Chapter 12

CATEGORY THEORY AS A FRAMEWORK FOR AN *IN RE* INTERPRETATION OF MATHEMATICAL STRUCTURALISM

Elaine Landry
University of Calgary
elandry@ucalgary.ca

The aim of this paper is to present category theory as a framework for an *in re* interpretation of mathematical structuralism. The use of the term 'framework' is significant. On the one hand, it is used in distinction from the term 'foundation'. As such, what I propose is that we consider category theory as *a philosophical tool* that allows us to *organize* what we say about the shared structure of abstract kinds of mathematical systems.[1] On the other hand, the term 'framework' is used in the sense of Carnap [1956]. That is, category theory is taken as *a language*[2] used to frame *what we say* about the shared structure of abstract kinds of mathematical systems, as opposed to being a "background theory" which constitutes what a structure is.[3]

12.1 Foundation versus Framework

In this section, I consider what it means to say that category theory is a framework for mathematical structuralism, though not a foundation for mathematics. I will show, contra Feferman [1977] and Mayberry [1994], that the

[1]This in contrast to viewing category theory as a mathematical foundation that provides us with the "atoms" (of meaning or reference) of mathematics itself, e.g., that it tells us what, or whether, a structure is.
[2]See Landry [1999; 2001] for further elaboration of what is meant by taking category theory as a language.
[3]In this sense, the use of a category-theoretic linguistic frame is in contrast, to, for example, Shapiro's [1997] ontological, *ante rem*, reading of the concept of structure which uses "structure theory" to frame the claim that mathematical structures exist both over and above systems that exemplify them and independently of language.

J. van Benthem, G. Heinzmann, M. Rebuschi and H. Visser (eds.), The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today, 163–179.

reason category theory cannot provide a foundation for mathematics is not that it depends on set theory as either an ontological or conceptual base. Rather it is that category theory cannot be construed as *being about* either objects or structures *qua* (actually or possibly) existing things. Relying on the work of Lawvere [1966] and McLarty [1990], we will see the Feferman's criticisms miss their mark, and, moreover, we will see that category theory satisfies Mayberry's criterion of being a "foundational sea" to the same degree that set theory does. Yet, while category theory cannot provide a foundation for mathematics, it remains, as Bell [1981] notes, "foundationally significant".

12.1.1 Categories as "Structures"

Since Lawvere's work with the category of categories has provided much grist for the foundational mill, let us consider what he says of his aims in this regard:

> [i]n the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of abstract structure rather than in terms of the elements which objects were thought to be made of. The question naturally arises whether we can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about and in particular in which classes and membership in classes do not play any role... (Lawvere quoted in Feferman, [1977], pp. 149-150).

It is as an answer to this challenge, then, that Lawvere [1966] "formulated a (first-order) theory whose objects are conceived to be arbitrary categories and functors between them". (Feferman, [1977], p. 150). It is held, by Feferman (and Bell [1981]), that the problem with such an account is that when it comes to accounting for categories as themselves abstract "structures" and/or using categories to account for *abstract kinds* of "structures", one must appeal to notions which fall outside the range of category theory.[4] As Feferman explains:

> when explaining the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc., we implicitly presume as understood the ideas of operation and collection; e.g., we say that a group consists of a collection of objects together with a binary operation satisfying such conditions ... when explaining the notion ... of functor for categories, etc., we must again understand the concept of operation ... (Feferman, [1977], p. 150).

[4]Specifically, Feferman claims that in either case, "[t]he logical and *psychological* priority if not primacy of the notion of operation and collection is ... evident" (Feferman, [1977], p. 150). And from this concludes that "[i]t is evidently begging the question to treat collections (and the operations between them) as a category which is supposed to be one of the objects of the universe of the theory to be formulated". (Feferman, [1977], p. 150.) Feferman's claim can be understood as follows: if we assume that mathematics is the study of abstract structure, then, insofar are categories themselves are structured (and, presumably, structured in terms of operation and collection), we need a general account of the very notion of structure itself.

That is, even if category theory can give a more general account of abstract kinds of structure than can set-theory,[5] we are still in need of a (meta) theory which makes "use of the *unstructured* notions of operation and collection to explain the structural notions to be studied". (Feferman, [1977], p. 150). To this end he provides a non-extensional type-free theory of operations and collections wherein "much of 'naïve' or 'unrestricted' category theory can be given an account..." (Feferman, [1977], p. 149).[6]

Now, if one had stopped one's inquiry here, one might be convinced, but as Feferman goes on to note, the schemes offer by Grothendieck Universes and the Gödel-Bernays theory of classes, readily offer the needed (meta) theory for category theory (though not in terms of operation and collection).[7] That is, Feferman himself concedes that

> there is no urgent or compelling reason to pursue foundations of unrestricted category theory, since the schemes... serve to secure all practical purposes... The aim in seeking a new foundation is mainly as a problem of logical interest motivated largely by aesthetic considerations (or rather by the inaesthetic character of the present solutions). (Feferman, [1977], p. 155)

To make his reasons compelling, then, Feferman needs to have demonstrated that the notions of operation and collection themselves are, in some sense, *constitutive* of the notion of structure, and he has not. Independently of Feferman offering-up these reasons, there are two possible, though not unrelated, responses to his claim that we yet need, to account for mathematics as the study of abstract structures, a non-extensional type-free theory of unstructured operations and collections. One that structuree' is not strictly a mathematical notion, and hence, such problems need not be resolved by providing a foundation for mathematics, but rather are best addressed by offering a philosophy for mathematics (see § 12.3). The second, though not unrelated, response is that a category *qua* a structured system is to be "algebraically" considered (see § 12.2). In either case, we note that category theory itself, i.e., without the

[5] Of the inadequacies of set theory as a foundation, Feferman says: "Since neither the realist (extensional) or the constructivist (intensional) point of view encompasses the other, there cannot be any present claim to *universal foundation* for mathematics..." (Feferman, [1977], p. 151.)

