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Completeness of $S4$ with Respect to the Lebesgue Measure Algebra Based on the Unit Interval

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Abstract We prove completeness of the propositional modal logic $S4$ for the measure algebra based on the Lebesgue-measurable subsets of the unit interval, [0, 1]. In recent talks, Dana Scott introduced a new measure-based semantics for the standard propositional modal language with Boolean connectives and necessity and possibility operators, $\Box$ and $\Diamond$. Propositional modal formulae are assigned to Lebesgue-measurable subsets of the real interval [0, 1], modulo sets of measure zero. Equivalence classes of Lebesgue-measurable subsets form a measure algebra, $\mathcal{M}$, and we add to this a non-trivial interior operator constructed from the frame of 'open' elements—elements in $\mathcal{M}$ with an open representative. We prove completeness of the modal logic $S4$ for the algebra $\mathcal{M}$. A corollary to the main result is that non-theorems of $S4$ can be falsified at each point in a subset of the real interval [0, 1] of measure arbitrarily close to 1. A second corollary is that Intuitionistic propositional logic (IPC) is complete for the frame of open elements in $\mathcal{M}$.

Keywords Measure algebra · Topological modal logic · Topological semantics · $S4$ · Completeness · Modal logic · Probabilistic semantics

1 Introduction

In 1944, Tarski and McKinsey proved that the propositional modal logic $S4$ is complete for any dense-in-itself metric space.[1] An important special case that

[1] See [5].

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has received much recent attention (see [1, 4, 6]) is completeness of $S4$ for (topological interpretations over) the real line. In this paper we study the real line from a measure-theoretic point of view, and prove that $S4$ is complete for a measure algebra, $\mathcal{M}$, based on the Lebesgue-measure structure of the real interval, [0, 1].

The algebra, $\mathcal{M}$, was introduced in recent talks by Dana Scott, not only as a model for the standard propositional modal language (with Boolean connectives and necessity, $\Box$, and possibility, $\Diamond$, operators), but also for higher order logic and a modal set theory.[2] $\mathcal{M}$ is the algebra of Lebesgue-measurable subsets of the real interval [0, 1], modulo sets of Lebesgue-measure zero (henceforth, null sets). Formulae are interpreted in $\mathcal{M}$ by assigning them to elements in $\mathcal{M}$, i.e. equivalence classes of Lebesgue-measurable sets. Thus, under a fixed valuation, each formula also gets assigned a probability, namely the probability corresponding to the measure of all sets in its equivalence class. This allows us to speak of the probability of a modal formula under a given valuation. Indeed, this was part of Scott's original motivation in introducing this new "probabilistic" semantics. "This provides rich ingredients," Scott writes, "for building many kinds of structures having non-standard random elements."[9]

My own motivation—and the motivation behind this paper—is in terms of completeness. Tarski and McKinsey's 1944 result can be thought of as a logical investigation of the topological structure of the real line. In the topological semantics, the modal operators $\Box$ and $\Diamond$ are interpreted as topological interior and closure, respectively. Here, in Scott's semantics, we construct a non-trivial interior operator for the algebra $\mathcal{M}$, based on the open subframe of $\mathcal{M}$— collection of elements of $\mathcal{M}$ with open representatives (see Definition 3.9). The modally expanded measure structure captures, in some sense, the Lebesgue measure structure of the real interval [0, 1] (subsets of the interval are identified just in case they differ by a set of measure zero), and its interior operator behaves quite differently from the topological interior operator, as we will see below. Just as Tarski and McKinsey investigated completeness of $S4$ for topological models, in particular topological models based on the reals, we can investigate completeness for this (modally expanded) measure structure. Indeed, there is no reason to restrict ourselves to Lebesgue-measure on the real interval. We can look at measure-structures more generally—those based on a given topological space with a measure defined on the Borel subsets—and ask whether completeness holds. (We should mention at this point that many such measure-structures are isomorphic to $\mathcal{M}$ by well known results, and thus that the algebra $\mathcal{M}$ is one of the most important measure algebras (see [7]). Thus the algebra $\mathcal{M}$ is one of the most important.)

The proof of completeness presented here uses ideas from more recent proofs of Tarski and McKinsey's 1944 result, but is essentially different

[2]The algebra itself is of course well known, but to my knowledge Scott was the first to interpret propositional modal formulae in $\mathcal{M}$ in the way described here.

in that measure-theoretic notions take center-stage. The proof proceeds by embedding the complete binary tree (defined below) in the algebra, $\mathcal{M}$. We transfer valuations over the tree to measurable subsets of the real line, and finally to equivalence classes of such subsets. A simple and known embedding of finite Kripke frames into the tree shows how such finite frames can be embedded in $\mathcal{M}$. Thus, completeness follows from completeness for the finite frames, or alternatively, completeness for the tree.

In Section 2, we recall the algebraic semantics for the standard propositional modal language. In Section 3, we introduce the measure algebra, $\mathcal{M}$. In Section 4, we explain the motivation for some of the constructions that follow. In Section 5, we construct a map that allows us to transfer completeness of $S4$ from the complete binary tree to the real interval [0, 1]. This map is different from other maps that appear in the literature, in that it has important measure-theoretic properties that provide the key to transferring completeness to $\mathcal{M}$. In Section 6, we prove our main result: completeness of $S4$ for the algebra, $\mathcal{M}$. By the Gödel translation from Intuitionistic propositional logic (IPC) to $S4$, this yields completeness of IPC for the frame of 'open' elements in $\mathcal{M}$ (defined below).

2 Topological Semantics from an Algebraic Point of View

Let the propositional modal language L consist of a countable set, $\mathbb{P} = \{P_i \mid \text{for all } i \in \mathbb{N}\}$, of atomic variables and be closed under binary connectives $\to, \vee, \wedge, \leftrightarrow$ and unary operators $\neg, \Box, \Diamond$. The modal logic $S4$ in the language L consists of some complete axiomatization of classical propositional logic PL, some complete axiomatization of the minimal normal modal logic K, and finally the two special $S4$ axioms:

$$4 : \Box P \to \Box\Box P$$

$$T : \Box P \to P$$

We are interested in algebraic models of the modal system $S4$, or topological Boolean algebras. A *topological Boolean algebra* (henceforth TBA) is a Boolean algebra with an *interior* operator, I, satisfying the following properties:

(I_1) $Ia \leq a$
(I_2) $I(a \wedge b) = Ia \wedge Ib$
(I_3) $IIa = Ia$
(I_4) $I(1) = 1$

A complete TBA is a TBA that is a complete lattice—i.e. every collection of elements has a supremum and infimum.

Example 1 (Topological field of sets) The set of subsets $\wp(\mathcal{X})$ of a topological space $\mathcal{X}$ with set-theoretic meets, joins and complements, and where Ia

denotes the (topological) interior of a, is a complete TBA and we denote it by $B(\mathcal{X})$. More generally, any Boolean algebra, $\mathcal{A}$, of subsets of a topological space $\mathcal{X}$ that is closed under topological interiors is a TBA ($\mathcal{A}$ need not contain all subsets of $\mathcal{X}$). We call any such algebra a *topological field of sets*. Note that we reserve the notation $B(\mathcal{X})$ for the topological Boolean algebra generated by *all* subsets of $\mathcal{X}$.

Let $\mathcal{A}$ be a TBA and let $f : \mathbb{P} \to \mathcal{A}$ be a function assigning propositional variables to arbitrary elements of the lattice $\mathcal{A}$. We call any such function a *valuation*. f can be extended to a function, h_f, on the set of all formulae by recursion as follows.

For any propositional variable P and any formulae ϕ and ψ let:

$$h_f(P) = f(P)$$

$$h_f(\phi \mathbin{\&} \psi) = h_f(\phi) \wedge h_f(\psi)$$

$$h_f(\phi \vee \psi) = h_f(\phi) \vee h_f(\psi)$$

$$h_f(\neg\phi) = -h_f(\phi)$$

$$h_f(\Box\phi) = I(h_f(\phi))$$

where symbols on the RHS denote (in order) the lattice meet, join and complement. (The remaining binary connectives $\{\rightarrow, \leftrightarrow\}$ and unary operator $\{\Diamond\}$ are defined in terms of the above in the usual way.)

Let $\mathcal{A}$ be a TBA. For any formula, ϕ, and valuation, f, over $\mathcal{A}$, we say ϕ is *satisfied by* f ($\mathcal{A}, f \models \phi$) iff $h_f(\phi) = 1_{\mathcal{A}}$ (the top element in the algebra); otherwise, ϕ is *falsified* by f. We say ϕ is *satisfied in* $\mathcal{A}$ ($\mathcal{A} \models \phi$) iff ϕ is satisfied by every valuation over $\mathcal{A}$. Finally, for any class C of TBA's, ϕ is *satisfied in* C ($\models_C \phi$) iff ϕ is satisfied in every TBA in C.[3]

We now define completeness in the usual way: A logic S is *complete* for a class, C, of TBA's if every formula that is satisfied in C is provable in S ($\models_C \phi \;\Rightarrow\; \vdash_S \phi$). An equivalent formulation is more useful in what follows: S is complete for C if any non-theorem of S is falsified in C ($\nvdash_S \phi \Rightarrow \not\models_C \phi$).

Note that if $\mathcal{A}$ is a topological field of sets, e.g. $B(\mathcal{X})$ for some topology $\mathcal{X}$, it makes sense to talk about truth *at a point* (much like truth at a world in Kripke semantics for the standard propositional modal language). For any formula ϕ, valuation $f : \mathbb{P} \to B(\mathcal{X})$, and point $x \in \mathcal{X}$, we can say that ϕ is true at x under f if

$$x \in h_f(\phi)$$

This ternary relation between a valuation, formula and point in the topological space has no place in the general algebraic semantics (i.e. where $\mathcal{A}$ need not be

[3]This semantics can be generalized by defining a set of designated elements, $D_{\mathcal{A}}$, of $\mathcal{A}$ and letting satisfaction by a valuation f over $\mathcal{A}$ be defined by: $h_f(\phi) \in D_{\mathcal{A}}$. The definition used in this paper is the special case where $D_{\mathcal{A}} = \{1_{\mathcal{A}}\}$.

a topological field of sets) and has no analog when it comes to the Lebesgue measure algebra, $\mathcal{M}$ (defined below). Formulae are evaluated to equivalence classes of Lebesgue-measureable subsets of [0, 1], but it makes no sense to speak of an individual point in [0, 1] belonging to an equivalence class, hence of a given formula being true at an individual point!

3 The Lebesgue Measure Algebra, $\mathcal{M}$

In this section we define our central object of study: the measure algebra, $\mathcal{M}$. We prove that $\mathcal{M}$ is a complete Boolean algebra, and that the sublattice of 'open' elements in $\mathcal{M}$ (see Definition 3.9) forms a complete Heyting algebra.

Definition 3.1 Let $\mathcal{A}$ be a Boolean algebra. We say that a non-empty subset $I \subseteq \mathcal{A}$ is an *ideal* if

1. For all $a, b \in I, a \vee b \in I$
2. If $a \in I$ and $b \leq a$, then $b \in I$

If I is closed under countable suprema, we say I is a σ*-ideal*.

