Reconstructing Hilbert to Construct Category Theoretic Structuralism

This paper considers the nature and role of axioms from the point of view of the current debates about the status of category theory and, in particular, its relation to the "algebraic"[1] approach to mathematical structuralism. I first consider the Frege-Hilbert debate with the aim of distinguishing between axioms as *assertions*, i.e., as statements that are used to express or assert truths about a unique subject matter, and an axiom system as a *schema* that is used to provide "a system of conditions for what might be called a relational structure" (Bernays [1967], p. 497) so that axioms, as *implicit definitions*, are about whatever satisfies the conditions set forth. I then use this inquiry to reevaluate arguments against using category theory to frame an algebraic structuralist philosophy of mathematics.

Hellman has argued that category theory cannot stand on its own as a "foundation" for a structuralist interpretation of mathematics because "the problem of the home address remains" (Hellman [2003], pgs. 8 & 15). That is, since the axioms for a category "merely tell us what it is to be a structure of a certain kind" and because "its axioms are not assertory" (Ibid. 7), we need a *background mathematical theory* whose axioms are assertory, i.e., a theory that assert truths about (possibly or actually) existing systems so structured.

With aims similar to mine but with a decidedly different conclusion, Shapiro [2005] has claimed that the Frege-Hilbert debate can be used to show that the current algebraic structuralist debates ought to be concerned with questions that consider the status of *meta-mathematical* axioms (as opposed to Hellman who considers the status of mathematical axioms). That is, Shapiro argues, even if we agree with the Hilbert-inspired algebraic structuralist that, at the mathematical level, "any given branch is 'about' any system that satisfies its axioms" (Shapiro [2005], p. 74), to give "criteria of acceptability" (of coherence, of consistency, of satisfiability) for such axioms or axiom systems themselves, we need a "foundation", as a *background meta-mathematical theory*, which is assertory, and so we cannot be algebraic structuralists all the way down.

According to Shapiro, our only other option, as proposed by Awodey [2004] is to "kick away the foundational ladder altogether, and take the meta-mathematical set-theory, structure theory, or whatever, itself to be an algebraic theory" (Shapiro [2005], p. 74). This option, however, is presented as a way not to be looked into because it supposedly has the unwanted consequence that

> mathematical logic is similarly liberated from theories… our theorist can hold… that satisfiability, consistency, or coherence implies existence, but she cannot maintain that any of these notions are mathematical matters (Ibid. 75).

The alleged result being that meta-mathematical analyses of these logical concepts are turned into non-mathematical, or, even worse, "philosophical", ones (see Shapiro [2005], pp. 74-75).

Against the claims of both Hellman and Shapiro, my aim is to show that category theory has as much to say about an algebraic consideration of meta-mathematical analyses of logical structure as it does about mathematical structure, *without* requiring either an assertory mathematical or meta-mathematical background theory, and too without turning meta-mathematical analyses of logical concepts into "philosophical" ones. Thus, we *can* use category theory to frame an interpretation of mathematics according to which we can be algebraic structuralists all the way down.

The Frege-Hilbert Debate

As is well known, Frege and Hilbert debated the nature of geometric axioms. Frege held that geometric axioms are *assertions*; that they are statements used to express or assert *truths*. On the other hand, Hilbert, like our modern-day algebraic structuralist, maintained that axioms are *implicit definitions*. Related to these differing views of axioms, Frege and Hilbert further disagreed on at least three points. First, *a theory*, for Hilbert, is not a set of truths about a "fixed" subject matter. As he explains,

> … every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and the basic elements can be thought of in any way one likes. If in speaking of points, I think of some systems of things, e.g., the system love, law, chimney-sweep… and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things…. [A]ny theory can always be applied to infinitely many systems of basic elements. (Hilbert [1899], pp. 40-41)

The second point of disagreement is that, for Hilbert, *a concept* is fixed axiomatically, or implicitly defined, only by its relation to other concepts. It is not constructively defined[2] (in the case of Frege logically defined) by its relation to independently existing (logical) objects; rather,

> … a concept can be fixed logically only by its relations to other concepts. These relations, formulated in certain statements I call axioms, thus arriving at the view that axioms …. are the definitions of concepts. (Correspondence [1900] 09/22)

The last point of disagreement is that Hilbert uses *consistency* to guarantee the "truth" of the axioms and hence to establish the *existence* of concepts, and not, as for Frege, the other way round;

> [a]s long as I have been thinking, writing and lecturing on these things, I have been saying the exact reverse [of Frege]; if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This [consistency] is for me the criterion of truth and existence. (Correspondence [1899] 12/29)

Underlying their disagreements and debates is the fact that Frege took mathematical axioms to be assertions about independently existing objects, that is, he held that mathematics has a fixed subject matter and this is what the axioms of its branches (arithmetic and geometry) are about. As Shapiro notes:

> Frege insisted that arithmetic and geometry each have a specific subject matter, space in the one case and the realm of natural numbers in the other. And the axioms express (presumably self-evident) truths about this subject matter. (Shapiro [2005], p. 67)

In contrast, Hilbert, as the precursor to the algebraic structuralist position, took the branches of mathematics (excepting, as we will see, finitary arithmetic[3]) to be about *any* system that satisfies its axioms[4]. As Bernays makes clear,