[6] While Feferman agrees with Mac Lane that work in elementary topos theory (ETS) shows the "formal" equivalence between ETS(Z) and ETS(ZF) and the theories of Z and ZF, respectively, he claims his point stands; because the "use of 'logical priority' refers not to the relative strength of formal theories but to the order of the definition of the concepts"... and "that the general concepts of operation and collection have logical priority with respect to structural notions (such as 'group', 'category', etc) because the latter are defined in terms of the former but are not conversely". (Feferman, [1977], p. 152.)

[7] Another scheme, offer by Bell [1986;1988], is to characterize (up to categorical equivalence) topoi as models of a higher-order, intuitionistically based, type theory; thus, allowing us to re-capture the sense in which set-theory and category theory are "formally" equivalent, i.e., by allowing for the specification of topoi as "local set theories".

background schemes, cannot provide a foundation any more than set theory: it cannot tell us what, or whether, structure is.[8]

12.1.2 The category of categories as a "Foundational Sea"

I now turn to consider Mayberry's claim that, because sets and their morphology are constitutive of the notion of structure, only set theory can provide a foundation for mathematics. My aim is to show that, while it can be agreed that "when we employ the axiomatic method we are dealing with structures",[9] it simply does not follow that "when we are dealing with mathematical structures, we are engaged in set theory". (Mayberry, [1990], p. 19) In particular, I will argue that there is no reason to hold that "each structure consists of a set or sets equipped with a morphology". (Mayberry, [1990], p. 19.)

Mayberry acknowledges that there are problems with his version of structuralism founded on an 'intuitive' set theory, viz., that it cannot be used to talk about the *large* categories,[10] for example, the category of *all* (small) groups. He further recognizes that

> to consider such categories seems a quite natural extension of ordinary structuralism, it appears to request the next level up in generality in which the notion under investigation is the notion of structure itself. (Mayberry, [1990], p. 35.)

His solution to this problem, however, is far from satisfying: it is to dismiss talk of such structures by simply denying that they are structures. He says

> [i]n fact, there can be no such structures, for the very notion of set is that of an extensional plurality limited in size, and the notion of set is constitutive of our notion of structure. (Mayberry, [1990], p. 35.)

The claim that the notion of set is that of an extensional plurality limited in size is both *ad hoc* and misleading: the only justification that Mayberry's privileging of 'intuitive' set theory has is that, given his claim that set is constitutive of our notion of structure, it makes his conclusion, that 'intuitive' set theory provides a foundation, follow. Consider, if, instead of defining a set intuitively as "an extensional plurality of determinate size, composed of definite

[8]Given set theory's inability to form the category of *all* structures of a given kind (groups, topological spaces, categories) and to form the category of all functors of any given category it cannot be used to 'foundationalize' category theory, and given category theory's inability to refer to *all* categories as 'objects' in the categories of categories, without making use of either Grothendieck Universes or a Gödel-Bernays theory of sets and classes, it cannot be seen as providing a foundation in and of itself. (See Feferman, [1977], pp. 154–155 for a brief but informative discussion of these issues.)

[9]That is, while it can be agreed that the aim of a structuralist foundation (or, more accurately, a structuralist philosophy) is to capture the belief that the subject matter of mathematics is structured systems and their morphology.

[10]Note, however, that it is not because it is large that the category of categories cannot be taken as a foundation. For a discussion of the various interpretations of large categories, (see McLarty, [1995], pp. 105–110).

property-distinguished objects" (Mayberry, [1990], p. 32) we define a category 'intuitively' as an object of *indeterminate* size,[11] composed of definite, functorially-distinguished objects. Then, following Lawvere, we could conclude that category theory provides a foundation for mathematics.[12]

While it seems clear, then, that neither set theory nor category theory can be a foundation in the sense of providing a *theory* which captures the idea that the subject matter of mathematics is structures and their morphology, it should also be clear that neither can it provide a foundation in the sense of providing "a sea in which structures swim".[13] Thus, while it is right to conclude that, on Feferman and Mayberry's "structuralist" criterion, category theory cannot provide a foundation for mathematics, this is not because it requires a prior notion of either operation or collection or 'intuitive' set theory. It is because, if it is to be counted as an "object language" for our talk of structures, it requires some prior, meta-theoretical, notion of structure that category theory itself cannot provide.[14]

12.2 Structures and structured systems

If we accept, then, that mathematics is the study of abstract structure, we must explain in what sense category theory provides the *philosophical tool* for organizing what we say about the shared structure of abstract kinds of mathematical systems. I begin first with Corry's [1996] historical investigation of the development of the 'algebraic' notion of structure. The aim here is to distinguish the set-theoretic path of the Bourbaki notion of structure from the algebraic path of the category-theoretic notion. Given this distinction, two observations can be made. The first, that the Bourbaki notion implicitly assumes an ontology out of which structures are made, i.e., assumes that types of structures are kinds of set-structured systems. The second, that this assumption leads to a reification of structure, i.e., leads to interpreting structures themselves as independently

[11]By 'indeterminate size' it is meant that we can define a category as large, either in the Gödel-Bernays sense, or in terms of Grothendieck Universes. That is, we do not have to restrict the size of a category by characterizing its objects and morphisms in terms of sets.