We can construct new Boolean algebras from existing ones by quotienting by an ideal. If $\mathcal{A}$ is a Boolean algebra and $I \subseteq \mathcal{A}$ is an ideal, we define the correspondence $\sim$ on $\mathcal{A}$ by:

$$x \sim y \text{ iff } (x \triangle y) \in I$$

where $\triangle$ denotes symmetric difference.[4] Letting $\mathcal{A}/I$ be the set of equivalence classes and $|x|$ be the equivalence class corresponding to $x \in \mathcal{A}$, the operations $\boldsymbol{\vee}$, $\boldsymbol{\wedge}$ and $\boldsymbol{-}$ on $\mathcal{A}/I$ are defined in the obvious way:

$$\begin{aligned} |x| \ \boldsymbol{\vee} \ |y| &= |x \vee y| \\ |x| \ \boldsymbol{\wedge} \ |y| &= |x \wedge y| \\ \boldsymbol{-}|x| &= |-x| \end{aligned} \tag{1}$$

It is easy to verify that $\mathcal{A}/I$ is a Boolean algebra with top and bottom elements $|1_{\mathcal{A}}|$ and $|0_{\mathcal{A}}|$, respectively. From the definitions of $\boldsymbol{\vee}$ and $\boldsymbol{\wedge}$ we can reconstruct the lattice order $\leq$: for any $|x|, |y| \in \mathcal{A}/I$,

$$|x| \leq |y| \ \text{ iff } \ |x| \ \boldsymbol{\wedge} \ |y| = |x|$$

[4]Note that differences and symmetric differences are defined in any Boolean algebra, not just in fields of sets. In particular, $x - y$ is $x \wedge -y$, and $x \triangle y$ is $(x - y) \vee (y - x)$.

Lemma 3.2 *Let $\mathcal{A}$ be a Boolean Algebra and I an ideal in $\mathcal{A}$. Then for any elements a, b in the quotient algebra $\mathcal{A}/I$, the following are equivalent:*

(i) $a \leq b$
(ii) *For any representatives A of a, and B of b, there exists some $N \in I$ with $A \leq B \vee N$.*
(iii) *For any representative A of a there exists a representative B of b with $A \leq B$.*

Proof

(i) → (ii) Suppose $a \leq b$ and let $a = |A|$, $b = |B|$. Then $|A \wedge B| = |A| \wedge |B| = |A|$, so $A \wedge B \sim A$. Thus $A - B = A - (A \wedge B) = N$ for some $N \in I$. It follows that $A \leq B \vee N$.
(ii) → (iii) This follows from the fact that $B \vee N \sim B$ for $N \in I$.
(iii) → (i) If $A \leq B$, then $|A| \wedge |B| = |A \wedge B| = |A|$, and $a = |A| \leq |B| = b$. □

We want to add measure-structure to Boolean algebras. The simplest such structures are Boolean algebras carrying a finitely additive measure. We are interested, however, in Boolean σ-algebras carrying a countably additive measure. The relevant definition is given below.

Definition 3.3 A *measure*, μ, on a Boolean σ-algebra,[5] $\mathcal{A}$, is a real-valued function μ on $\mathcal{A}$ that satisfies countable additivity: If $\{F_n\}_{n\in\mathbb{N}}$ is a countable collection of elements in $\mathcal{A}$ with $F_n \wedge F_m = 0_{\mathcal{A}}$ for all $n, m \in \mathbb{N}$, then

$$\mu\left(\bigvee_n F_n\right) = \sum_n \mu(F_n)$$

We say that a measure, μ, on a Boolean σ-algebra is *normalized* if $\mu(1) = 1$. We say that μ is *positive* if $\mu(a) = 0$ iff $a = 0_{\mathcal{A}}$.

Definition 3.4 (Halmos) A measure algebra is a Boolean σ-algebra, $\mathcal{A}$, together with a positive, normalized measure, μ, on $\mathcal{A}$.

Fact 3.5 *Let μ be a normalized measure on a Boolean σ-algebra, $\mathcal{A}$, and let U be the set of elements $a \in \mathcal{A}$ with $\mu(a) = 0$. Then,*

(i) *U is a σ-ideal in $\mathcal{A}$*
(ii) *The quotient $\mathcal{A}/U$ is a Boolean σ-algebra.*
(iii) *There exists a unique measure ν on $\mathcal{A}/U$ defined by*

$$\nu(|a|) = \mu(a)$$

Moreover, ν is positive and normalized.

[5] A Boolean σ-algebra is a Boolean algebra that is closed under countable joins (and meets).

Proof

(i) If $a \leq b$, and $\mu(b) = 0$, we write $b = a \vee (b - a)$. But then $\mu(a) \leq \mu(b)$, by additivity of μ, so $\mu(a) = 0$. If $\{a_n \,|\, n \in \mathbb{N}\}$ is a countable collection of elements in $\mathcal{A}$ with $\mu(a_n) = 0$ for all $n \in \mathbb{N}$, then by countable subadditivity of μ, $\mu(\bigvee_n a_n) \leq \sum_n \mu(a_n) = 0$.

(ii) We need to show that the quotient algebra $\mathcal{A}/U$ is closed under countable joins. Let $\{a_n \,|\, n \in \mathbb{N}\}$ be a collection of elements in $\mathcal{A}/U$, with $a_n = |A_n|$ for each $n \in \mathbb{N}$. We claim $\bigvee_n a_n = |\bigvee_n A_n|$. Clearly $|\bigvee_n A_n|$ is an upper bound on $\{a_n \,|\, n \in \mathbb{N}\}$. If $b = |B|$ is an upper bound on $\{a_n \,|\, n \in \mathbb{N}\}$, then $|A_n| = a_n \leq |B|$, and $A_n \leq B \vee N_n$ for some $N_n \in U$ (see Lemma 3.2). But then $\bigvee_n A_n \leq B \vee \bigvee_n N_n$, and $\bigvee_n N_n \in U$ (since U is a σ-ideal). So $|\bigvee_n A_n| \leq |B| = b$.

(iii) The proof can be found in, e.g., [2]. □

Definition 3.6 (The Lebesgue Measure Algebra, $\mathcal{M}$) Let $Leb([0, 1])$ be the σ-algebra of Lebesgue-measurable subsets of the unit interval $[0, 1]$, and let $Null$ be the set of subsets of $[0, 1]$ with Lebesgue-measure zero. Then by Fact 3.5, the quotient algebra, $Leb([0, 1])/Null$, is a measure algebra. We denote it by $\mathcal{M}$ and refer to it as the Lebesgue measure algebra.

In what follows, we use uppercase letters $A, B, C...$ to denote subsets of $[0, 1]$ and lower-case letters $a, b, c...$ to denote elements of $\mathcal{M}$. Equivalence classes of measurable sets are denoted with a bar above the relevant set (e.g. $a = \overline{A}$, $0_{\mathcal{M}} = \overline{\emptyset}$, $1_{\mathcal{M}} = \overline{[0, 1]}$). We use 'measure (A)' or simply '$m(A)$' to denote the measure of the set A. The definitions in (1) give: for any elements $a, b \in \mathcal{M}$ with $a = \overline{A}$ and $b = \overline{B}$: $a \vee b = \overline{A \cup B}$, $a \wedge b = \overline{A \cap B}$ and $-a = \overline{[0, 1] - A}$.

Lemma 3.7 *For any sets $A, B \in Leb([0, 1])$,*

$$A \sim B \quad iff \quad \overline{A} \leq \overline{B} \text{ and } m(A) = m(B)$$

Proof The left–to–right direction is obvious. For the right–to–left direction, suppose $\overline{A} \leq \overline{B}$ and $m(A) = m(B)$. Then $A \subseteq B \cup N$ for some $N \in Null$, so $m(A - B) = 0$. Furthermore,

$$m(B - A) = m(B) - m(B \cap A) = m(A) - m(B \cap A) = m(A - B)$$

and we have $m(B - A) = 0$. Thus $A \triangle B \in Null$ and $A \sim B$. □

Lemma 3.7 tells us that all representatives of a given element in $\mathcal{M}$ have the same measure. This allows us, in evaluating formulae to $\mathcal{M}$, to speak of the probability of a given formula under a fixed valuation.

Proposition 3.8 *$\mathcal{M}$ is a complete Boolean algebra*

Proof[6] We show that any well-ordered subset S of $\mathcal{M}$ has a least upper bound. The proof is by transfinite induction on the order type of S. Let S have order type α and write $S = \{p_\gamma \mid \gamma < \alpha\}$. For $\beta < \alpha$, let $q_\beta = sup\,\{p_\gamma \mid \gamma < \beta\}$ (existence follows from the inductive hypothesis). If α is a limit ordinal then $\{q_\beta \mid \beta < \alpha\}$ is a non-decreasing sequence of elements in $\mathcal{M}$ and $\{m(q_\beta) \mid \beta < \alpha\}$ is a non-decreasing sequence of reals. But note that there are only *countably* many distinct reals in this sequence (for each "jump" between two reals in the sequence, there is a distinct rational number.) It follows from Lemma 3.7 that there are only *countably* many distinct elements 'q_β' in the sequence $\{q_\beta \mid \beta < \alpha\}$. But $\mathcal{M}$ is closed under countable suprema (see Fact 3.5 (ii)), so $sup\, S = sup\,\{q_\beta \mid \beta < \alpha\}$ exists. □

By contrast, *Leb* ([0, 1]) is *not* a complete Boolean algebra. If, e.g., S is a non-measurable subset of [0, 1], then the collection $\{\{x\} \mid x \in S\}$ has no supremum in *Leb* ([0, 1]). Note that the Lebesgue measure, μ_L, on *Leb* ([0, 1]) is *not* a positive measure - any non-empty countable set has measure zero, but is not equal to the bottom element, $\emptyset$. Indeed, it is proved in [2] that every (positive, normalized) measure algebra is complete.

Definition 3.9 We say an element $a \in \mathcal{M}$ is *open* if some representative A of a is an open subset of [0, 1]. We denote the set of open elements in $\mathcal{M}$ by $\mathcal{G}$.

The next proposition states that not all elements of $\mathcal{M}$ are open.

Proposition 3.10 $\mathcal{M} \neq \mathcal{G}$

Proof The proof is postponed until Section 5.3, where we introduce 'thick' Cantor sets. □

In the next proposition we show that open elements in $\mathcal{M}$ form a complete Heyting algebra. Recall that a complete Heyting algebra is a complete lattice that satisfies the following infinite distributive law: For any $x \in A$ and $\{a_i \mid i \in I\} \subseteq A$,

$$x \wedge \bigvee_i a_i = \bigvee_i (x \wedge a_i) \tag{2}$$

[6]This proof was suggested to the author by Dana Scott. In fact, the more general claim that every (positive, normalized) measure algebra is complete is proved in [2]. The proof procedes by showing that an algebra is complete iff it satisfies the countable chain condition, and that any measure algebra so defined satisfies this condition.