> [a] main feature of Hilbert's axiomatization of geometry is that the axiomatic method is presented and practiced in the spirit of the abstract conception [the algebraic structuralist conception] of mathematics that arose at the end of the nineteenth century and which has been adopted in modern mathematics. It consists in… understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation… for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure". (Bernays [1967], p. 497)

Hallett, making clearer the relation between Hilbert's use of the axiomatic method and our current rational reconstruction of Hilbert as an algebraic structuralist, says:

> In this case [in the case of the collection of Dedekind cuts exhibiting the properties that the axiom system for real number demands], axiomatization really uncovers certain structural relations that in general will be common to various structures… Indeed, the formulation of axioms then becomes one natural means of attempting to isolate structure. (Hallett [1994], p. 174)

We must now pause to consider the distinction between mathematics and meta-mathematics. For Frege, logic for arithmetic and our Kantian intuition of space for geometry is what "founds" our claims about the "truths" of their respective subject matter. Yet, even if we allow for an underlying role for set-theory in Frege's account of arithmetic, there is no obviously discernible distinction to be had between what we would now call mathematics and meta-mathematics.[5]

For Hilbert, by contrast, there is a clear-cut distinction. At the *mathematical level*, where we undertake a *conceptual analysis*, i.e., where we talk about the objects of various interpretations of the branches of mathematics as concepts in terms of anything that satisfies the axioms[6], no "founding", other than the organizational role afforded to the axiomatic method itself, is required. The implicit definitions of the concepts, and relative consistency or independence proofs act to guarantee the "truth" or "necessity" of the chosen axioms and, thereby establish the existence of such concepts. At the *meta-mathematical level*, however, where we talk about proofs themselves as objects, we must undertake a *contentual analysis*, i.e., we must rely upon our intuition to "found" claims about the "truth" of those finitary arithmetical axioms which provide those "irrefutable" logical principles that underpin the underlying meta-mathematical proof theory used for an absolute consistency proof of, say, arithmetic[7]. That is, as Shapiro notes

> [f]initary proof theory has its own unique subject matter, related to natural numbers and formal syntax, and it is ultimately *founded* on something in the neighbourhood of Kantian intuition. (Shapiro [2005], p. 70; italics added.)[8]

Shapiro's Criticism

As noted above in brief, Shapiro's recent criticism of the algebraic structuralist's use of category theory uses the Hilbert-Frege debate to point out that even if, at the mathematical level, category theory can be used to argue for an algebraic account of mathematics, where the category-theoretic axioms act as implicit definitions of the concepts of mathematics, at the meta-mathematical level, the category-theoretic axioms, themselves, need be assertions. According to Shapiro, the category-theoretic algebraic structuralist, if he is to avoid the "philosophical" pitfall similar to that faced by Hilbert, is forced to "found" his meta-mathematical category-theoretic axioms on some assertory theory, i.e., on some theory that asserts truths and so can "answer legitimate foundational questions" (Shapiro [2005], p. 71).

More pointedly, Shapiro's claim is that meta-mathematical analyses of those notions of "acceptability", like coherence, consistency, and satisfiability, themselves require "founding" by an assertory theory of sets or structures. This with the consequence that we must *either* accept that we cannot be algebraic structuralists all the way down and go "foundational" (by accepting a meta-mathematical assertory background theory) *or* we must reject foundations and, like Hilbert's appeal to intuition, go "philosophical". My aim, then, is to show, at least in the category-theoretic case, this dichotomy is false.

Shapiro first presents us with the three "foundational" options that can be used to "save" the mathematical structuralist: Hellman's modal set theory, Shapiro's model-theoretically motivated structure theory, or McLarty's category theory, as framed by either the ETCS or CCAF[9] axioms. Shapiro next claims that, to fulfill its "foundational" role of meta-mathematically analyzing those logical criteria of acceptability, and, thereby, to be able to use these criteria to guarantee the existence of structures or systems so structured, each must be taken as an *assertory* meta-mathematical background theory;

> [t]o be sure, if a category-based theory is to play this role, then its axioms must be assertory…each of them (the category based set theory, modal set theory, structure theory) is not just another theory, providing an implicit definition of some structures, or isomorphism types. The reason for this is that …. [each] has a foundational role to play concerning the coherence of definitions. And this last is an assertory matter. (Shapiro [2005], pgs. 73 & 74)

Shapiro then notes that, in contrast to the above "foundational" options, for category-theoretically minded structuralists, there is also the *purely algebraic*, "non-foundational", alternative. This option is claimed by Shapiro to be in line with Awodey [2004] where we

> …kick away the foundational ladder altogether, and take the meta-mathematical set-theory, structure theory, or whatever, itself to be an algebraic theory. On this view, set theory does not directly serve as a court of appeal for matters of coherence and thus existence…. The axioms of set theory are just implicit definitions that, if coherent, characterize a structure or a class of structures. The same goes for structure theory, modal set theory, and the various topos theories. (Shapiro [2005], p. 74)

The problem, as Shapiro sees it, is that

> [o]n this view, everything in mathematics is algebraic. So if there is to be an assertory canonical backdrop – a non-algebraic theory of coherence, consistency, mathematical existence, whatever – it will be regulated outside of mathematics, perhaps to philosophy… [otherwise] we will go back to the plan executed in Hilbert's *Grundlagen*, and settle for the analogue of relative consistency proofs. (Shapiro [2005], p. 74)

But oddly, while considering the possibility of our having to settle for relative consistency proofs, Shapiro continues on to give us the tools for the construction of his own demise. He first says of the "second theoretical option", which includes Awodey's, algebraic, non-foundational account:

> [n]otice that we have no formal assurance that our [background theory] is itself coherent… (Shapiro [2005], p. 74),

but then goes on to say of his own model/set-theoretically motivated structure-theoretic foundational account

> [o]n the first theoretical option, where the meta-theory is assertory, we likewise have no theoretical assurance that set theory is *true*. Again we have no safety net, and do not really need one. (Shapiro [2005], p. 74)

We are here left asking: Why is it that the advocate of the "foundational" option needs no assurance that this background meta-mathematical theory is *true*, yet the proponent of non-foundational option is required to show that his background theory is *coherent*?