[12]To see this, in the following quote by Mayberry, simply replace 'set' with 'category' and 'universe of sets' with 'category of categories'. "The *fons et origo* of all confusion here is the view that set theory is just another axiomatic theory and that the universe of sets is just another mathematical structure … The universe of sets is not a structure; it is the world that all mathematical structures inhabit, the sea in which they all swim." (Mayberry, [1990], p. 35.)

[13]And this fact cannot be altered by claiming that either stands along the shore of these issues since it is needed to provide a semantics for mathematics. As McLarty notes, "Mayberry … has simply confused his own head with Lawvere's. [By claiming that "the idea of denying intuitive set theory its function in the semantics of the axiomatic method never entered Lawvere's head in his treatment of the categories of categories". (Mayberry, [1977]).] Lawvere believes 'intuitive' categories, and spaces, and other structures are just as real (or, more accurately, just as ideal) as 'intuitive' sets." (McLarty, [1990], p. 364.)

[14]For example, even though the category of categories can be used to talk about the shared structure of categories *qua* kinds of structured systems, it cannot be used to axiomatically define (all) categories *qua* structures.

existing things. In contrast to such set-theoretic and/or ontological readings of what structure is, I will use this history to point to a category-theoretic, *schematic*[15] *interpretation* of types of structured systems.

12.2.1 What *kind* structures number systems?

In the development of Abstract Algebra,[16] the use of kinds of "structures", as tool and/or unifying concepts, is evident. This development has its beginning in the various attempts at answering the question: "What structures number systems?". For Dedekind, the subject matter of "algebra"[17] may be considered in two different ways. On the one hand, we may consider the properties of number systems *qua* collections, wherein we overlook *the nature* of the elements involved. The *tools* which Dedekind used to talk about the algebraic structure of number systems, considered as such, were groups, ideals and modules. On the other hand, we may consider the properties of the elements of number systems and the interrelations among 'rational domains' contained in it. For Dedekind, the *unifying concept* for such an analysis was thought to be that of a field.[18]

Hilbert, continuing this "algebraic" analysis of number systems, maintained the distinction between properties of numbers systems (though not *qua* collections) and properties of the elements of number systems and their interrelations: he used invariants, ideas, rings, groups and fields as *tools* to talk about the latter. To talk about properties of number systems, he took a geometric turn, and considered them *qua* postulational systems. The *unifying concepts* for talking about number systems as such were the 'logical' (or meta-mathematical) properties of axiom systems themselves, namely, independence and consistency. In addition to this "algebraic" investigation was Hilbert's meta-mathematical analysis, which took axiom systems and their properties as objects of study in their own right. Thus, while we had, with Dedekind, that, in some sense, number systems themselves were the basis for algebraic analysis, the question at hand was "Could in his [Hilbert's] view the conceptual order be turned around

[15]I use the term 'schematic' in the sense of Goldfarb [2001].

[16]The reader is strongly encouraged to read Corry's [1996] insightful and informative account of this. While I stop short of fully accepting his account of the category theory's 'significance', I note here a debt to, and reliance on, his presentation of the 'facts' of the development of the notions of kinds and types of algebraic and mathematics structures.

[17]The term 'algebra' is placed in quotes since at the time this was not a well defined field. It may be characterized as the "theory of solving equations" (see Hasse, [1954], p. 11.)

[18]As Dedekind, himself, explains: "... I have attempted to introduce the reader to a higher domain, in which algebra and the theory of numbers interconnect in the most intimate matter ... I got convinced that studying the algebraic relationship of number is most conveniently based on a concept that is directly connected with the simplest arithmetic principles. I have originally used the term "rational domains". Which I later changed to "field". (*Werke*, p. 400)... The term [field] should denote here, in a similar fashion as in the natural sciences, in geometry, and in the social life of men, a system possessing a certain completeness, perfection and comprehensiveness, by mean of which it appears as a natural unity". (Dedekind, [1894], p. 452.)

so that the system of real numbers be dependent on the results of [the axiomatic analysis of] algebra rather than being the basis for it?" (Corry, [1996], p. 172)[19].

Appreciating the "foundational value" of the axiomatic method, Noether applied this shift in conceptual priority to Dedekind's subject matters. That is, to the properties of number systems *qua* collections (again, overlooking the nature of the elements) she proposed ideals, modules, groups and rings as *tools* for talking about their algebraic structure. Such tools, in light of Hilbert, where themselves now considered *qua* axiom systems. In a similar vein, for systems of abstract elements of any axiom system, the *unifying concept* was thought to be abstract rings, or the axiomatic presentation of rings themselves. Whereas Dedekind had considered properties of concrete elements of number systems and the field-theoretic interrelations between them as unifying, Noether considered the properties of abstract elements of abstract rings *qua* axiom systems as unifying. In this manner the unifying power is taken out of concrete number systems and put into *an abstract kind* of axiomatically presented structured system. As Corry explains:

