-time and the world, which, following Lewis, he calls 'pointillisme' (Butterfield 2006).

[16]For instance, we have just seen that there is a strong argument against intrinsic properties of space-time points, reducing therefore their ontological weight: at best, they are on the same ontological footing as space-time relations, their identity being entierly determined by relational properties, see Esfeld and Lam 2007.

[17]Indeed, it seems that it is exactly what it is done in the recent 'sophisticated' substantivalist and relationalist positions that seek to account for the hole arguement, see Rickles and French 2007 and references therein.

[18]This amounts to reducing the 'ontological space' of local entities to zero, i.e. rejecting them from our ontology.

[19]However this does not entail that GR is either false or incomplete, see Earman 1996.

[20]A maximal ideal of a commutative algebra $\mathcal{A}$ is the largest proper subset of - indeed a subgroup of the additive group of - $\mathcal{A}$ closed under multiplication by any element of $\mathcal{A}$. The corresponding maximal ideal of $C^\infty(M)$ to a point $p \in M$ is the set of all vanishing functions at p.

[21]The original idea is due to Geroch 1972.

[22]There are indeed several algebraic approaches to GR. For instance, according to the Abstract Differential Geometry program of Mallios and Raptis, space-time singularities are merely direct artifacts of our mathematical (C^∞-)representation of space-time: indeed, they simply disappear once GR is written in purely algebraic (sheaf-theoretic) terms, see Mallios and Raptis 2003. In the following, we rather brielfy consider the (less radical) approach of Heller, which emphasizes some interesting points for our discussion.

[23]In the algebraic formalism, vector fields are abstract 'derivations'.

[24]The hole argument has been recently discussed in (Bain 2003) within the framework of the algebraic formulation of GR.

[25]And this does not even take into account the fact that, within sheaf-theoretic or non-commutative generalizations, the very concept of a point may be challenged at the fundamental level.