Surely, given their difference of opinion as to the nature of axioms of their chosen theories (respectively, as either assertions or as implicit definitions), these "acceptability problems" are either equally pointed (the axioms-as-assertions foundationalist faces the "truth problem" to the same extent that the axioms-as-implicit-definitions non-foundationalist the "coherence problem") or they equally dissolve (the non-foundationalist can likewise claim that he does not have, or need, a "safety net").

In this light, before continuing on to make the case for category theory, I have three things to note. First, Shapiro's structure theory, even if cast in the frame of the axioms for ZF (see Shapiro [1997][10]), appears to be more "philosophical" than any category-theoretic option; structure-theory is clearly *not* a mathematical or meta-mathematical theory. Second, Shapiro himself has been far more concerned with the "coherence" of his structure theory axioms than with their truth; even going as far as to include a "coherence axiom" (again, see Shapiro [1997]). Finally, and perhaps most problematic, Shapiro has, in making the category-theoretic "foundational" versus "non-foundational" distinction, conflated two category-theoretic levels. That is, even if ETCS axioms are claimed to "found" branches of mathematics, it is only the CCAF axioms that are claimed to "found" category theory.

The Case for Category-Theory

To understand what is at issue here, I begin with an *abstract* definition of a category.

> ***Definition*****:** A cat-structured system C (a category) is an abstract system of two abstract kinds; objects X, Y,… and morphisms f, g,… such that
>
> ***Eilenberg - Mac Lane (EM) Axioms:***
>
> a) Each morphism f has an object X as a domain and an object Y as a codomain, indicated by writing f: $X \rightarrow Y$.
>
> b) If g is any morphism g: $Y \rightarrow Z$ with domain Y (the codomain of f) and codomain Z, there is a morphism $h = g \circ f$ called the *composition* of f and g.
>
> c) For each object X there is a morphism 1_x: $X \rightarrow X$ called the *identity* morphism of X.
>
> d) These objects and morphisms satisfy:
>
> i) *Associativity*: $f \circ (g \circ h) = (f \circ g) \circ h$
>
> ii) *Identity*: For all X the domain of 1_x = codomain of 1_x = X and for all f,
>
> $f \circ 1_x = f$, $1_y \circ f = f$

The claim of the category-theoretic algebraic structuralist is that the above cat-structured system acts as an *abstract* Hilbertian axiom system[11]; it provides an *abstract schema* for organizing the mathematical structure of *both* the concepts of the branches of mathematics *and* the concept of a category itself.

In the former case, for example, the following categories allow us to organize the mathematical structure of the concepts: set, group, topological space, etc.,

Set – where we take sets as objects, functions as morphisms,

Grp – where we take groups as objects, homomorphisms as morphisms,

Top – where we take topological spaces as objects, continuous functions as morphisms,

Diff – where we take differential manifolds as objects, smooth maps as morphisms,

Lat and **Bool** – where we take lattices and Boolean algebras as objects, respectively, and ($\top$, $\perp$, $\wedge$, $\vee$) homomorphisms, as morphisms

Heyt – where we take Heyting algebras as objects and ($\top$, $\perp$, $\wedge$, $\vee$, $\rightarrow$) homomorphisms as morphisms

Rings – where we take rings as objects and ring homomorphisms, i.e., (0, 1, +, ×) homomorphisms, as morphisms.[12]

The ETCS axioms, begin with the above abstract Eilenberg-Mac Lane (EM) axioms and applies them to sets as objects and functions as morphisms so that the axioms are satisfied; for example, every function *f* goes from a unique set *X* to a unique set *Y*, every set *X* has an identity function, etc. Thus the ETCS axioms, as the ZF axioms, can be used to analyze the *mathematical* or *logical* structure of concepts that are organized set-theoretically[13] (except, or course, the category **Set** of all sets**, Grp** of all groups, etc.[14]). More pointedly, the ETCS axioms, as Shapiro intends of set- or structure-axioms, can be used *meta-mathematically* to analyze those logical concepts (of consistency, satisfiability, independence) used as "criterion of acceptability" for axiom systems themselves.

To talk about the *mathematical* structure of categories themselves, including the *meta-mathematical* structure of the category **Set** of all sets as framed by the ETCS axioms[15], we can use the CCAF axioms; where now, in the abstract definition above, categories are objects and functors are morphisms. In so doing, we use **Cat**[16] as a Hilbertian axiom scheme for the concept of a category itself. Thus, we have a means of talking about *both* the *meta-mathematical* structure of the concepts of the branches of mathematics that are organized in category-theoretic terms, **Set** included, and for talking about the *mathematical* structure of the concept of a category itself.