> Noether's abstractly conceived concepts provide a natural framework in which conceptual priority may be given to the axiomatic definitions [of concepts] over the numerical systems considered as concrete mathematical entities. With Noether, then, the balance between the genetic and the axiomatic point of view begins to shift more consciously in favour of the latter. (Corry, [1996], p. 250)

These developments in the analysis of the algebraic structure of number systems gave rise to the independent branch of study of Abstract Algebra, wherein the focus of analysis was now the shared structure of the *abstract kinds* of algebraic systems (e.g., groups, rings fields) considered in themselves (typically considered qua axiom systems).[20] That is, those very tools and/or concepts that were once useful or unifying when talking about the algebraic structure of concrete, number, systems are now seen as systems of study in there own right.[21]

---

[19]Hilbert responded to such a query by distinguishing between the *genetic* and the *axiomatic* method, and, at least as regards the 'foundations' of mathematics, he held a preference for the latter: he says, "In spite of the high pedagogic value of the genetic method, the axiomatic method has the advantage of providing a conclusive exposition and full logical confidence to the contents of our knowledge." (Hilbert, [1900], p. 184) and "When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of the science. The axioms so set up are at the same time the definition of those elementary ideas, and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps." (Hilbert, [1902], p. 447.)

[20]Exemplifying this shift is van der Waerden's *Modern Algebra*, in which "... different mathematical domains are considered as individual instances of algebraic structures, and therefore undergo similar treatments; they are abstractly defined, they are investigated by recurrently using a well-defined collection of key concepts, and a series of questions and standard techniques is applied to all of them." (Corry, [1996], p. 252.)

[21]As Hasse witnesses: "It is characteristic of the *modern* development of algebra that the tools specified about [i.e., groups and fields] have given rise to far-reaching autonomous theories which are more and more

12.2.2 What *type* structures abstract kinds of mathematical systems

Given this structural approach to abstract algebraic systems, the next question that arose was: What is the tool and/or unifying concept that allows us to talk about such *abstract kinds* of systems as instances of the same mathematical *type*? I begin first with Ore for whom the *type* which structures the various kinds of algebraic systems is the lattice. More specifically, what "structures" kinds of algebraic systems are the (union and cross-cut) properties of the lattice of certain subsystems of any given system. Here, then, is where we note both Hilbert's axiomatic influence and Noether's "set-theoretic"[22] influence. What is new, however, is that, in addition to overlooking the nature of the elements, we overlook too their *existence*. As Ore explains:

> In the discussion of the structure of algebraic domains, one is not primarily interested in the elements of these domains but in the relations of certain *distinguished sub-domains*. . . For all these systems there are defined *two operations* of union and cross-cut satisfying the ordinary axioms. This leads naturally to the introduction of new systems, which we shall call *structures*, having these two operations. The elements of the structure correspond isomorphically with respect to union and cross-cut to the distinguished subdomains of the original sub-domain while the elements of the original domain are completely eliminated in the structure. (Ore, [1935], p. 406.)

It is in this sense that the lattice-theoretic properties were taken as the unifying concepts for algebra; lattice theory, itself, was taken by Ore as *the formal tool* for providing a general structural account of the various kinds of algebraic systems, and, quite possibly, as having "foundational significance" insofar as it may further provide a structural account of the various kinds of mathematical systems as well.[23]

In contrast to Ore, for Bourbaki a type of structure is a system of elements that has a set-structure, that is, one overlooks the specific nature of the elements in favour of their algebraic, order or topological structure. As Shapiro notes:

---

replacing the basic problem of *classical* algebra . . . Thus in the modern interpretation algebra is no longer merely the theory of solving equations, but the *theory of formal calculating* domains, as fields, groups, etc.: and its basic problem has now become that of obtaining an insight into the structure of such domains. . . " (Hasse, [1954], p. 11.)

[22]To explain the reading I give to Noether's use of the term 'set-theoretic', I point to Corry's telling remark that: "[t]he expression "purely set-theoretic considerations", in Noether's usage, does not refer to concepts nowadays related to the theory of sets (membership, power, etc.). It denotes arguments for proof in algebra, which do not rely on the properties of the operation defining the [system] under inspection, but rather properties of the inclusions and intersections of sub-[systems] of it. (Corry, [1996], p. 244) . . . Such an approach would certainly correspond to the problem, mentioned by Alexandrov, of "axiomatizing the notion of a group from its partition into cosets as the fundamental concept" (Corry, [1996], p. 248.)

[23]As Corry notes: "At that opportunity [the 1936 International Congress of Mathematicians at Oslo] Ore claimed that the guidelines of his program, although originating with algebra, should not be limited to that domain alone, and he envisioned that they would be applied in additional fields of mathematics as well." (Corry, [1996], p. 276.)

> According to Bourbaki, there are three great types of structures, or "mother structures": algebraic structures, such as group, ring, field; order structures, such as partial order, linear order, and well order; and topological structures [which provides a formalization of the concepts of limit, neighbourhood and continuity]. . . (Shapiro, [1997], p. 176.)

Yet, as types of set-structured systems, one does not overlook the existence of elements: what structures the elements of kinds of mathematical systems into their respective types are the relations that hold between such systems *qua* set-theoretically presented axiom systems. Like Hilbert and Noether, Bourbaki's attention was focused on the axiomatic method. Unlike Hilbert, who focused on the logical properties of axiom systems, or Noether and Ore who focused on the properties of the inclusions and intersections of subsystems, Bourbaki used various set-theoretic types of structures *qua* axiom systems to unify what could be said of the various kinds of mathematically structured systems.