Proposition 3.11 *$\mathcal{G}$ is a complete Heyting algebra.*[7]

Proof We need to show that $\mathcal{G}$ is a complete lattice. Let $\{a_i \mid i \in I\} \subseteq \mathcal{G}$, and let $a_i = \overline{A_i}$ for each $i \in I$, with A_i an *open* representative of a_i. Let $\{(p_n, q_n) \mid n \in \mathbb{N}\}$ be the collection of open rational intervals (open intervals with rational endpoints) contained in some (or other) A_i. We claim that $\bigvee_i a_i = \overline{\bigcup_n (p_n, q_n)}$. Clearly RHS is an upper bound on $\{a_i \mid i \in I\}$ (this follows from the fact that each open set, A_i is equal to the union of rational intervals contained in it). Suppose $b = \overline{B}$ is an upper bound on $\{a_i \mid i \in I\}$ with $b \in \mathcal{G}$.[8] For each $i \in I$, choose $N_i \in Null$ such that $A_i \subseteq B \cup N_i$. For each $n \in \mathbb{N}$, choose $i(n)$ such that $(p_n, q_n) \subseteq A_{i(n)}$. We have: $\bigcup_n (p_n, q_n) \subseteq \bigcup_n A_{i(n)} \subseteq B \cup \bigcup_n N_{i(n)}$, where $\bigcup_n N_{i(n)} \in Null$. So $\overline{\bigcup_n (p_n, q_n)} \leq \overline{B} = b$, proving the claim. This shows that every collection of elements in $\mathcal{G}$ has a supremum. What about infima? Consider now the collection of $\{b_j \mid j \in J\}$ of lower bounds in $\mathcal{G}$[9] on $\{a_i \mid i \in I\}$. This collection has a supremum, b. We claim that $b = \bigwedge_i a_i$. The proof is similar to the previous and is left to the reader.

Note that the proof shows that $\bigvee_i a_i = \overline{\bigcup_i A_i}$, where A_i is any open representative of a_i (for $i \in I$). We use this fact to show that $\mathcal{G}$ satisfies the distributive law (2), as follows. Let $x = \overline{X}$, with X an *open* representative. Then,

$$\begin{aligned} x \wedge \bigvee_i a_i &= \overline{X} \wedge \overline{\bigcup_i A_i} \\ &= \overline{X \cap \bigcup_i A_i} \\ &= \overline{\bigcup_i (X \cap A_i)} \\ &= \bigvee_i (\overline{X \cap A_i}) \\ &= \bigvee_i (x \wedge a_i) \end{aligned}$$

□

[7] In general, infima in $\mathcal{G}$ and $\mathcal{M}$ do not coincide. Example: For each $n \in \mathbb{N}$, let K_n denote the set of points belonging to "remaining intervals" at the n-th stage of construction of K (defined in Section 5.3). Then $\overline{K_n} \in \mathcal{G}$ for each $n \in \mathbb{N}$, but $inf_{\mathcal{M}}\{\overline{K_n} \mid n \in \mathbb{N}\} = \overline{K}$, and $inf_{\mathcal{G}}\{\overline{K_n} \mid n \in \mathbb{N}\} = \overline{\emptyset}$ (where $inf_{\mathcal{M}}$ and $inf_{\mathcal{G}}$ denote infima in $\mathcal{M}$ and $\mathcal{G}$, respectively).

[8] The reader can verify that the condition $b \in \mathcal{G}$ does no work in the proof. Indeed, this shows that suprema in $\mathcal{M}$ and $\mathcal{G}$ coincide. This is not the case for infima (see footnote 6).

[9] It is crucial that we take lower bounds in $\mathcal{G}$ and not in the larger $\mathcal{M}$. In general, the set of lower bounds in $\mathcal{G}$ and $\mathcal{M}$ do not coincide! See footnote 6.

We want to transform the Boolean algebra, $\mathcal{M}$, into a TBA. To do this, we equip $\mathcal{M}$ with the unary operator I defined as follows:

$$Ia = sup\left\{b \text{ open} \,|\, b \leq a\right\} \tag{3}$$

for any element $a \in \mathcal{M}$. A natural question is: Why not define the operator I in terms of the interior operator on underlying sets (just as Boolean operations on $\mathcal{M}$ are defined in terms of Boolean operations on underlying sets):

$$I(\overline{A}) = \overline{I(A)} \tag{3*}$$

A simple example shows that definition (3*) is not correct (i.e. not well-defined). Let $A = [0, 1] - \mathbb{Q}$. Then $A \sim [0, 1]$. But Interior $(A) = \emptyset$, and Interior $([0, 1]) = [0, 1]$. So according to (3*), $\overline{[0, 1]} = I(\overline{A}) = \overline{\emptyset}$.

Indeed, the example shows that the interior operator in the topological fields of sets Leb([0,1]) and $B([0, 1])$ behaves quite differently from the interior operator in $\mathcal{M}$. This is crucial in what follows, where, despite this difference, we aim to transfer valuations over $B([0, 1])$ to $\mathcal{M}$.

Proposition 3.12 *I is an interior operator.*

Proof Let $a, b \in \mathcal{M}$. Axiom (l_1) is obvious. For (l_2), note that $I(a \wedge b) \leq I(a)$ and $I(a \wedge b) \leq I(b)$. So $I(a \wedge b) \leq Ia \wedge Ib$. For the reverse inequality, note that $Ia \leq a$ and $Ib \leq b$. Thus $Ia \wedge Ib \leq a \wedge b$. Moreover, $(Ia \wedge Ib) \in \mathcal{G}$. It follows that $Ia \wedge Ib \leq sup\,\{c \in \mathcal{G} \,|\, c \leq a \wedge b\} = I(a \wedge b)$. For (l_3) note that $Ia \in \mathcal{G}$, and $Ia \leq Ia$, giving $Ia \leq sup\,\{c \in \mathcal{G} \,|\, c \leq Ia\}$. By (l_1) we also have $IIa \leq Ia$. Finally for (l_4), note that $\overline{[0, 1]} \in \mathcal{G}$. Thus $I\,\overline{[0, 1]} = sup\,\{c \in \mathcal{G} \,|\, c \leq \overline{[0, 1]}\} = \overline{[0, 1]}$. □

Corollary 3.13 *The Measure Algebra, $\mathcal{M}$, with unary operator I is a TBA.*

Proof Immediate from Propositions 3.8 and 3.12. □

In general, there is no easy way to calculate the supremum of an uncountable collection of elements in $\mathcal{M}$, as indicated by the non-constructive proof of Proposition 3.8. However, when we calculate Ia, we take the supremum of a collection of *open* elements, and arbitrary joins of open elements are well-behaved (see proof of Proposition 3.11). The following proposition shows how to calculate the interior operator in $\mathcal{M}$ in terms of underlying sets.

Proposition 3.14 *Let $a \in \mathcal{M}$ and let $\{(p_n, q_n) \,|\, n \in \mathbb{N}\}$ be an enumeration of open rational intervals (open intervals with rational endpoints) contained in some (or other) representative A of a. Then, $Ia = \overline{\bigcup_n (p_n, q_n)}$.*

Proof The proof is similar to the proof of Proposition 3.11. We need to show that $\overline{\bigcup_n (p_n, q_n)} = sup\,\{c \in \mathcal{G} \,|\, c \leq a\}$. Suppose that $c \in \mathcal{G}$ and $c \leq a$. Then $c = \overline{C}$ for some open representative C and $C \subseteq A$ for some representative A of a (see

Lemma 3.2). Since C is open, C can be written as the union of open rational intervals contained in C. Each such interval is also contained in A, so $C \subseteq \bigcup_n (p_n, q_n)$, and $c \leq \overline{\bigcup_n (p_n, q_n)}$. This shows $\overline{\bigcup_n (p_n, q_n)}$ is an upper bound on $\{c \in \mathcal{G} \mid c \leq a\}$. Now suppose that $b = \overline{B}$ is an upper bound on $\{c \in \mathcal{G} \mid c \leq a\}$. Then, for each $n \in \mathbb{N}$, $\overline{(p_n, q_n)} \leq b$, and $(p_n, q_n) \subseteq B \cup N_n$ for some $N_n \in Null$. So $\bigcup_n (p_n, q_n) \subseteq B \cup \bigcup_n N_n$ and $\overline{\bigcup_n (p_n, q_n)} \leq b$. This shows that $\overline{\bigcup_n (p_n, q_n)}$ is the least upper bound on $\{c \in \mathcal{G} \mid c \leq a\}$. □

We state without proof an obvious corollary which represents the interior in $\mathcal{M}$ in terms of open sets rather than rational intervals:

Corollary 3.15 *For any $a \in \mathcal{M}$,*

$$Ia = \overline{\bigcup \{O \text{ open} \mid O \subseteq A \text{ for some representative } A \text{ of } a\}}$$

Note from Corollary 3.15 that $Ia \in \mathcal{G}$ for any $a \in \mathcal{M}$. Thus, as expected, boxed formulae (i.e. formulae of the form $\Box\phi$ for some $\phi \in L$) are evaluated to open elements in $\mathcal{M}$.

4 Motivation

Our aim is to prove completeness of $S4$ for $\mathcal{M}$. Can we leverage completeness of $S4$ for the class of finite topologies—or, better yet, can we leverage completeness of $S4$ for the real interval $[0, 1]$—to this end? Here is a natural first thought. If α is a non-theorem of $S4$ then, by completeness of $S4$ for the unit interval, there is a valuation $f : \mathbb{P} \to B([0, 1])$ that falsifies α (i.e. $h_f(\alpha) \neq [0, 1]$). So long as $f(P)$ is Lebesgue-measurable for every $P \in \mathbb{P}$, we can define the corresponding valuation, $\overline{f}$, over $\mathcal{M}$ by

$$\overline{f} = q \circ f$$

where $q : Leb([0, 1]) \to \mathcal{M}$ is the quotient map (thus, $\overline{f}(P) = \overline{f(P)}$, for all $P \in \mathbb{P}$). The obvious question is: Does $\overline{f}$ falsify α in $\mathcal{M}$?

In general, the answer is no. Here are two problems:

(A_1) **Null sets vanish in** $\mathcal{M}$. For example, if $f(P) = [0, 1] - \mathbb{Q}$, then $\overline{f}(P) = \overline{f(P)} = \overline{[0, 1] - \mathbb{Q}} = \overline{[0, 1]}$. Thus P is falsified by f at each rational point in the interval $[0, 1]$, but $\overline{f}$ does not falsify P in $\mathcal{M}$.

(A_2) **The quotient map, q, does not preserve interiors**. The same counterexample as above illustrates the point. If $f(P) = [0, 1] - \mathbb{Q}$, then $h_f(\Box P) = \text{Interior}(h_f(P)) = \emptyset$, but $h_{\overline{f}}(\Box P) = I(h_{\overline{f}}(P)) = I(\overline{h_f(P)}) = I(\overline{[0, 1] - \mathbb{Q}}) = \overline{[0, 1]}$. So while f falsifies $\Box P$ at *every* point in the interval $[0, 1]$, $\overline{f}$ does not falsify $\Box P$ in $\mathcal{M}$.

Thus for any propositional variable, P, we have (by definition)

$$h_{\overline{f}}(P) = \overline{h_f(P)}$$

but we do *not* have, for arbitrary formula ϕ in the language L,

$$h_{\overline{f}}(\phi) = \overline{h_f(\phi)}$$

The examples above show that the simple idea of taking valuations, f, that falsify α in $B([0, 1])$ and composing with the quotient map does not always produce valuations that falsify α in $\mathcal{M}$. To get around the difficulties, we need, for each non-theorem, α, of $S4$, a valuation $f : \mathbb{P} \to B([0, 1])$ with $f(P)$ Lebesgue-measurable (for each $P \in \mathbb{P}$) that satisfies:

(B_1) $\quad m(h_f(\neg\alpha)) > 0$
(B_2) $\quad h_{\overline{f}}(\phi) = \overline{h_f(\phi)}$

for each formula ϕ in the language L. (B_2) poses by far the greatest challenge, and in fact motivates much of the work that follows.