We must now turn to ask: What "foundational" role are the category axioms, both those of ETCS and CCAF, intended to play and what is meant by 'foundation'? To answer these questions, I refer to the writings of McLarty [2005] who notes that there are two senses of the term 'foundation', arising from two uses of 'axioms'.[17] In the first, Aristotelian sense, an axiom (whether it is an assertion or an implicit definition) is that which itself must not admit of any proof. In the second sense, an axiom is that which must be independently plausible to a reasonably sophisticated mathematician[18]. Resulting from the first sense of 'axiom', a foundation must account for the *privileged status* of such "proof-less" axioms. By contrast, and in line with the second sense of 'axiom', a foundation, as considered by Mac Lane [1986] for example, can be seen as a proposal for the *structural organization* of mathematics via the axiomatic method[19]. Further witnessing the Hilbertian heritage of the resulting category-theoretic structuralist consideration of an axiom system *qua* relational structure, Mac Lane states:

> …a structure is essentially a list of operations and relations and their required properties, commonly given as axioms, and often so formulated as to be properties shared by a number of possibly quite different specific mathematical objects… a mathematical object 'has' a particular structure when specified aspects of the objects satisfy the (standard) list of axioms for the structure. This notion of 'structure' is clearly an outgrowth of the widespread use of the axiomatic method in mathematics [as exemplified by Hilbert's *Grundlagen*]. (Mac Lane [1996], pp. 174 & 176)

Thus, it is in this second sense, then, that McLarty suggests category theory as a foundation and adds that, even if it is not the last word, "[i]t is the latest and currently best word in the structuralist organization of mathematics" (McLarty [2005], p. 45). More to the point, however, it is only the CCAF axioms, and *not* the ETCS axioms, that are used to respond to Hellman [2003] by showing that category theory requires no "home address"; simply its axioms, as elements of an axiom system *qua* schema for the structural organization of the concept of a category, are not assertory[20]. So McLarty concedes to Shapiro the point that

> [the Awodey way] is a fine way to work for some purposes [for *abstractly* organizing concepts in category-theoretic terms] but Hellman is right that we also have foundational concerns [of organizing the concept of a category itself]. When we pursue these we cannot be satisfied with Awodey's equation, where he says 'the question of whether the conditions [for the acceptance of a given theory] are ever satisfied' is just the question of 'whether they are consistent'. (McLarty [2005], p. 53)

Yet, McLarty further clarifies, and in so doing responds to Hellman's concern:

> [t]he key point to grasp here is precisely that categorical foundations for category theory are not set-theoretical foundations for category theory. When we axiomatize a meta-category of categories, by the axioms of CCAF, the categories are not 'anything satisfying the algebraic axioms of category theory'- i.e., the Eilenberg-Mac Lane axioms. *They are anything whose existence follows from the CCAF axioms*. They are precisely not sets satisfying the Eilenberg axioms. They are categories as described by Lawvere's CCAF axioms. (McLarty [2005], p. 52; italics added.)

The CCAF axioms, then, *are* intended to be foundational in both a *mathematical* and *meta-mathematical*, yet *non-assertory*, sense, i.e., in the sense that they organize what we say about the concept category itself, and in the sense that are about *any* object that *is* a category, including the category **Set** as organized the ETCS axioms. Yet too they are foundational not in the Aristotelian sense that they are accepted because they account for the claim that the axioms are *privileged* but in the sense that they are accepted because they are *organizational* of both the mathematical structure of categories *and* the meta-mathematical structure of anything that is a category. Thus, we can clearly use the CCAF axioms to respond to Hellman's criticisms. The question that remains, however, is whether we can give a category-theoretic account of meta-mathematical analyses of those *logical* criteria of acceptability that does not rely, as McLarty suggests, on what is "plausible to a reasonably sophisticated mathematician" and too that does not require, as Shapiro claims, either a "foundation", as an assertory meta-mathematical background theory, or our turning to "philosophy".

To make the above situation more perspicuous, and, in so doing, set the stage for my response to Shapiro's criticisms, let us return to perhaps glean more from the algebraic reconstruction of Hilbert. Hilbert too can be seen as having made the distinction between using 'axiom', and so 'foundational', in the organizational versus the Aristotelian sense. Reconstructing Hilbert along these lines, we see also two distinct components of Hilbert's "foundational" project; the *mathematical* project of founding, in the organizational sense, mathematics (indeed, and all scientific thought) on the axiomatic method[21] and the *meta-mathematical* project of founding, in the Aristotelian sense, the axioms of arithmetic and proof theory by finitary, intuitive, means. That is, this founding was needed so that the axioms of finitary proof theory could be taken as privileged (as "irrefutable" as so not requiring proof) with the result that proofs themselves can be taken as "objects" of logical analysis to be then used to prove, by finitary means, the consistency of infinitary arithmetic. Merging these components with Shapiro's criticisms we come to three aspects of Hilbert's "foundational" programme, i.e., the *conceptual*, *logical* and *meta-mathematical*, which can now be put to use to reconstruct some of the details of Hilbert's algebraic structuralism:

a) When *conceptually analyzing* the *mathematical* structure of a given branch of mathematics, we have axioms as implicitly defining concepts; here our task is to present an axiom system *qua* conceptual schema for the facts of any given interpretation (which provides a domain of objects for these concepts) in such a way as to organize what can be mathematically asserted about such objects as concepts.[22]

b) When *logically analyzing* axioms or axiom systems themselves, we have logical criteria, e.g., *completeness, independence* and *consistency*; here our task is to give an account of those axioms that are necessary[23] and prove the consistency of these axioms relative to, for example, the theory of arithmetic, and thereby establish the *existence* of such concepts.