What remains open for discussion is whether, and in what sense, Bourbaki intended the theory of sets to be *constitutive* of the concept of structure, i.e., intended it as an answer to the question: "What is structure?". Whatever their intention might have been, the tension between the account of set theory as a *formal* language and the heuristic role of the formally, though implicitly, defined concept of structure was pulling at the seams of their 'algebraic' structuralism.[24] In any case, whether set-theory was intended to be used foundationally or heuristically, what appears to be true is that the efforts of Bourbaki were interpreted, both mathematically and philosophically, as providing a set-theoretically constitutive account of what structure is and, in so doing, shifted from the algebraic tradition's attempts to overlook *the nature of the elements* of kinds of mathematical systems in favour of abstractly characterizing their shared structure. As Bell explains:

> With the rise of abstract algebra. . . the attitude gradually emerged that the crucial characteristic of mathematical structure is not their internal constitution as set-theoretical entities but rather the relationship among them as embodied in the network of morphisms. . . However, although the account of mathematics they [Bourbaki] gave in their *Eléments* was manifestly structuralist in intention, actually they still defined structures as sets of a certain kind, thereby failing to make them truly independent of their 'internal constitution'. (Bell, [1981], p. 351.)

For the Bourbaki structuralist what unifies kinds of mathematical systems are types, and, more significantly, what appears to *make* these types *"powerful tools" for unification*, is the constitutive character of set theory.[25] In this manner,

[24]As they, themselves, note: "[t]he reader may have observed that the indications given here [of the concept of structure] are left rather vague; they are not intended to be other than heuristic, and indeed it seems scarcely possible to state general and precise definitions for structure outside the framework of formal mathematics". (Bourbaki, [1968], p. 347, footnote.), (Corry, [1996], p. 326.)

[25]Speaking to this "constitutive" reading, we note the following quotes of Bourbaki: "Each structure carries with it its own language, freighted with special intuitive references derived from the theories which the

types, or "structures", as set-structured systems are turn into "things". In contrast, the category-theoretic structuralist holds that what unifies kinds are types as cat-structured systems, yet, what makes these types tools for unification, is the *schematic* use of categories, in particular, and the organizational role of category theory, in general. Wherein, then, lies this distinction? It is that, nothing, in particular, is constitutive of what a category *is*. As Mac Lane explains:

> [i]n this description of a category, one can regard "object", "morphism", "domain", "codomain", and "composites" as *undefined terms or predicates*. (Mac Lane, [1968], p. 287, italics added.)

Like Bourbaki, we thus characterize the shared structure of abstract kinds of mathematical systems *qua* a type of structured system.[26] Yet unlike Bourbaki we need not take set, or, indeed, any particular kind of set, to be constitutive of what these types are themselves types of (though, of course, we might). Again, as Mac Lane explains:

> Bourbaki's concepts defined "mathematical structures" by taking an abstract set and appending to it an additional construct, in category theory there is no subordination of "mathematical structures" to sets, and this is the source of the supremacy of this theory over Bourbaki. (Mac Lane, [1980], p. 382.)

Moreover, in the spirit of Lawvere [1966], we can use Cat (or CAT) as the type used to talk about what structures these kinds of cat-structured systems, again, without having to appeal to set as constitutive of what this type is a type of. What we must note, however, is that, contra Lawvere, we, like our set-theoretic cousins, cannot use category theory as a *formal language*, or foundation. That is, we cannot use it to answer the question: "What is a mathematical structure *qua* a either a kind or type of category?". As Corry explains,

> [i]n no sense, however, has category theory provided, to this day, a definite, or even a provisionally satisfactory answer to the question of what is a "mathematical structure" .... Neither does category theory provide ultimate foundations for mathematics. (Corry, [1996], p. 389, italics added.)

---

axiomatic analysis ... has derived the structure... Mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does the intuition dominate the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of the great type of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method ..." (Bourbaki, [1950], pp. 227–228) and further that "... whereas in the past it was thought that every branch of mathematics depended on its own particular intuitions which provided its concepts and primary truths, nowadays it is known to be possible, logically speaking, to derive practically the whole of mathematics from a single source, the theory of sets." (Bourbaki, [1968], p. 9.)

[26]For example, Set, Top, Group are types, i.e., kinds of cat-structured systems, that allow us to talk about the shared structure of abstract kinds of mathematical systems in terms of their being instances of the same type.

12.3 A schematic *in re* interpretation of mathematical structuralism

The final section of this paper brings together the above investigations to present a category-theoretically framed *in re* interpretation of philosophically positioned mathematical structuralism. The objective of this section is to show that it is in following the Bourbaki tradition too closely and, thereby, not appreciating the algebraic alternative, that philosophically interpreted mathematical structuralism has most failed us. Seen in this light, my aim is to first argue that category-theoretic analysis ought to be best seen as answering "What are the types that "structure" abstract kinds of structured systems?" (as opposed to speaking to the foundational/ontological claim that "structures" *are*) and, second, to separate these analyses from those which end with claims that types of structured systems, or "structures", are set-structured (or place-structured) "things".