Our strategy is this. It is known that $S4$ is complete for the complete binary tree, T_2, defined below (the proof is by unraveling finite Kripke frames onto T_2). We construct a map, $\Phi : [0, 1] \to T_2$, with "nice" topological properties, and "nice" measure-theoretic properties. The nice topological properties ensure that Φ is "truth-preserving" (see Section 5.1). The nice measure-theoretic properties ensure that any valuation $f : \mathbb{P} \to B([0, 1])$ *constructed as a* Φ-*pullback* of a valuation $f' : \mathbb{P} \to B(T_2)$[10] does satisfy (B_1) and (B_2). Together this ensures that the valuation $\overline{f} = q \circ f$ falsifies α in $\mathcal{M}$.

It should be noted that there is nothing special about the map, Φ, (constructed in Section 5.4) from a topological point of view *alone*. Truth-preserving maps that pull back valuations from finite Kripke frames (or from the complete binary tree) to the real interval [0, 1] can be found in several places in the literature. Tarski and McKinsey effectively did this when they showed how, given a finite topological space $\mathcal{X}$, to define a continuous, open function from $\mathcal{X}$ onto [0,1]. More recent constructions appear in [1, 3, 4]. The point of defining a *new* map, then, is to ensure the right measure-theoretic properties. Indeed, this is what makes the proof of completeness of $S4$ for $\mathcal{M}$ different from proofs of completeness for $B([0, 1])$.

5 Completeness Transfer from the Complete Binary Tree to the Unit Interval

Our goal in this section is to construct a map, $\Phi : [0, 1] \to T_2$, from the unit interval to the complete binary tree (defined below), with 'nice' topological properties and 'nice' measure-theoretic properties. We begin (Section 5.1) by studying 'truth-preserving' maps between topological fields of sets. In

[10] I.e. $f = [\Phi^{-1}] \circ f'$.

Section 5.2, we define the complete binary tree, and in Section 5.3, we prove some key properties of the thick Cantor set—a set that figures heavily in the construction of Φ. Finally, in Section 5.4, we construct the map $\Phi : [0, 1] \to T_2$, and in Section 5.5, we prove that Φ has the desired topological and measure-theoretic properties.

5.1 Truth Preserving Maps

In this section we study 'truth-preserving' maps. These maps allow us to transfer completeness of $S4$ from one TBA to another. In the special case where we deal with topological fields of sets, the key notion is that of an interior, surjective map. The key notion in the more general algebraic semantics is that of a TBA isomorphism. The definitions are given below.

Definition 5.1 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be TBA's. A function $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ is a *TBA homomorphism* if it preserves Boolean and Interior operations:

$$\pi(-a) = -\pi(a)$$

$$\pi(a \wedge b) = \pi(a) \wedge \pi(b)$$

$$\pi(Ia) = I(\pi(a))$$

(where in the final equation, 'I' on the LHS is the interior operator in A_1 and 'I' on the RHS is the interior operator in A_2. We occasionally leave out subscripts for purposes of notational simplicity.) We say that $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ is a *TBA isomorphism* of $\mathcal{A}_1$ into $\mathcal{A}_2$ if π is a TBA homomorphism and is injective. Note that this somewhat non-standard definition of isomorphism does not require surjectivity. See [8].

Lemma 5.2 *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are TBA's and that $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ is a TBA homomorphism. Let $f' : \mathbb{P} \to \mathcal{A}_1$ be any valuation over $\mathcal{A}_1$ and define the valuation $f : \mathbb{P} \to \mathcal{A}_2$ by $f(P) = \pi \circ f'(P)$. Then for any formula α in the propositional modal language, L,*

$$h_f(\alpha) = \pi \circ h_{f'}(\alpha)$$

If π is a TBA isomorphism, then (also)

$$h_{f'}(\alpha) = 1_{\mathcal{A}_1} \ \mathit{iff}\ h_f(\alpha) = 1_{\mathcal{A}_2}$$

Proof The proof is by induction on the complexity of α. The base case is true by definition of f, and we prove only the modal clause:

$$\begin{aligned} h_f(\Box\phi) &= I(h_f(\phi)) \\ &= I(\pi \circ h_{f'}(\phi)) \quad \text{(by inductive hypothesis)} \\ &= \pi(I(h_{f'}(\phi))) \quad \text{(since } \pi \text{ a TBA homomorphism)} \\ &= \pi \circ h_{f'}(\Box\phi) \end{aligned}$$

For the second part of the lemma (where π is an TBA isomorphism), note that if $h_f(\alpha) = 1_{\mathcal{A}_2}$, then by the previous part, $\pi \circ h_{f'}(\alpha) = 1_{\mathcal{A}_2}$. But since π is injective, $h_{f'}(\alpha) = 1_{\mathcal{A}_1}$. Conversely, if $h_{f'}(\alpha) = 1_{\mathcal{A}_1}$, then $h_f(\alpha) = \pi \circ h_{f'}(\alpha) = \pi \circ 1_{\mathcal{A}_1} = 1_{\mathcal{A}_2}$. □

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. Recall that a map, $f : \mathcal{X} \to \mathcal{Y}$ is *continuous* if the inverse image of every open set in $\mathcal{Y}$ is open in $\mathcal{X}$. f is *open* if the image of any open set in $\mathcal{X}$ is open in $\mathcal{Y}$. A map that is both open and continuous is *interior*.

Lemma 5.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and form the corresponding topological field of sets $B(\mathcal{X})$ and $B(\mathcal{Y})$. If $g : \mathcal{X} \to \mathcal{Y}$ is interior and surjective, then $[g^{-1}] : B(\mathcal{Y}) \to B(\mathcal{X})$*[11] *is a TBA isomorphism.*

Proof Suppose $S_1, S_2 \in B(\mathcal{Y})$, with $S_1 \neq S_2$. WLOG, let $y \in S_1$, $y \notin S_2$. Then since g is surjective, there exists $x \in \mathcal{X}$ with $g(x) = y$. But then $x \in [g^{-1}](S_1)$ and $x \notin [g^{-1}](S_2)$, proving that $[g^{-1}]$ is injective.

We need to show that $[g^{-1}]$ preserves lattice operations. The Boolean operations are straightforward and we prove only the modal clause: i.e. for any $a \in B(\mathcal{Y})$,

$$\left[g^{-1}\right](Ia) = I\left(\left[g^{-1}\right](a)\right)$$

By continuity of g, we know that $[g^{-1}](Ia)$ is open in $\mathcal{X}$. Moreover, since $Ia \subseteq a$, we have $[g^{-1}](Ia) \subseteq [g^{-1}](a)$. Thus $[g^{-1}](Ia)$ is an open subset of $[g^{-1}](a)$. To see that it is the largest such subset, suppose $O \subseteq [g^{-1}](a)$ is open in $\mathcal{X}$. Then, since g is open, $g(O)$ is an open subset of a, hence $g(O) \subseteq Ia$. But then $O \subseteq [g^{-1}](Ia)$. □

Proposition 5.4 *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces and $g : \mathcal{X} \to \mathcal{Y}$ is an interior, surjective map. Let $f' : \mathbb{P} \to B(Y)$ be a valuation function and define $f = ([g^{-1}] \circ f')$. Then for every formula α of the modal language L we have:*

$$h_f(\alpha) = \left[g^{-1}\right] \circ h_{f'}(\alpha)$$

and

$$h_{f'}(\alpha) = 1_{B(\mathcal{Y})} \; iff \; h_f(\alpha) = 1_{B(\mathcal{X})}$$

Proof Immediate from the previous two lemmas. □

5.2 Complete Binary Tree

In this section we define the complete binary tree, T_2, a topological space for which $S4$ is complete. The tree is an extension of the (better known) infinite

[11]The map $[g^{-1}]$ is defined on $B(\mathcal{Y})$. It takes subsets of Y to their pullbacks in X (where X and Y are the underlying sets of $\mathcal{X}$ and $\mathcal{Y}$, respectively)—i.e. for $S \subseteq Y$, $[g^{-1}](S) = \{x \in X \mid g(x) \in S\}$.

binary tree: in addition to finite nodes, it contains limit nodes that can be thought of as infinite branches of the infinite binary tree. In Section 5.4, we will construct an interior, surjective map, $\Phi : [0, 1] \to T_2$, that allows us to transfer completeness from T_2 to $B([0, 1])$.

Definition 5.5 (Complete Binary Tree) Take alphabet $\Sigma = \{0, 1\}$ and construct the set $\Sigma^f(\Sigma^c)$ of all finite (countable) strings over Σ. For any $s \in \Sigma^f$, and $t \in \Sigma^c$, let $s * t$ denote the concatenation of s with t. For any $s \in \Sigma^f$, let $U_s = \{s * t \mid t \in \Sigma^c\}$, i.e. the set of all (possibly infinite) strings with initial segment s. We include as a finite string the empty string, or 'root', of T_2, and denote it by $\langle\cdot\rangle$. Let $B = \{U_s \mid s \in \Sigma^f\} \cup \{\emptyset\}$. Note that B is closed under finite intersections (For any $s, t \in \Sigma^f$ either $U_s \subseteq U_t$, $U_t \subseteq U_s$ or $U_s \cap U_t = \emptyset$) and contains the empty set and the whole space, hence is a basis for some topology $\mathcal{J}$ over Σ^c. We let $T_2 = (\Sigma^c, \mathcal{J})$ and refer to it as the *complete binary tree*. (See Fig. 1).

For any two nodes s and t of T_2, we say that t is an *ancestor* of s if $s = t * t'$ for some (possibly empty) $t' \in \Sigma^c$. We also say s is a *descendant* of t. (If t' is the empty string, then $s = t$ and s is both an ancestor and descendant of t.) If $s = t * 0$ or $s = t * 1$ we say s is an *immediate* descendant of t. We refer to points in Σ^f as *finite* nodes of the tree, and to points in $\Sigma^c - \Sigma^f$ as *limit* nodes.