Having undertaken both a) and b) for the branches of mathematics, we thereby establish, via the axiomatic method, a *conceptual foundation* for mathematics, where 'foundation' is taken in the *organizational* sense of the term.

c) When *meta-mathematically analyzing* the *logical structure* of proofs, we take proofs themselves as objects[24] (as finitely intuited signs[25]); here, for example, our task is to establish a *contentual foundation* for finitary mathematics from which we can then securely extend, by a proof of consistency, to infinitary mathematics, where now 'foundation' is taken in the Aristotelian sense of the term.

So, at the *mathematical level*, Hilbert was happy to let the axioms speak for themselves, modulo certain logical "criteria of acceptability" (completeness, independence, and consistency). At the *meta-mathematical* level, however, he required a *philosophical story* that used intuition to further "found" the "truth" of those "logical principles" underlying the proof-theoretic axioms themselves. This with the aim of *both* showing that the axiomatic method applies to logic itself[26] and of providing a "natural"[27] account of infinitary arithmetic, analysis, set-theory, etc.[28]

Thus, accepting (as Shapiro does) the lesson of Gödel, that proofs of absolute consistency by finitary means, is a way not to be looked into, there are three ways the category-theoretic mathematical structuralist can use this rational reconstruction of Hilbert to answer Shapiro's meta-mathematical challenge, and neither requires our taking category-theoretic axioms as assertions or our turning to "philosophy"[29]. These are:

a) When *conceptually analyzing* the *abstract* structure of any given branch of mathematics, we have the EM axioms as implicitly defining the abstract concept of a category; here our task is present an axiom system *qua* an *abstract conceptual schema* for the facts of any given interpretation (which provides a domain of objects, i.e., 'objects' and 'morphisms', for these concepts) in such a way as to organize what can be mathematically asserted about such objects as abstract cat-structured concepts.

   i) When conceptually analyzing the branches of mathematics that are themselves organized set-theoretically, the category theorist can take the ETCS axioms as conceptual scheme for organizing, in category-theoretic terms, what we say about the mathematical or logical structure of these *set-structured concepts*.

   ii) When conceptually analyzing the concept of a category itself, the category theorist can take the CCAF axioms as a *meta-mathematical* conceptual scheme for organizing, in category-theoretic terms, what we say about the mathematical or logical structure of *categories themselves* as 'objects'.

b) When *logically analyzing* axioms or axiom systems themselves, either at the abstract (EM), mathematical (ETCS) or meta-mathematical (CCAF) level, the category theorist can make use of the resources of the many categorical logics to organize what we say about those logical concepts, like completeness, independence, consistency, coherence, satisfiability, etc, that are used as "acceptability criteria" for axioms or axioms systems themselves.

It is with respect to Hilbert's third way that Shapiro believes that he has one more sword to swing at the category-theoretic algebraic structuralist. As noted, Hilbert's meta-mathematical proof-theoretic language takes proofs as objects; it is concerned not with the logical relations that bear between concepts like sets, groups, etc., but rather with the logical relations that bear between proofs themselves. Thus, according to Shapiro, even if we do not kick away the foundational ladder and so maintain category theory, as defined by the CCAF axioms, as a meta-mathematical foundation in the organizational sense of the term, we are we still left to face the problematic consequence that

> mathematical logic is similarly liberated from theories… As a structuralist, our theorist can hold –in assertory *philosophy*– that satisfiability, consistency, or coherence implies existence, but she cannot maintain that any of these notions are mathematical matters. There are simply no distinctly mathematical objects, and so theories, deductions, and interpretations are not mathematical. But perhaps we should not quibble over labels. (Shapiro [2005], p. 75; italics added.)

In response, I note that there is nothing to quibble about. Regardless of labels, on the category-theoretic algebraic structuralist view, a meta-mathematical analysis of logical structure does not require a non-mathematical, philosophical, analysis. Indeed, as Marquis claims, "this is the very first moral: the distinction between mathematics and meta-mathematics more or less evaporates in a category-theoretical framework" (Marquis, personal correspondence)[30]. More pointedly, when considering a meta-mathematical *semantic* analysis of the various *model-theoretic* concepts of satisfiability, interpretation, truth, relative consistency, etc., as Marquis explains,

> it is easy to define these notions in the appropriate categories and these are nothing more than a generalization of Tarski's notions. With a bonus: it is easy and natural to do this for multi-sorted languages…Standard references: Makkai & Reyes [1977], Johnstone [2002]. (Marquis, personal correspondence)

So, the 'sorts' need not be taken as sets, though of course, they could be; in which case, as noted, we can use the ETCS axioms to provide a meta-mathematical analysis of the various *model-theoretic* concepts of satisfiability, interpretation, truth, relative consistency in so far as these concepts are themselves organized set-theoretically. And too, in line with Hilbert's meta-mathematical proof-theoretic analysis, category theory allows us to describe categories in terms of deductive systems and so we can employ categorical methods for proof-theoretical purposes. For example, one can analyze proof-theoretic structure itself by using **Ded**, the category of deductive systems, which takes 'objects' as *formulas*, 'morphisms' as *proofs* or *deductions*, and operations on morphisms as *rules of inference* (See Lambek & Scott [1986]).[31] Finally, in line with Hilbert's preference for finitistic reasoning, we can use topos theory, as a meta-mathematical theory, to logically analyze various aspects of constructive mathematics, including constructive set theory, the concepts of recursiveness, independence, and models of higher-order type theories generally.[32]