12.3.1 Levels, interpretations and varieties of mathematical structuralism

Mathematical structuralism can be construed as the philosophical position that the subject matter of mathematics is structured systems and their morphology,[27] so that mathematical objects are nothing but "positions in structured systems" and mathematical theories aim to describe such objects and systems via their shared structure. At the level at which we consider *concrete kinds* of structured systems, *i.e.*, the level where 'system' means 'model', we have objects as positions in models and can use either isomorphisms or embeddings to talk about the shared structure of such kinds. For example, the theory of natural numbers aims to describe concrete systems of the natural-number structure, as characterized by the Peano axioms, so that its objects may be seen as von Neumann ordinals, Zermelo numerals, or any other object which shares the same structure, or morphology. If all systems that share this structure are isomorphic, we say that the natural-number structure and its morphology determine its objects up to isomorphism. Analogous, then, to the shift in levels that one finds in the mathematical history of the development of the notion of algebraic structure, at the next level of philosophical analysis one finds the question: "What structures *abstract kinds* of structured systems"? In answer to this question, in the philosophical literature, one finds two interpretations of

[27]Note here that I have changed the slogan of structuralism from "mathematics is about structures and their morphology" to "mathematics is about structured systems and their morphology". This shift is intentional, it means to indicate that the aim of the structuralist is to account for the shared structure of mathematical systems in terms of kinds or types, as opposed to answering the question: "What is a structure?", or "What are the kinds or types that are constitutive of what a structure is?" This shift is further discussed in § (12.3.2).

mathematical structuralism: *ante rem and in re*. The latter is aligned with a realist view of structures insofar as it holds that "structures exist as legitimate objects of study in their own right. According to this view, a given structure exists independently of any system that exemplifies it ..." (Shapiro, [1996], pp. 149-150). *In re* structuralism, in contrast, is aligned with a nominalist view of structures insofar as it eliminates talk about structures in favour of talk about systems: "it does not countenance mathematical objects, or structures for that matter, as *bona fide* objects ... Talk of structure generally is convenient shorthand for talk about systems of objects". (Shapiro, [1996], p. 150.)

To further inform this debate, I rely on Aristotle's distinction between *prior in place* and *prior in definition*.[28] Against the *ante rem* structuralist, a category-theoretically framed *in re* interpretation of mathematical structuralism implies that there are no "structures", *qua* "things", over and above kinds of structured systems. As such, structures are not prior in place. Against the *in re* structuralist, categories *qua* schema are prior in definition insofar as they are needed, as an organizational tool (see Mac Lane [1992]), to talk about the shared structure of abstract kinds of structured systems as instances of the same type. Category theory, then, defines *what* a type of structured system is, but remains silent as to the claim *that* structure is.

Failing to heed Resnik's counsel (see Resnik, [1996], p. 96) that structuralism is not committed to asserting the existence of structures, yet, in response to this worry, three varieties of mathematical structuralism have been proposed, these are: the *set-theoretic*, the *sui generic*, and the *modal*.[29] In essence, these are suggested as "background theories" that allow us to talk about "structures" as either actually or possibly existing "things": they allow us to answer that either set-theory, structure-theory, or modal logic, provide the conditions for the actuality (or possibility) of a system being a "structure" of the appropriate kind.

12.3.2 The Bourbaki versus the "Algebraic" tradition

I now turn to my claim is that it is in following the Bourbaki tradition (which takes structures as set-structured "things") too closely and, thereby, not appreciating the algebraic alternative of mathematical structuralism that philosophically interpreted mathematical structuralism has most failed us. Witnessing this is Dummett's remark that:

[28]See the last two books, viz., M, N, of the Metaphysics (1076a5 – 1093b30), where Aristotle discusses mathematical objects and Ideas, and the manner in which these are prior in definition yet not, contra the Platonist, prior in place. See also *Metaphysics* Book V (1018b9–1019a14) where he discusses the various ways in which something can be correctly called prior to another.

[29]See (Hellman [2001]) for an excellent overview of these varieties and the problems associated with each.

> There is an unfortunate ambiguity in the standard use of the word 'structure', which is often applied to an algebraic or relational system - a set with certain operations or relations defined on it, perhaps with some designated elements; that is to say, a model considered independently of any theory which it satisfies. This terminology hinders a more abstract use of the word 'structure'; if, instead we use 'system' for the forgoing purpose, we may speak of two systems as having an identical structure, in this more abstract sense, just in case they are isomorphic. The dictum that mathematics is the study of structure is ambiguous between these two senses of 'structure'. If it is meant in the less abstract sense, the dictum is hardly disputable, since any model of a mathematical theory will be a structure in this sense. It is probably usually intended in accordance with the more abstract sense of 'structure'; in this case, it expresses a philosophical doctrine that may be labelled 'structuralism'. (Dummett, [1991], p. 295.)

While Dummett's analysis is, in some sense, helpful, it conflates two things: algebraic and set-theoretic accounts of types of structured systems, and concrete and abstract accounts of kinds of structured systems. Systems qua models can be used to account for the shared structure of a *concrete kind* of structured system, i.e., for the shared structure of the elements and/or properties of natural numbers *qua* set-structured systems. However, as we will see, algebraically read systems *qua* schematic types, as opposed to Bourbaki read "structures" *qua* set-theoretic types, may also be used to account for the shared structure of *abstract kinds* of structured systems. Instead, then, on focusing on the clarification of, and providing background theories for, the notion of structure as a "thing", I will focus on clarification of, and providing a framework for, the notion of a system as a *schema*. Thus, my aim as an algebraic structuralist is not the analysis of the constitutive character or modal status of "structures", but the analysis of the shared structure of abstract kinds of structured systems.[30]