Proposition 5.6

(i) Σ^c*, the underlying set of* T_2*, is uncountable,*
(ii) T_2 *is non-Alexandroff.*

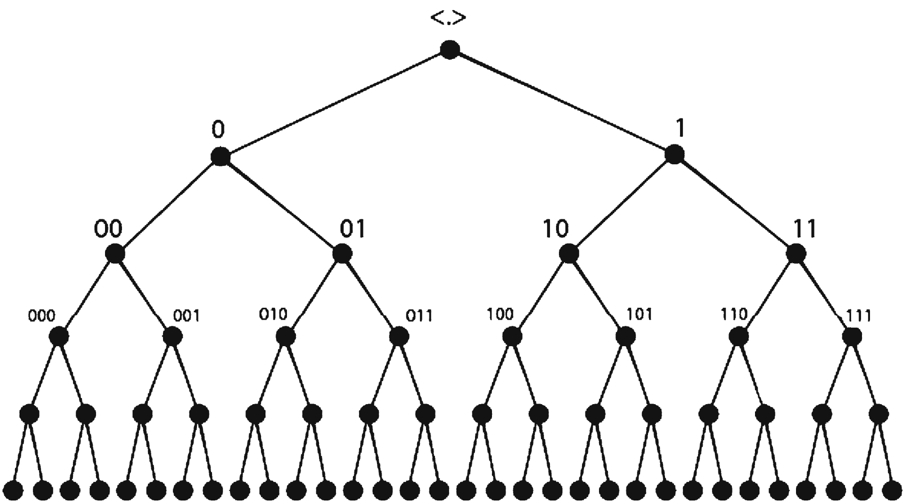

Fig. 1 First few levels of the complete binary tree, T_2. Each node, t, has a 'left' successor (t*0) and a 'right' successor (t*1)

Separation axioms:

(iii) *T_2 is $\mathcal{T}_0$,*
(iv) *T_2 is not $\mathcal{T}_1$ (hence non-Hausdorff and non-metrizable)*

Proof

(i) By a bijection between the set of infinite strings over $\Sigma = \{0, 1\}$ and the interval $[0, 1]$;
(ii) The intersection of basic opens $U_0, U_{00}, U_{000}, \ldots$ (i.e. the countable sequence 000...) is not open;
(iii) Let $s, t \in \Sigma^c$, $s \neq t$. The interesting case is where $s = t * t'$ or $t = s * t'$ for some $t' \in \Sigma^c$. Let $s = t * t'$. Then either t' is a limit or a finite node. If t' is finite, then $s \in U_s$ and $t \notin U_s$. If t' is a limit node, then there exists a finite node, t'', 'between' t and s with $s \in U_{t''}$ and $t \notin U_{t''}$;
(iv) Take, for instance, $t = 0$ and $s = 00$: there is no open set containing t that does not contain s. □

$T_2 = (\Sigma^c, \mathcal{J})$ is a topological space, and we can construct the corresponding topological field of sets $B(T_2)$. Let $f : \mathbb{P} \to B(T_2)$ be a valuation over T_2. Suppose that for some formula ψ, and some limit node t, $t \in h_f(\psi)$ and $s \in h_f(\psi)$ for only finitely many ancestors, s, of t. Then we say that f is a *degenerate* valuation. The following proposition states that we can ignore degenerate valuations over T_2 for the purposes of completeness.

Proposition 5.7 *S4 is complete for T_2. In particular, if α is a non-theorem of S4, then α is falsified at the root of T_2 by a non-degenerate valuation ($\langle\cdot\rangle \notin h_f(\alpha)$ for some non-degenerate $f : \mathbb{P} \to B(T_2)$).*

Proof The proposition follows from the unraveling of finite Kripke frames onto the complete binary tree, and the fact that all non-theorems are falsified at the root of some finite frame. The unraveling map assigns limit nodes of the complete binary tree to worlds that label infinitely many nodes of the associated branch. For this reason the relevant valuations over the tree are non-degenerate. See [4]. □

5.3 Thick Cantors

Recall the construction of the Cantor set. We begin with the interval $[0, 1]$. At stage $n = 0$ of construction, we remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving "remaining intervals" $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. At stage $n = 1$, we remove the open middle thirds of each of these intervals, leaving remaining intervals $\left[0, \frac{1}{9}\right]$, $\left[\frac{2}{9}, \frac{1}{3}\right]$, $\left[\frac{2}{3}, \frac{7}{9}\right]$ and $\left[\frac{8}{9}, 1\right]$, etc. The Cantor set, C, is the set of points remaining after infinitely many stages of construction.

An easy argument due to Dana Scott shows that the decision to remove middle *thirds* (as opposed to fourths, fifths, etc.) in the construction of C plays

Fig. 2 First five stages of construction of the Smith–Volterra–Cantor set, K

no role in the fact that C has measure zero. Let C_n be the set resulting from removing open middle intervals of proportional length $1/n$ at each stage of construction. After removing the first middle interval we produce scaled copies of C_n on the intervals $\left[0, \frac{n-1}{2n}\right]$ and $\left[\frac{n+1}{2n}, 1\right]$, giving,

$$m\left(C_n\right) = 2\,\frac{n-1}{2n}\,m\left(C_n\right)$$

and $m\left(C_n\right) = 0$.

We can, however, construct a set that is 'Cantor-like' (sharing all topological properties with C) and yet has non-zero measure. The trick is to remove successively smaller portions of the remaining intervals. The set we end up with is sometimes called a 'thick' or 'fat' Cantor set. The particular version of it below has measure $= 1/2$, but this is not necessary—sets of arbitrary positive measure can be constructed in similar fashion.[12]

Definition 5.8 Begin with the interval $[0, 1]$, and at stage $n = 0$ of construction, remove the open middle interval of length $\frac{1}{4}$, leaving remaining intervals $\left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$. At stage $n = 1$, remove open middle $\frac{1}{16}$'s from each interval, leaving $\left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right]$, etc. In general, at stage n of construction, remove open middle intervals of length $\left(\frac{1}{4}\right)^{n+1}$ from each remaining interval. The set of points remaining after infinitely many stages of construction is the Smith–Volterra–Cantor set. We call it the 'thick' Cantor set and denote it by K.[13] (See Figs. 2 and 3).

What is the measure of K? Note that at each finite stage of construction of K, 2^n intervals of length $\left(\frac{1}{4}\right)^{n+1}$ are removed, so the total measure of points removed is

$$\sum_{n\geq 0} 2^n \left(\frac{1}{4}\right)^{n+1} = \sum_{n\geq 0} \left(\frac{1}{2}\right)^{n+2} = \frac{1}{2}$$

and $m(K) = 1 - 1/2 = 1/2$.

Proposition 5.9 *Let O be an open set with $K \cap O \neq \emptyset$. Then $K \cap O$ has non-zero measure.*

[12]To construct a thick Cantor set with measure $1-\epsilon$, remove middle intervals of length $2\epsilon\left(\frac{1}{4}\right)^{n+1}$. Over the course of the construction we remove a total measure of $2\epsilon \sum_{n\geq 0} 2^n \left(\frac{1}{4}\right)^{n+1} = 2\epsilon \sum_{n\geq 0} \left(\frac{1}{2}\right)^{n+2} = 2\epsilon\left(\frac{1}{2}\right) = \epsilon$.

[13]Figures 2 and 3 are licensed by Creative Commons.

Fig. 3 The set K. After white intervals have been removed, the black points which remain make up K

Proof Let O be open and $x \in K \cap O$. Then, since $x \in K$, x is in a remaining interval at each stage of construction of K. Let $R_{n,x}$ denote the remaining interval containing x at stage n of construction. The length of remaining intervals tends to zero, so for N large enough, $R_{N,x} \subseteq O$. But, by symmetry, $m\left(K \cap R_{N,x}\right) = \left(\frac{1}{2}\right)^{N+2} > 0$. (At stage N of construction, there are 2^{N+1} remaining intervals and they split the measure of K equally). Thus

$$m\left(K \cap O\right) \geq m\left(K \cap R_{N,x}\right) > 0 \qquad \square$$

We can construct a 'scaled copy' of K by starting from the interval $[a, b]$ instead of $[0, 1]$, and removing successively middle segments of length $(b-a)\left(\frac{1}{2}\right)^{2n+2}$. In fact, we can carry out the construction of K on any closed, open, or half-open interval $[a, b]$, (a, b), $[a, b)$, $(a, b]$. If we start from the open interval (a, b), the resulting set is not closed (compact, etc.) and hence differs in important properties from K. Nevertheless, with slight abuse of notation, we refer to all such constructions as 'scaled copies' of K. Clearly the measure of a scaled copy of K on any of the intervals $[a, b]$, (a, b), $[a, b)$, $(a, b]$ is just $\frac{1}{2}(b-a)$.

We state without proof an obvious corollary to Proposition 5.9:

Corollary 5.10 *Let K^* be a scaled copy of K. If O is open and $O \cap K^*$ is non-empty, then $O \cap K^*$ has non-zero measure.*

We are now in a position to prove Proposition 3.10, which states that $\mathcal{M} \neq \mathcal{G}$ (see Section 3). The example is due to Dana Scott, but we give a different proof here.

Proof of Proposition 3.10 We claim that $\overline{K} \notin \mathcal{G}$ (and thus $\mathcal{M} \neq \mathcal{G}$). We need to show that for any open set O, $K \nsim O$. Suppose $O \subseteq [0, 1]$ is open and $O \sim K$. We know $O \cap K \neq \emptyset$ (else $K \subseteq O \triangle K$ and $m(K) > 1$). Let $x \in O \cap K$. By the proof of Proposition 5.9, there exists $N \in \mathbb{N}$ with $R_{N,x} \subseteq O$ (where $R_{n,x}$ is, again, the remaining interval at stage n of thick Cantor construction containing x). But at stage $n+1$ of construction of K, we remove from $R_{N,x}$ an open interval, I, of non-zero measure. So $I \subseteq O - K \subseteq O \triangle K$ and $O \nsim K$. $\perp$. $\square$

Lemma 5.11 *Let I be any interval (closed, open, half-open) and let $K(I)$ be the scaled copy of K on I. Then $K(I)$ approaches both endpoints of I.*

Proof We assume I is an open interval and write $I = (a, b)$ (the proof for the other cases is identical). Consider the 'left-most' remaining interval at each finite stage of construction of $K(I)$, and denote it by $(a, r_n]$. Then $r_n \to a$ (since the length of remaining intervals tends to zero) and $r_n \in K(I)$ for all $n \in \mathbb{N}$. For the proof that $K(I)$ approaches b, simply replace 'left-most' with 'right-most' remaining intervals and make appropriate substitutions. □

Lemma 5.12 *Let I be any interval (open, closed, half-open), and let $K(I)$ be the scaled copy of K on I. Then $I - K(I)$ is a disjoint union of open intervals (i.e. intervals removed at finite stages of construction of $K(I)$). Let L_I (R_I) be the set of left (right) endpoints of these intervals. Then, for any $x \in K(I)$, points in L_I (R_I) approach x.*

Proof Fix $\epsilon > 0$. Again, since $x \in K(I)$, x belongs to a 'remaining interval', $R_{n,x}$, at each finite stage, n, of construction. As before, the length of $R_{n,x}$ tends to zero, so for some $N \in \mathbb{N}$, *length* $(R_{N,x}) < \epsilon$. But at stage $N + 1$ of construction, we remove an interval, (l, r), from $R_{N,x}$. Moreover, $l \in L(I)$, and $r \in R(I)$, and $|x - l| < \epsilon$ and $|x - r| < \epsilon$. □

5.4 Construction of the Map $\Phi : [0, 1] \to T_2$

We construct the map, $\Phi : [0, 1] \to T_2$, in stages, much like the Cantor set. The function is based quite heavily around thick Cantor sets. Indeed, the pullback, under Φ, of the root, $\langle\cdot\rangle$, of T_2, is just the thick Cantor set, K, with midpoints (defined below), and the pullback under Φ of any finite node of T_2 is a countable disjoint union of scaled copies of K (with midpoints).

In what follows, we say that an interval, $I \subseteq [0, 1]$, is *uniformly labeled* by a function $f : [0, 1] \to T_2$ if there exists a node $s \in T_2$ such that $f(x) = s$ for all $x \in I$. We say that an interval, I, is a *maximal, uniformly labeled* (MUL) interval under f if I is uniformly labeled by f and there does not exist $I' \supsetneq I$ with I' uniformly labeled by f. (For the purposes of this paper, intervals are always non-trivial.)