Clearly then Shapiro's is mistaken in his claim that meta-mathematical analyses of those logical concepts used as "criteria of acceptability", like coherence, consistency, satisfiability, deductive system, themselves require "founding" by an assertory theory of sets or structures. And so the category-theoretic structuralist is in a position to reject the consequence that we must *either* accept that we cannot be algebraic structuralists all the way down and go "foundational" (by accepting a meta-mathematical assertory background theory) *or* we must reject foundations and, like Hilbert's appeal to intuition, go "philosophical".[33] I have shown we need not do either. *Contra* both Hellman and Shapiro, we do not have to give up the Hilbertian, and now algebraic structuralist, notion of 'existence in virtue of acceptability'[34] in favor of adding either a "foundation" as an assertory mathematical or meta-mathematical background theory. Simply, the *mathematical structure* of the concepts of the branches of mathematics, either abstractly or set-theoretically organized, and the concept of a category itself can be organized by the various category-theoretic axiom systems, i.e., by the EM, ETCS and CCAF axioms, respectively. And the *logical structure* of the cat-structured axiom systems themselves can be *meta-mathematically* organized, at the semantic or syntactic levels[35], by means of the various categorical logics.

Category theory, then, has as much to say about an algebraic consideration of a meta-mathematical analysis of logical structure as it does about the conceptual analysis of mathematical structure, *without* requiring either an assertory meta-mathematical or mathematical background theory, and too *without* turning logical issues into "philosophical" ones. Thus, we *can* use category theory to frame an interpretation of mathematical structuralism according to which we *can* be algebraic structuralists all the way down.

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[1] For a precise account of what I intend by the "algebraic" approach, see Landry and Marquis [2005].

[2] Throughout his writings, Hilbert was expressedly against both Frege and Dedekind's "construction" of numbers. For example, he characterized the method of defining concepts via construction as the "genetic method" and held this in sharp contrast to his preferred "axiomatic method" (See Hilbert 1900a). As well see Hallett [1994], p. 174, who claims "[t]he central difference [between Frege and Hilbert] is that the construction will no longer be a *definition* by construction, but rather only "interpretation" by construction…". Finally, see Hallett [1994] and Ewald [1999] for a more detailed explanation and discussion of this difference.

[3] As we will see, things are not so straightforward as to what "branches" Hilbert would have included in his axiomatic treatment of mathematics. As Hallett notes "[j]ust prior to this [1896] he seems to have held a version of the 'Dirichlet thesis' that all of higher analysis will in some sense 'reduce' to the theory of natural numbers, a thesis which is stated without challenge in the *Vorwort* to Dedekind's 1888 monograph… In the 1920s, [however] Hilbert stated decisively his rejection of the Dirichlet thesis…." (Hallett [2007], p. 34). Indeed, as early as 1918 (Ewald [1999], p. 1109), Hilbert explicitly includes arithmetic as an axiomatic theory just as any other mathematical or, indeed, scientific, theory, e.g., just as geometry, mechanics, radiation theory, or thermodynamics.

[4] See Hallett [1994] for an excellent overview of Hilbert's "reference free" [variation in reference across interpretations] account of mathematics and for an explanation of how this is related to Hilbert and Frege's "differing attitudes to logic and to the laws of thought" (p. 163).

[5] One position is that because Frege's logic is universal there is no room outside of it for metatheory (see, for example, Goldfarb [1979]). The other position (see Antonelli and May [2002]) is that, in light of Frege's [1906] account of geometry where he showed how to construct independence proofs for the axioms of geometry, it is possible to give a rational reconstruction of at least some meta-logical notions.

[6] For Hilbert the interpretation, and so the objects, may be taken from *any*, scientific, domain of knowledge, including, for example, both mathematics and physics. As Hilbert explains, "[a]ccording to this point of view, the method of axiomatic construction of a theory presents itself as the procedure of the mapping of a domain of knowledge onto the framework of concepts, which is carried out in such a way that to the objects

of the domain of knowledge there now corresponds the concepts, and to the statements about the objects there corresponds the logical relations between concepts" (Hilbert 1921/1922 in Hallett [2007], p. 9).

[7] So the "truths" of finitary arithmetic, for Hilbert, are not true in the Fregean sense of being about a unique subject matter but rather are true in the Kantian *a priori* sense (see Hilbert 1931a, 1930b in Ewald [1999]) that the intuitive operations and principles that give rise to the axioms for finitary arithmetic/proof theory are irrefutable because they are founded on Kantian preconditions for (pure) reasoning itself. That is, the intuitive operations and principles that underlie our symbolic reasoning about natural numbers as signs, i.e., as sequences of strokes, give rise to the logical structure of the axioms of finitary arithmetic and these to those logical principles that underlie our symbolic reasoning about formulas and formal proofs as signs. This precondition, as Zach explains, is: '[i]n order to carry out the task of providing a secure foundation for infinitary mathematics, access to finitary object [as signs] must be immediate and certain". (Zach [2006], p. 423) But it was the resulting proof-theoretic formalism and not the contentual reasoning, nor the "philosophy" underlying it, which did the meta-mathematical work. So, for example, as Zach notes "Hilbert and Bernays developed the ε-calculus as their definitive formalism for axioms systems for arithmetic and analysis, and the so-called ε-substitution method as the preferred approach to giving consistency proofs" (Ibid., 417).