I begin, then, with an *abstract* notion of a *system*, since, as we will see, this is where we find our corresponding notion of a *cat-structured* system. In its most general sense, a cat-structured system, then, has 'objects' and 'morphisms' as its abstract kinds which are structured by the category-theoretic axioms. So that, the schema for *a type* of structured system, *i.e.*, for a kind of mathematical system *qua* a category is

> ...anything satisfying these axioms. The objects need not have 'elements', nor need the morphisms be 'functions'... We do not really care what non-categorical

[30]We note, however, that Hellman [2002], does appreciate the distinction between the algebraic-schematic use of categories (what he calls the 'algebraico-structuralist perspective', p. 9), but his suggestion that the "problem of the 'home address' remains" (p. 8, p. 15), clearly indicates that he is stilling thinking of "structures" (be they categories of toposes) as 'things' requiring 'conditions for the possibility of existence'. In fact, however, if, on the algebraic approach, the aim of structuralism is to account for the shared structure of kinds of mathematical systems in term of schematic types, as opposed to answering "What is (or where is!) a structure?" then why should we be troubled by the fact that "[b]y themselves they [the category-theoretic axioms] assert nothing. They merely tell us what it is to be a structure of a certain kind" (p. 7) and thus are "unlike the axioms of set theory, [in that] its axioms are not assertory." (p. 7.)

> properties the objects and morphisms of a given category may have; that is to say, we view it 'abstractly' by restricting to the language of objects and morphisms, domains and codomains, composition, and identity morphisms. (Awodey [1996], p. 213.)

At once we see important differences: on the category-theoretic view, not only are there are no "objects" as either sets-with-structure (see Dummett, [1991], p. 295) or places-with-structure (see Shapiro, [1997, pgs. 73, 93]), there are no "structures" as either (equivalence types of) systems-with-structure or "the abstract form of a system, highlighting the interrelationships among the objects..." (Shapiro [1997], p. 74.) What this means is that the Bourbaki conception of a system (of a system whose "objects" are "positions in a set-structure",[31] or "places in a structure"[32]) is to be considered as a *kind* of structured system: it is not the archetype of either the concept 'system' or the concept 'structure'. A category, too, neither *constitutes* a privileged system or structure: it is a schematic type. It functions as a *philosophical tool* used to organize what we can say about the shared structure of the various abstract kinds of mathematically structured systems. The value, then, of this schematic notion of a cat-structured system is that it can be used to capture the shared structure of abstract kinds of structured systems, *independently* of its specific set-structure (independently of what its kinds are).[33]

We have shown, then, that if category theory is taken as the framework for what we say about the shared structure of abstract kinds of mathematical systems, then, we can account for a *schematic in re interpretation* of mathematical structuralism.[34] Against the *ante rem* structuralist, this category-theoretically framed *in re* interpretation of mathematical structuralism implies that there are no "structures'", *qua* "things" over and above kinds of structured systems.

[31]We can, however, present the underlying structure of a Bourbaki system, or equivalently present the *kind* of any set-structured system as a kind of cat-structured, by taking our objects to be sets and our morphisms to be functions. The result is the type of structured system called Set. But this does not mean that objects *are* sets and morphisms *are* functions, it means *in this type* of system propositions that talk about objects and morphisms can be interpreted as being about kinds of sets and functions.

[32]Shapiro's structure-theory itself is framed by ZF+ Coherence axiom.

[33]For example, in the kind of category called Top, we present the *topological-structure* by taking objects as kinds of topological spaces and morphisms as kinds of continuous mappings, independently of what those kinds are kinds of. As Awodey explains: "...suppose we have somehow specified a particular kind of structure in terms of objects and morphisms ... Then that category characterizes that kind of mathematical structure, independently of the initial means of specification. For example, the topology of a given space is determined by its continuous mappings to and from the other spaces, regardless of whether it was initially specified in terms of open sets, limit points, a closure operator, or whatever. The category Top thus serves the purpose of characterizing the notion of 'topological structure'." (Awodey [1996], p. 213.)

[34]Simply put, to talk about the shared structure of abstract kinds of mathematical systems in terms of kinds of cat-structured systems, there is no need for either set theory or structure theory or modal logic over and above category theory: a category acts as a schematic type that can be used to frame what we say about the shared structure of abstract kinds of mathematical systems, (in terms of types of cat-structured systems like Set, Group, or Top), *and* for kinds of cat-structured systems, (in terms of the types Cat or CAT). And, more significantly, it does so *without* our having to specify what these kinds are kinds of.

As such, categories as structures *are not* prior in place. Against the typical *in re* structuralist, however, categories as schema are prior in definition insofar as they are needed, as an organizational tool (Mac Lane [1992]), to talk about the shared structure of abstract kinds of structured systems as instances of the same type. Herein, then, lies the "foundational significance" (Bell, [1981]) of using category theory to frame an *in re* structuralist philosophy of mathematics: while the notion of a cat-structured system is privileged as a schema (is prior in definition) it is not reified as a constituting a structure (is not prior in place). Category theory, then, can act as *the other theoretical language* (see Carnap [1956]) because it permits us to *talk about* abstract kinds of structured systems *qua* cat-structured systems without our having to claim that category theory is either a "thing language" or that Cat (or CAT) is a "thing world". Thus, to be an algebraic *in re* structuralist about abstract kinds of mathematical systems, we need not provide a "background theory", that provides the conditions for the actuality (or possibility) of what, or whether, a category *qua* a structure is.