For any interval I, let $K(I)$ be the scaled copy of K on I. Then $I - K(I)$ is a disjoint union of open intervals $\{(a_n^I, b_n^I) \mid n \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ we let $A_n^I = \left(a_n^I, \frac{b_n^I - a_n^I}{2}\right)$ and $B_n^I = \left(\frac{b_n^I - a_n^I}{2}, b_n^I\right)$. (i.e. A_n^I is 'open left half' and B_n^I is 'open right half' of (a_n^I, b_n^I)). Let $A(I) = \bigcup_n A_n^I$ and $B(I) = \bigcup_n B_n^I$. (Think of $A(I)$ as the open left half and $B(I)$ as the open right half of the thick Cantor complement). (See Fig. 4.)

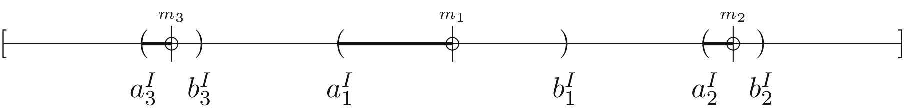

Fig. 4 Three intervals (a_i^I, b_i^I) ($i = 1, 2, 3$) in $I - K(I)$ for some interval I. Midpoints m_1, m_2 and m_3 belong to the set M. The shaded region of the diagram belongs to $A(I)$

We construct Φ in stages, as follows.

Definition 5.13 (Stagewise labeling functions, $\Phi_n : [0, 1] \to T_2$)
Let $\Phi_0(x) = \langle\cdot\rangle$ for all $x \in [0, 1]$.
For any $x \in [0, 1]$ and any $n \in \mathbb{N}$:

$$\Phi_{n+1}(x) = \begin{cases} \Phi_n(x) * 0 & \text{if } x \in A(I) \text{ for some } I \text{ that is a MUL interval under } \Phi_n) \\ \Phi_n(x) * 1 & \text{if } x \in B(I) \text{ for some } I \text{ that is a MUL interval under } \Phi_n) \\ \Phi_n(x) & \text{otherwise} \end{cases}$$

Thus, e.g.:

- Φ_0 labels all points in the interval [0, 1] by the root, $\langle\cdot\rangle$, of T_2.
- Φ_1 labels all points in $A([0, 1])$ by 0 ('left successor' of the root), and labels all points in $B([0, 1])$ by 1 ('right successor' of the root). It leaves all other points labeled by the root.
- Φ_2 labels as follows: If I is a MUL interval under Φ_1 and points in I are labeled by 0 (under Φ_1), then Φ_2 labels all points in $A(I)$ by 00 ('left successor' of 0), and all points in $B(I)$ by 01 ('right successor' of 0).
  If I is a MUL interval under Φ_1 and points in I are labeled by 1 (under Φ_1), then Φ_2 labels all points in $A(I)$ by 10 ('left successor' of 1), and all points in $B(I)$ by 11 ('right successor' of 1).
  Φ_2 leaves all other points labeled as they were by Φ_1.
  etc.

Note that for any interval I, midpoints of the intervals $\left(a_n^I, b_n^I\right)$ are in neither $A(I)$ or $B(I)$. Thus, if I is a MUL interval under Φ_n, whose points are labeled by node t, then under Φ_{n+1}, midpoints of the intervals $\left(a_n^I, b_n^I\right)$ *remain labeled by t*. We denote the set of 'midpoints' introduced at finite stages of labeling by M (M should not be confused with our notation, $\mathcal{M}$, for the Measure Algebra). Note that M is *countable* (only countably many points are added to M at each finite stage of labeling, Φ_n), and thus, from a measure-theoretic point of view, negligible. In what follows, we say that a node, t, labels all points in a set A *plus midpoints*, to mean that t labels all points in A and some points in M.

Some points $x \in [0, 1]$ "stabilize" over successive labelings and some do not. More precisely, some but not all points satisfy the following condition:

$$\exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \Phi_n(x) = \Phi_N(x) \qquad (*)$$

(As an example of a point that stabilizes, consider $\frac{1}{2}$, or any point in M.) Our final labeling function agrees with stagewise labeling functions on points that stabilize, but assigns limit nodes of T_2 to all points that do not stabilize. We define the function, $\Phi : [0, 1] \to T_2$ as follows.

Definition 5.14 'Final' labeling function, $\Phi : [0, 1] \to T_2$.

$$\Phi(x) = \begin{cases} \Phi_N(x) & \text{if } x \text{ satisfies } (*) \\ t & \text{otherwise} \end{cases}$$

where N is the stage at which x stabilizes, and t is the unique countable sequence over $\{0, 1\}$ that has $\Phi_n(x)$ as initial segment for each $n \in \mathbb{N}$.

The following observation sums up the construction, Φ.

Observation Level-n nodes in T_2 label points in $[0, 1]$ for the first time at stage n. If t is any level-n node ($n \geq 1$), then at stage n, t labels all points in a countable disjoint union of open intervals. If I is a MUL interval under Φ_n whose points are labeled by t, then at stage $n+1$, only points in $K(I)$ (plus midpoints) remain labeled by t. These points remain labeled by t under successive labeling functions. Thus, under the final labeling function, Φ, the pullback of any finite node is a countable disjoint union of scaled copies of K (plus midpoints).

5.5 Proof of Key Lemmas

Over the course of the next several lemmas, we prove that Φ is an interior, surjective map, and that it has a key measure-theoretic property, stated in Corollary 5.18. The proof of these results is quite technical, and the reader can skip ahead to the next section, where completeness of $S4$ for $\mathcal{M}$ is proved.

Again, let M denote the set of midpoints in $[0, 1]$ (see Section 5.4), and note that M is countable, hence $M \in Null$.

Lemma 5.15 *Let t be any finite node of T_2. Let $O \subseteq [0, 1]$ be an open set with $O \cap (\Phi^{-1}(t) - M) \neq \emptyset$. Then $O \cap \Phi^{-1}(t)$ has non-zero measure.*

Proof This follows from the observation that $\Phi^{-1}(t) - M$ is a countable disjoint union of scaled copies of K. If $O \cap (\Phi^{-1}(s) - M) \neq \emptyset$, then O intersects one of these scaled copies of K. By Corollary 5.10, $O \cap (\Phi^{-1}(s) - M)$ has non-zero measure, hence also $O \cap \Phi^{-1}(s)$ has non-zero measure. □

In the next two lemmas, we refer to 'open intervals' in $A(I)$, $B(I)$ and $I - K(I)$, and to left and right 'endpoints' of $A(I)$, $B(I)$, and $I - K(I)$. The meaning should be clear: $A(I)$, e.g., can be written uniquely as a disjoint union of open intervals and 'open intervals in A(I)' are *these* intervals. 'Endpoints' of $A(I)$ are simply endpoints of these intervals. (Likewise for $B(I)$ and $I - K(I)$.)

Lemma 5.16 *Let I be a MUL interval labeled by t at some finite stage n, and again let $K(I)$ be the scaled copy of K on I. Then $\Phi^{-1}(t*0) - M$ approaches all left endpoints of $I - K(I)$ and $\Phi^{-1}(t*1) - M$ approaches all right endpoints of $I - K(I)$.*

Proof Left endpoints of $I - K(I)$ are simply left endpoints of $A(I)$, and right endpoints of $I - K(I)$ are simply right endpoints of $B(I)$. $\Phi^{-1}(t*0) - M$ is a thick Cantor on each open interval in $A(I)$, hence, by Lemma 5.11, approaches endpoints of $A(I)$. Likewise, $\Phi^{-1}(t*1) - M$ is a thick Cantor on each open interval in $B(I)$, hence approaches endpoints of $B(I)$. □

Lemma 5.17 *Let s and t be finite nodes of T_2 with s a descendant of t. If $O \subseteq [0, 1]$ is open with $O \cap (\Phi^{-1}(t) - M) \neq \emptyset$, then $O \cap (\Phi^{-1}(s) - M) \neq \emptyset$.*

Proof We assume $s \neq t$ (the case where $s = t$ is trivial). We show that $\Phi^{-1}(s) - M$ approaches any point in $\Phi^{-1}(t) - M$. Assume first that s is an immediate successor of t. WLOG $s = t*0$, and we let $x \in \Phi^{-1}(t) - M$. Then x belongs to an interval I first labeled by t at some finite stage n and $x \in K(I)$. By the previous lemma, $\Phi^{-1}(s) - M$ approaches left endpoints of $I - K(I)$, and by Lemma 5.12, these endpoints in turn approach all points in $K(I)$. (If we had let $s = t*1$, then $\Phi^{-1}(s) - M$ would approach right endpoints of $I - K(I)$, which also approach all points in $K(I)$.) By induction on the "distance" in T_2 from t to s, the claim is true for any finite descendant s of t. □

Corollary 5.18 *Let s and t be finite nodes of T_2 with s a descendant of t. If $O \subseteq [0, 1]$ is open with $O \cap (\Phi^{-1}(t) - M) \neq \emptyset$, then $O \cap \Phi^{-1}(s)$ has non-zero measure.*

Proof Immediate from Lemmas 5.15 and 5.17. □

Lemma 5.19 Φ *is continuous.*

Proof Let U be a basic open set in T_2. Then U is generated by some finite node s. Suppose $x \in \Phi^{-1}(U)$. It follows that $\Phi(x) = s*t$ where $t \in \Sigma^c$. By the construction of Φ, this means that at some stage n of labeling, x was in an open MUL interval, I, that got labeled by s. But then $x \in I \subseteq \Phi^{-1}(U)$ and $\Phi^{-1}(U)$ is open. □

Lemma 5.20 Φ *is open.*

Proof Let $O \subseteq [0, 1]$ be open and let $t \in \Phi(O)$. We show that if t is a finite node, then $U_t \subseteq \Phi(O)$ and if t is a limit node, then for some (finite) ancestor t' of t, $U_{t'} \subseteq \Phi(O)$. The proof is fairly intricate and is left to the Appendix.

Lemma 5.21 Φ *is surjective.*

Proof This follows from the fact that Φ is open, and the root, $\langle\cdot\rangle$, is in the range of Φ—e.g. $\Phi\left(\frac{1}{2}\right) = \langle\cdot\rangle$.

Proposition 5.22 *Let $f' : P \to B(T_2)$ be any valuation over T_2 and let $f : \mathbb{P} \to B([0, 1])$ be defined by $f(P) = [\Phi^{-1}] \circ f'(P)$ for all $P \in \mathbb{P}$. Then for any formula, ϕ, in the modal language, L,*

$$h_f(\phi) = \left[\Phi^{-1}\right] \circ h_{f'}(\phi)$$

Moreover,

$$h_{f'}(\phi) = 1_{B(T_2)} \quad iff \quad h_f(\phi) = 1_{B([0,1])}$$

Proof Immediate from Lemmas 5.19, 5.20, 5.21 and Proposition 5.4.

The following lemma and proposition tell us that the Φ-pullback of any subset of T_2 is a Lebesgue-measurable set. Moreover, although the finite nodes are only a countable subset of an uncountable tree, the pullback of these nodes takes up *all* the measure of the real interval [0, 1]!