[8] As I will show, what is at stake here is just what is meant by the term 'founded'. It suffices to point out here, however, that, as explained in the endnote above, it is not intuition that founds the subject matter of proof theory; it is the "irrefutability" of finitistic reasoning about signs that does the founding as a precondition for reasoning. That is, 'intuition' is used here not in the Kantian-Fregean sense, as in intuition of space or time, as a *precondition for the construction of concepts*; rather, it is used in the sense as a Kantian *precondition for pure thought itself*. As Hilbert explains, "The *a priori* is here nothing more and nothing less than a fundamental mode of thought, which I also call the finite mode of thought… (Hilbert 1931a, in Ewald [1999], p. 1150)

[9] These are, respectively, the axioms of the Elementary Theory of the Category of Sets and the Category of Categories as a Foundation. For a more precise, formal, account of the ETCS axioms see, for example, Lawvere [1964] and Mac Lane [1986]. For a more precise, formal, account of the CCAF axioms, see, for example, Lawvere [1966] and McLarty [1991].

[10] In this regard Shapiro himself claims: "[m]y own structure theory (Shapiro [1997], Chapter 3) was meant to play the same assertory, foundational, role as set theory, and, indeed, structure theory is a notation variant of set theory". (Shapiro [2005], p. 73)

[11] As Mac Lane explains: [i]n this description of a category, one can regard "object", "morphism", "domain", "codomain", and "composites" as *undefined terms* or *predicates*. (Mac Lane [1968], p. 287; italics added).

[12] These examples are taken from Marquis [2007] which provides a more detailed list of categories for various mathematical concepts

[13] Basically, the ETCS axioms plus an adjoined axiom scheme of replacement yields a set theory equivalent to ZF. See McLarty [2007] for more details.

[14] There are ways, of course, as there are for the set-theorist, of bypassing these problems; for example, by appealing to Grothendieck universes. As McLarty notes: "[o]n ZF foundations a Grothendieck universe is a set satisfying all the ZF axioms. In ETCS foundations it is a set of sets which, together with all the functions between them, satisfy the ETCS axioms. Either way a Grothendieck universe proves the consistency of its set theory, so that neither ZF nor ETCS proves there are universes". (McLarty [2007], p. 11)

[15] This because the CCAF axioms prove a theorem scheme of unbounded set for **Set**; again see McLarty [2007] for details.

[16] See McLarty [2007], especially pgs. 13-18 for details of the CCAF axioms.

[17] See McLarty [2005], p. 44, footnote #3.

[18] My aim (see page 15) will be to provide a middle-ground between the Aristotelian "proof-less" account of an axiom and McLarty's "plausibility" account, which, in appealing to what is "independently plausible to a reasonably sophisticated mathematician", seems to allow for a socially constructed component that is both not wanted and not warranted. The idea here will be to show that an axiom or axiom system is taken as "plausible" *in service of* its foundational role, that is, because it structurally organizes concepts, there are, of course, *logical* criteria of acceptability which, thought these criteria are themselves variable, e.g., may be chosen from a semantic or syntactic perspective and for various tasks, like organizing constructive mathematics, they are not either social or, to borrow Shapiro's term, "philosophical" criteria. As Marquis

notes, in the category-theoretic setting, an axiom or axiom system is "plausible" to the extent that "it is what is basically required for a conceptual framework to work in the way in should… For instance, a homology theory, as axiomatized by Eilenberg and Steenrod, is basically any functor between two categories satisfying their axioms. It is not that the axioms are true, not that they are coherent (we already know they are in this case), but they specify *norms* for such a theory to qualify as a homology theory and these norms are basic properties found in all homology theories." (Marquis, personal correspondence)

[19] Note that Mac Lane himself was no mathematical structuralist. See Landry and Marquis [2005] for a more detailed account of Mac Lane's position.

[20] As McLarty points out elsewhere, the question of the existence of categories is not a question of whether its axioms are assertory: "Indeed category theory *per se* has no such [assertory] axioms, but that is no lack, since category theory *per se* is a general theory applicable to many structures. Each specific categorical foundation offers various quite strong existence axioms" (McLarty [2004], p.43).

[21] In addition to endnote 6, and witnessing this "organizational" aspect, see, for example, Hilbert's claim that "[e]very science takes its starting point from a sufficiently coherent body of facts as given. It takes form, however, only by *organizing* this body of facts. This organization takes place though the *axiomatic method*, i.e., one constructs a *logical structure of concepts* so that the relationship between the concepts corresponds to the relationship between the facts to be organized. There is an arbitrariness in the construction of such a structure of concepts; we, however, *demand* of it: 1) completeness, 2) independence, 3) consistency." (Hilbert 1902 in Hallett and Majer [2004])

[22] Where the assertion is no longer dependent upon the *intuitive construction* of concepts, i.e., constructions made on the basis of either our Kantian intuition of space or time, as it was thought to for Frege's conception of geometry or Peano's conception of arithmetic, or on the *logical construction* of concepts, as was both Dedekind and Frege's construction of the concept of number and for Russell and Whitehead's construction of the concept of set.