References

Alexandrov, P.S. and Hopf, H., [1935], *Topologie*, Springer, Berlin.

Awodey, S., [1996], "Structure in Mathematics and Logic: A Categorical Perspective", *Philosophia Mathematica*, (3), Volume 4, 209–237.

Bell, J.L., [1981], "Category Theory and the Foundations of Mathematics", *Brit. J. Phil. Sci.* 32, 349–358.

Bell, J.L., [1986], "From Absolute to Local Mathematics", *Synthese*, 69, 409–426.

Bell, J.L., [1988], *Toposes and local set theories*, Oxford University Press, Oxford.

Bourbaki, N., [1950], "The Architecture of Mathematics", *AMM* 67, 221–232.

Bourbaki, N., [1968], *Theory of Sets*, Hermann, Paris.

Carnap, R., [1956], "Empiricism, Semantics, and Ontology", in Benacerraf, P. and Putnam, H., (eds.), [1991], *Philosophy of Mathematics*, (2nd ed.), Cambridge University Press, Cambridge.

Corry, L., [1996], *Modern Algebra and the Rise of Mathematical Structures*, Springer Verlag, New York.

Dedekind, R., [1930-1932], *Gesammelte mathematische Werke*, 3 vols. Fricke, R., Noether, E., and Ore., O., (eds.), Braunschweig. (Chelsea reprint, [1969], New York).

Dummett, M., [1991], *Frege Philosophy of Mathematics*, Harvard University Press, Massachusetts.

Eilenberg, S., and Mac Lane, S., [1945], "The general theory of natural equivalence", *Transactions of the American Mathematical Society*, 55, 231–94.

Feferman, S., [1977], "Categorical Foundations and Foundations of Category Theory", in *Foundations of Mathematics and Computability Theory*, Butts, R., and Hintikka, J., (eds.), Reidel Publishing Company, Dordrecht-Holland.

Goldfarb, W., [2001], "Frege's Conception of Logic", in, J. Floyd, J., and S. Shieh, S., (eds.), *Futures Past: Reflections on the History and Nature of Analytic Philosophy*, Harvard University Press, Massachusetts.

Hasse. H., [1954], *Higher Algebra*, Fredrick Ungar, New York. (English trans. of the 3rd ed. Of Hasse [1926] by Benac, T.J.).

Hellman, G., [2001], "Three Varieties of Mathematical Structuralism", *Philosophia Mathematica*, (3), Volume 9, 184–211.

Hellman, G., [2002], Does Category Theory Provide a Framework for Mathematical Structuralism", *Philosophia Mathematica*, (3), Volume 9, 184–211.

Hilbert, D., [1900], "Über den Zahlenbegriff", *JDMV* 8, 180–184.

Hilbert, D., [1902], "Mathematical Problems", *BAMS* 8, 437–479 (English trans. Of Hilbert [1901], by Newson, M.W.).

Landry, E., [1999], "Category Theory: The Language of Mathematics", *Philosophy of Science* 66 (Proceedings), S14–S27.

Landry, E., [2001], "Logicism, Structuralism and Objectivity", *Topoi*: Special Issue – Mathematical Practice, Volume 20, 79–95.

Lawvere, F.W., [1966], "The Category of Categories as a Foundation of Mathematics", Proc. Conference Categorical Algebra (LaJolla 1965), Springer Verlag, New York, 1–20.

Mac Lane, S., [1968], "Foundations of Mathematics: Category Theory", in *Contemporary Philosophy: A Survey*, Klibansky, R., (ed.), Firenza, la Nuova Italia Editrice, 286–294.

Mac Lane, S., [1980], "The Genesis of Mathematical Structures", *Cahiers Topol. Geom. Diff*. 21, 353–365.

Mac Lane, S., [1992], "The Protean Character of Mathematics", in *The Space of Mathematics*, Echeverra, J., Ibarra, A., and Mormann, J., (eds.), de Gruyter, New York, 3–12.

Mayberry, J., [1994], "What is Required of a Foundation for Mathematics?", *Philosophia Mathematica 3*, Volume 2, Special Issue, "Categories in the Foundations of Mathematics and Language", Bell, J.L., (ed.), 16–35.

McLarty, C., [1990], "The Uses and Abuses of the History of Topos Theory", *Brit. J. Phil. Sci*., 41, 351–375.

McLarty, C., [1995], *Elementary Categories, Elementary Toposes*, Clarendon Press, Oxford.

Noether, E., [1921], "Idealtheorie in Ringbereichen", *MA* 83, 24–66.

Noether, E., [1926], "Abstrakter Aufbau der Idealtheorie in alebraischen Zahlun Funktionskörper", *MA* 96, 26.61.

Ore, O., [1935], "On the foundations of Abstract Algebra, I", *AM* 36, 406–437.

Resnik, M.D., [1996],"Structural Relativity", *Philosophia Mathematica*, (3), Volume 4, Special Issue, "Mathematical Structuralism", Shapiro, S., (ed.), 83–99.

Shapiro, S., [1996], "Space, Number and Structure: A Tale of Two Debates", *Philosophia Mathematica, 3*, Volume 4, Special Issue, "Mathematical Structuralism", Shapiro, S., (ed.), 148–173.

Shapiro, S., [1997], *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press, Oxford.

Waerden, B.L. van der, [1930], *Modern Algebra, 2 vols*, Springer, Berlin.