Lemma 5.23 *Let F be the set of finite nodes of T_2. Then $\Phi^{-1}(F)$ is measurable, and*

$$m\left(\Phi^{-1}(F)\right) = 1$$

Proof Note that for any finite node t, $\Phi^{-1}(t) - M$ is Borel (since it is a countable disjoint union of scaled copies of K). Also, $\Phi^{-1}(t) \cap M$ is countable, hence Borel. It follows that $\Phi^{-1}(t)$ is Borel. Finally, $\Phi^{-1}(F) = \bigcup_{t\in F} \Phi^{-1}(t)$ is a countable union of Borel sets, hence Borel (and measurable).

We say a finite node t is a level-n node if it is n steps from the root, $\langle\cdot\rangle$, or, equivalently, is a string of length n in Σ^f. For the second part of the lemma, note that there are 2^n level-n nodes for each $n \in \mathbb{N}$. Let E_n = measure $(\Phi^{-1}(t))$, and let S_n = measure $(\Phi^{-1}(U_t))$ where t is any level-n node. (By symmetry, these quantities do not depend on which level-n node we pick.) Then

$$\text{measure}\left(\Phi^{-1}(F)\right) = \sum_{n\geq 0} 2^n E_n$$

To calculate E_n, note that:[14]

(i) $E_n = S_n/2$
(ii) $S_0 = 1$ and $S_{n+1} = S_n/4$

[14]Why? (i) follows from the fact that for every MUL interval, I, labeled by t at stage n, only $K(I)$ (plus midpoints) remains labeled by t in subsequent stages. (ii) follows from the fact that $\Phi^{-1}(U_t)$ is a disjoint union of MUL intervals, I, labeled by t at stage n, and $\Phi^{-1}(U_{t*0} \cup U_{t*1})$ is the set $I - K(I)$ (minus midpoints) on each such interval, I. We know $m(I - K(I)) = \frac{1}{2}m(I)$, and $\Phi^{-1}(U_{t*0}) = \Phi^{-1}(U_{t*0})$, giving (ii).

Solving, we get $S_n = \left(\frac{1}{2}\right)^{2n}$ and $E_n = \left(\frac{1}{2}\right)^{2n+1}$. Finally:

$$\text{measure}\left(\Phi^{-1}(F)\right) = \sum_{n\geq 0} 2^n E_n = \sum_{n\geq 0} 2^n \left(\frac{1}{2}\right)^{2n+1} = \sum_{n\geq 0} \left(\frac{1}{2}\right)^{n+1} = 1$$

□

Proposition 5.24 *Let S be any subset of T_2. Then $\Phi^{-1}(S)$ is Lebesgue-measurable.*

Proof From the proof of Lemma 5.23, we know that $\Phi^{-1}(t)$ is Borel for any $t \in F$. Since $S \cap F$ is countable, $\Phi^{-1}(S \cap F) = \bigcup_{t\in S\cap F} \Phi^{-1}(t)$ is a countable union of Borel sets, hence Borel. Moreover, $\Phi^{-1}(S \cap L)$ has Lebesgue-measure zero (again, by Lemma 5.23). So $\Phi^{-1}(S) = \Phi^{-1}(S \cap F) \cup \Phi^{-1}(S \cap L) = B \cup N$ for some Borel set B and $N \in \mathit{Null}$. □

6 Completeness of *S*4 for the Lebesgue Measure Algebra $\mathcal{M}$

We want to transfer valuations over $B([0, 1])$ to the Measure Algebra, $\mathcal{M}$. In Section 4, we reasoned as follows. For any valuation $f : \mathbb{P} \to B([0, 1])$ we can define $\overline{f}$ over $\mathcal{M}$ by letting: $\overline{f}(P) = \overline{f(P)}$ (so long as $f(P)$ is Lebesgue-measureable for each $P \in \mathbb{P}$). If, for any formula ϕ, of the modal language L,

$$h_{\overline{f}}(\phi) = \overline{h_f(\phi)} \tag{4}$$

we would have a proof of completeness for $\mathcal{M}$ along the following lines. For any non-theorem α of $S4$, let $f : \mathbb{P} \to B([0, 1])$ be a valuation that falsifies α at each point on a Lebesgue-measurable set of non-zero measure (i.e. $m\left(h_f(\alpha)\right) < 1$). Then by (4), $h_{\overline{f}}(\alpha) = \overline{h_f(\alpha)}$, but since $m\left(h_f(\alpha)\right) < 1$, $\overline{h_f(\alpha)} \neq \overline{[0, 1]}$ and α is falsified in $\mathcal{M}$ by $\overline{f}$.

The problem with this approach is that in general, Eq. 4 is not true (see (A_2) in Section 4). The construction of Φ was aimed at getting around this problem. Because of the special measure-theoretic properties of Φ, any valuation $f : \mathbb{P} \to B([0, 1])$ defined as a Φ-pullback of a valuation $f' : \mathbb{P} \to B(T_2)$ *does* satisfy Eq. 4. More precisely,

Lemma 6.1 *Let $f' : \mathbb{P} \to B(T_2)$ be a non-degenerate valuation over T_2, and let $f(P) = \Phi^{-1} \circ f'(P)$ and $\overline{f}(P) = \overline{f(P)}$. ($\overline{f}$ is well-defined by the fact that $f(P)$ is a measurable set for each propositional variable P—see Lemma* 5.24.) *Then for any formula α of the modal language L,*

$$h_{\overline{f}}(\alpha) = \overline{h_f(\alpha)}$$

Proof By induction on the complexity of α. The Boolean cases are straightforward, and we prove only the modal clause. For simplicity of notation we let $h' = h_{f'}$, $h = h_f$ and $\overline{h} = h_{\overline{f}}$, and show that

$$\overline{h}(\Box\phi) = \overline{h(\Box\phi)} \tag{5}$$

We already know that

$$\begin{aligned}
h(\Box\phi) &= \text{Interior}\,(h(\phi)) = \bigcup\{O \text{ open } \mid O \subseteq h(\phi)\}, \text{ and}\\
\overline{h}(\Box\phi) &= I\,(\overline{h}(\phi)) = \sup\{c \in \mathcal{G} \mid c \leq \overline{h}(\phi)\}\\
&= \sup\{c \in \mathcal{G} \mid c \leq \overline{h(\phi)}\} \quad \text{(by inductive hypothesis)}\\
&= \overline{\bigcup\{O \text{ open } \mid O \subseteq X \text{ for some representative } X \text{ of } \overline{h(\phi)}\}}\\
&\quad \text{(by Proposition 3.14)}\\
&= \overline{\bigcup\{O \text{ open } \mid O \subseteq h(\phi) \cup N \text{ for some } N \in \mathit{Null}\}}
\end{aligned}$$

Thus, proving Eq. 5 amounts to showing:

$$\overline{\bigcup\{O \text{ open } \mid O \subseteq h(\phi) \cup N \text{ for some } N \in \mathit{Null}\}} = \overline{\bigcup\{O \text{ open } \mid O \subseteq h(\phi)\}}$$

The inequality ($\geq$) is obvious. We prove ($\leq$). We need to show that for some $S \in \mathit{Null}$,

$$\bigcup\{O \text{ open } \mid O \subseteq h(\phi) \cup N \text{ for some } N \in \mathit{Null}\} \subseteq \text{ Interior}\,(h(\phi)) \cup S$$

In particular, we let $S = M \cup \Phi^{-1}(L)$ (i.e. the union of the set of midpoints and the points labeled by limit nodes of T_2). S is a null set because both M and $\Phi^{-1}(L)$ are null sets (see Proposition 5.23). Note on notation: In the remainder of this paper we sometimes write A^c for $[0, 1] - A$ (where A is a subset of $[0, 1]$).

Let $O \subseteq [0, 1]$ be open with $O \subseteq h(\phi) \cup N$ for some $N \in \mathit{Null}$ and let $x \in O$. Suppose (toward contradiction) that $x \notin \text{Interior}\,(h(\phi)) \cup S$. Letting $t = \Phi(x)$, we know (since $x \notin S$) that t is a finite node. Since $x \notin \text{Interior}\,(h(\phi))$, we know $x \notin h(\Box\phi)$. But then by Proposition 5.22, $t \notin h'(\Box\phi)$, and so $t \notin \text{Interior}\,(h'(\phi))$. It follows from the topology on T_2 that there is some descendant, s, of t with $s \notin h'(\phi)$. Again by Proposition 5.22, $\Phi^{-1}(s) \subseteq (h(\phi))^c$. We can assume s is finite by the fact that f' is non-degenerate. But now $(\Phi^{-1}(t) - M) \cap O \neq \emptyset$ (since $x \in O$, $x \notin M$). So by Corollary 5.18, $\Phi^{-1}(s) \cap O$ has non-zero measure. We have,

$$\begin{aligned}
&\Phi^{-1}(s) \cap O \subseteq h(\phi) \cup N \quad \text{and}\\
&\Phi^{-1}(s) \cap O \subseteq (h(\phi))^c
\end{aligned}$$

giving $\Phi^{-1}(s) \cap O \subseteq N$ and contradicting the fact that LHS has non-zero measure. $\perp$. □

Corollary 6.2 *Let f' be a non-degenerate valuation over T_2, and let $f(P) = \Phi^{-1} \circ f'(P)$ and $\overline{f}(P) = \overline{f(P)}$, as above. Then for any formula, α, of the modal language, L,*

$$h_{\overline{f}}(\alpha) = \overline{[0,1]} \ \textit{iff} \ m\left(h_f(\alpha)\right) = 1$$

Proof The corollary follows immediately from Lemma 6.1 and the fact that for any Lebesgue-measurable set $A \subseteq [0,1]$, $A \sim [0,1]$ iff $m(A) = 1$. □

Theorem 6.3 *$S4$ is complete for $\mathcal{M}$.*

Proof Let α be a non-theorem of $S4$. Then α is falsified at the root, $\langle\cdot\rangle$, of T_2 by some non-degenerate valuation $f' : \mathbb{P} \to B(T_2)$ - i.e. $\langle\cdot\rangle \notin h_{f'}(\alpha)$. Define the valuations f, $\overline{f}$ as follows:

$$f(P) = \Phi^{-1} \circ f'(P)$$

$$\overline{f}(P) = \overline{f(P)}$$

Then $\langle\cdot\rangle \in h_{f'}(\neg\alpha)$, and by Proposition 5.22, $\Phi^{-1}(\langle\cdot\rangle) \subseteq h_f(\neg\alpha)$. Thus, $K \subseteq \Phi^{-1}(\langle\cdot\rangle) \subseteq (h_f(\alpha))^c$, so $m(h_f(\alpha)) < 1$. By Corollary 6.2, $h_{\overline{f}}(\alpha) \neq \overline{[0,1]}$, and α is falsified in $\mathcal{M}$. □

We know, from Tarski's proof of completeness of $S4$ for the reals, that any non-theorem, α, of $S4$ can be falsified at a point in the real interval, [0, 1] (i.e. there is a valuation, $f : \mathbb{P} \to B([0,1])$, and point $x \in [0,1]$ with $x \notin h_f(\alpha)$). The next corollary states that if α is a non-theorem of $S4$, there exists a valuation, $f : \mathbb{P} \to B([0,1])$, that falsifies α at each point in a subset of [0, 1] of measure *arbitrarily close to 1*.

Corollary 6.4 *Suppose α is a non-theorem of $S4$. Then for any $\epsilon > 0$, there exists a valuation, $f : \mathbb{P} \to B([0,1])$, and a measurable set, S, with $m(S) < \epsilon$ and $h_f(\alpha) = S$. Likewise, for any $\epsilon > 0$ there exists a valuation $g : \mathbb{P} \to \mathcal{M}$, and a measurable set, S, with $m(S) < \epsilon$ and $h_g(\alpha) = \overline{S}$.*

Proof Sketch Let $\epsilon > 0$. We define a function $\Phi^* : [0,1] \to T_2$, using thick Cantor sets of measure $1 - \epsilon$, but otherwise identical to Φ. Let K^* be the thick Cantor set of measure $1 - \epsilon$. Then stagewise labeling functions, Φ^*_n, are constructed as in Definition 5.13, but using K^* instead of K. Again, let Φ^* be the limit of stagewise labeling functions, Φ^*_n (see Definition 5.14). Then the root of T_2 is pulled back, under Φ^*, to K^* (plus midpoints), a set of measure $1-\epsilon$. We now simply repeat the argument given in the proof of Theorem 6.3. If α is a non-theorem of $S4$, let $f' : \mathbb{P} \to B(T_2)$ be a non-degenerate valuation that falsifies α at the root, $\langle\cdot\rangle$, of T_2. Defining $f : \mathbb{P} \to B([0,1])$ as the Φ^*-pullback of f' ($f = [\Phi^{*-1}] \circ f'$), we know that $K^* \subseteq \Phi^{-1}(\langle\cdot\rangle) \subseteq (h_f(\alpha))^c$. Letting $S = h_f(\alpha)$, we have $m(S) \leq \epsilon$. Furthermore, defining $g : \mathbb{P} \to \mathcal{M}$ by $g(P) = \overline{f(P)}$, we have $h_g(\alpha) = \overline{h_f(\alpha)} = \overline{S}$.[15] $\square$

As a final corollary, we prove that Intuitionistic propositional logic (IPC) is complete for the frame $\mathcal{G}$. Let the propositional language L_0 consist of a countable set, $\mathbb{P} = \{P_n \mid n \in \mathbb{N}\}$, of atomic variables and be closed under binary connectives $\to, \vee, \wedge, \leftrightarrow$ and unary operator $\neg$. Recall that $\mathcal{G}$ is a complete Heyting algebra. In particular, for any elements $x, y \in \mathcal{G}$, there exists an element, $x \Rightarrow y \in \mathcal{G}$, called the *relative pseudo-complement* of x with respect to y and defined by:

$$sup\,\{c \in \mathcal{G} \mid c \wedge x \leq y\}$$

Semantics For any valuation $f : \mathbb{P} \to \mathcal{G}$ assigning propositional variables to arbitrary elements of $\mathcal{G}$, we define the extension h_f by: $h_f(\phi \to \psi) = h_f(\phi) \Rightarrow h_f(\psi)$. h_f is defined in the usual way on $\{\&, \vee\}$. ('$\neg\phi$' abbreviates '$\phi \to \bot$' and '$\phi \leftrightarrow \psi$' abbreviates '$\phi \to \psi \,\&\, \psi \to \phi$'.)

For any formula $\Phi \in L_0$, let $T(\phi)$ be the Gödel–Tarski translation of ϕ given inductively as follows:

$$\begin{aligned} T(P) &= \Box P \text{ for all propositional variables } P \\ T(\bot) &= \bot \\ T(\phi \vee \psi) &= T(\phi) \vee T(\psi) \\ T(\phi \wedge \psi) &= T(\phi) \wedge T(\psi) \\ T(\phi \to \psi) &= \Box(T(\phi) \to T(\psi)) \end{aligned}$$

Gödel and Tarski showed that $\vdash_{IPC} \alpha$ iff $\vdash_{S4} T(\alpha)$ for any formula $\alpha \in L_0$. Moreover, for any valuation $f : \mathbb{P} \to \mathcal{M}$, we can define the valuation,

[15] Lemma 5.23 still holds under this construction. The new recursive equations for S_n and E_n are as follows: $S_0 = 1$, $S_{n+1} = \frac{\epsilon}{2} S_n$ and $E_n = (1-\epsilon) S_n$. Solving, we get: $E_n = (1-\epsilon)\left(\frac{\epsilon}{2}\right)^n$. So

$$\text{measure}\left(\Phi^{-1}(F)\right) = \sum_{n\geq 0} 2^n E_n = \sum_{n\geq 0} 2^n (1-\epsilon)\left(\frac{\epsilon}{2}\right)^n = (1-\epsilon)\sum_{n\geq 0}\epsilon^n = 1$$

$f_I : \mathbb{P} \to \mathcal{G}$, by $f_I(P) = h_f(\Box P)$. It is easy to show[16] that for any formula, $\alpha \in L_0$, $T(\alpha) \in L_1$ and

$$h_{f_I}(\alpha) = h_f(T(\alpha))$$

In particular, $h_f(T(\alpha)) \in \mathcal{G}$ for each $\alpha \in L_0$ (the Gödel translation of any formula is evaluated to an open element).

Corollary 6.5 *IPC is complete for* $\mathcal{G}$.

Proof Suppose $\nvdash_{IPC} \alpha$. Then $\nvdash_{S4} T(\alpha)$. By completeness of $S4$ for $\mathcal{M}$, there is a valuation $f : \mathbb{P} \to \mathcal{M}$ with $h_f(T(\alpha)) \neq \overline{[0,1]}$. But letting f_I be defined as above, we have $h_{f_I}(\alpha) = h_f(T(\alpha)) \neq \overline{[0,1]}$, so α is falsified under f_I in $\mathcal{G}$. □

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Appendix

Proof of Lemma 5.20 The proof makes use of the following two claims:

Claim 1 If I is a MUL interval under Φ_n, whose points are labeled by a finite node, t, then $U_t \subseteq \Phi(I)$.

Claim 2 If I is any interval, then $I - K(I)$ is a countable disjoint union of open intervals $\{(a_k, b_k) \mid K \in \mathbb{N}\}$. If $x \in K(I)$ or x is an endpoint of I, then for any open O with $x \in O$, $(a_k, b_k) \subseteq O$ for some $k \in \mathbb{N}$.

(The proof of Claim 1 can be found in [4], and Claim 2 follows quite easily from construction of the thick Cantor set.)

[16]The proof is by induction on the complexity of α. For propositional variables, P, the claim is true by definition of f_I. The interesting case is $\phi \to \psi$, which we prove as follows:

$$\begin{aligned}
h_{f_I}(\alpha) &= h_f(T(\alpha)) \\
&= h_{f_I}(\phi) \Rightarrow h_{f_I}(\psi) \\
&= sup_{\mathcal{G}}\,\{c \in \mathcal{G} \mid c \wedge h_{f_I}(\phi) \leq h_{f_I}(\psi)\} \\
&= sup_{\mathcal{G}}\,\{c \in \mathcal{G} \mid c \wedge h_f(T(\phi)) \leq h_f(T(\psi))\} \\
&= sup_{\mathcal{M}}\,\{c \in \mathcal{G} \mid c \leq -h_f(T(\phi)) \vee h_f(T(\psi))\} \\
&= sup_{\mathcal{M}}\,\{c \in \mathcal{G} \mid c \leq h_f(T(\phi) \to T(\psi))\} \\
&= I(h_f(T(\phi) \to T(\psi))) \\
&= h_f(\Box(T(\phi) \to T(\psi)))
\end{aligned}$$

where $sup_{\mathcal{G}}$ and $sup_{\mathcal{M}}$ denote suprema in $\mathcal{G}$ and $\mathcal{M}$ respectively. In fact, suprema coincide in $\mathcal{G}$ and $\mathcal{M}$, so subscripts are unnecessary (see footnote 7).

There are two cases: $t(=\Phi(x))$ is a limit node, and t is a finite node. If (case 1) t is a limit node then, by construction of Φ, for each $n \in \mathbb{N}$, x belongs to some MUL interval $I_{n,x}$ under Φ_n, with $m(I_{n,x}) \to 0$. Since O is open, we know that for large enough N, $I_{N,x} \subseteq O$. Letting $t' = \Phi_N(x)$ (i.e. x's label at stage N), we know that t' is a (finite) ancestor of t. Moreover, by Claim 1, $U_{t'} \subseteq \Phi(I_{N,x}) \subseteq \Phi(O)$. We have

$$t \in U_{t'} \subseteq \Phi(O)$$

as needed.

If (case 2) t is a finite node, then for some $n \in \mathbb{N}$, x belongs to a MUL interval I under Φ_n whose points are labeled by t. Moreover, $x \notin A(I)$ and $x \notin B(I)$ (else $\Phi(x)$ a strict descendant of t). There are two subcases: $x \in K(I)$ or $x \notin K(I)$.

- If (subcase 1) $x \in K(I)$, then, letting $I - K(I)$ be the disjoint union of open intervals $\{(a_k, b_k) \mid k \in \mathbb{N}\}$, we know (by Claim 2) there exists $k \in \mathbb{N}$ with $(a_k, b_k) \subseteq O$. By construction of the finite labeling functions, $\left(a_k, \frac{b_k-a_k}{2}\right)$ is a MUL interval under Φ_{n+1}, all of whose points are labeled by t^*0. Similarly, $\left(\frac{b_k-a_k}{2}, b_k\right)$ is a MUL interval under Φ_{n+1}, all of whose points are labeled by t^*1. So by Claim 1,

  $$U_{t^*0} \subseteq \Phi\left(a_k, \frac{b_k - a_k}{2}\right) \subseteq \Phi(O)$$

  and

  $$U_{t^*1} \subseteq \Phi\left(\frac{b_k - a_k}{2}, b_k\right) \subseteq \Phi(O)$$

  It follows that $U_t \subseteq \Phi(O)$.
- If (subcase 2) $x \notin K(I)$, then x is the midpoint of some interval (a_k, b_k) in $I - K(I)$. Let $A_k^I = \left(a_k, \frac{b_k-a_k}{2}\right)$, and let $B_k^I = \left(\frac{b_k-a_k}{2}, b_k\right)$. Note first that

  $$t^*0, t^*1 \in \Phi(O)$$

  (Why? $K\left(A_k^I\right) \subseteq \Phi^{-1}(t^*0)$ and points in $K\left(A_k^I\right)$ approach endpoints of A_k^I (by Lemma 5.11), hence approach x.) Second, note that

  $$U_{t^*00}, U_{t^*01}, U_{t^*10}, U_{t^*11} \subseteq \Phi(O)$$

  (Why? Letting $A_k^I - K(A_k^I)$ be the disjoint union of open intervals $\{(c_j, d_j) \mid j \in \mathbb{N}\}$, we know, by Claim 2, that for some $j \in \mathbb{N}$, $(c_j, d_j) \subseteq O$. But $\left(c_j, \frac{d_j-c_j}{2}\right) \subseteq A\left(A_k^I\right)$ and $A\left(A_k^I\right)$ is a MUL interval under Φ_{n+2} whose points are labeled by t^*00. But then, by Claim 1, $U_{t^*00} \subseteq \Phi(O)$. By symmetry, the same holds for U_{t^*01}, U_{t^*10}, and U_{t^*11}.)

  We conclude that $U_t \subseteq \Phi(O)$. □

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