[23] See Hallett [2007] for a detailed discussion of the search for the "necessary" axioms of geometry as an example of Hilbert's attempt to reach an epistemological "purity of method", a method equally free from both intuitive assumptions (giving rise to the mistaken belief in the "truth" of the parallel axiom) and analytic assumptions (giving rise to the mistaken belief in the "truth" of the continuity axiom), so that an

axiom systems, as a framework for concepts, when reduced to its necessary axioms could then be used to conceptually organize mathematics, and, indeed, all of scientific thought. So by 'necessary' is meant needed to frame about all possible interpretations. As Hilbert explains: "Nevertheless [in spite of it being free of a particular interpretation] this framework of concepts has a meaning for knowledge of the actual world, because it represents a 'possible form in which things are actually connected'. It is the task of mathematics to develop such conceptual frameworks in a logical way, be it that one is led to them by experience or by systematic speculation". (Hilbert 1921/1922 in Hallett and Majer [2004])

[24] As Hilbert says: "[t]o conquer this [meta-mathematical] field we must, I am persuaded, make the concept of specifically mathematical proof itself into an object of investigation." (Hilbert 1922a in Ewald [1999], p. 1115)

[25] As Ewald explains "[m]athematical proofs were to be translated into a special formal language; this language was then itself to be the object of a *mathematical investigation*, which would culminate in a proof that a formal contradiction could never be derived within the system. (Ewald [1999], p. 1091; italics added.)

[26] See Hilbert 1930b in Ewald [1999], especially, p. 1159)

[27] That is, an account that would "do full justice to the constructive tendencies, to the extent that they are natural" (Hilbert 1922a in Ewald [1999], p. 1119) and in so doing avoid the "unnatural" and problematic accounts of Kronecker and his followers, Weyl and Brouwer.

[28] Note then that his aim was not to secure any one theory, arithmetic for example, as a foundation, but rather his goal with his "new [proof-theoretic] grounding of mathematics" was to "rid the world of the question of the foundations of mathematics once and for all by making every mathematical statement into a formula that can be concretely exhibited and rigorously derived, and thereby bring mathematical concept-formations and inferences into such a form that they are irrefutable and yet furnish a model of the entire science. (Hilbert 1931a, in Ewald [1999], p. 1152) That is, by showing that mathematical thought "takes place parallel to speaking and writing; by the formation and placing together of sentences. And for justification I need neither God, like Kronecker, nor the assumption of a special capacity of our understanding directed towards the principle of complete induction, like Poincaré, nor some ur-intuition like Brouwer, nor, like Whitehead and Russell, the axioms of infinity and reducibility, which are real,

contentual presuppositions, not compensated by proofs of consistency, and of which the latter is not even plausible….” (Hilbert 1930b, in Ewald [1999], p. 1157) Hilbert claims to have “fully attained what I desired and promised: The world has thereby been rid, once and for all, of the question of the foundations of mathematics as such” (Ibid.)

[29] In brief, my argument is as follows: Shapiro, to avoid an infinite regress of using stronger (higher-order cardinal) set theories to prove the consistency of the lower (set) theory, must, at some point take, one of these theories to be true (hence take its axioms/theorems as assertions). We are both committed to some type of statement like “If theory X is consistent (or acceptable), then ...”. Where we differ is that I deny the claim that statements of consistency, etc., are assertory in the sense that at some point the “If …, then…” dissolves because some true theory stops the regress. Put otherwise, all we have is relative consistency; the statement of which is assertory in the stronger theory, i.e., in the theory which we take as “acceptable” for the purpose of proving relative consistency, but which is not assertory in the sense that we take the stronger theory as true.

[30] As Marquis [2007] explains in detail, the reason that there is no distinction between mathematics and meta-mathematics is that the resources of the various categorical logics can be used to analyze logical concepts as considered from within the those systems that are organized by either the EM, the ETCS or the CCAF axioms.

[31] As Marquis [2007] notes “It is therefore legitimate to think of a category as an algebraic encoding of a deductive system. This phenomenon is already well-known to logicians, but probably not to its fullest extent. An example of such an algebraic encoding is the Lindenbaum-Tarski algebra, a Boolean algebra corresponding to classical propositional logic. Since a Boolean algebra is a poset, it is also a category… Thus far we have merely a change of vocabulary. Things become more interesting when first-order and higher-order logics are considered.”

[32] As Marquis explains” “… Hilbert’s [finitary] program is getting new fuel from categorical logic! There is some fascinating work done by mathematicians on constructive proofs of classical results using at their core geometric logic and basic theorems of preservation in the topos-theoretical setting. This is a beautiful example of what can be done in this framework”. (Marquis, personal correspondence). See Marquis [2007]

for a detailed list of references and for a brief sketch of the history and current uses of topos theory. For more on the history of topos theory, see McLarty [1992].

[33] Note, however, that there are several rational reconstructions of the finitistic aspect of Hilbert's programme that are mathematical, so that, against Shapiro, even for Hilbert, "going philosophical" need not be the only alternative to "going assertory". Here I have in mind Tait's [1981] claim that finitistic reasoning is just primitive recursive reasoning, so that appeals to Kantian intuition can be dispensed. See Zach, however, for criticisms of this view and for the presentation of various other alternatives, both mathematical, e.g., Kreisel's [1960], and philosophical, e.g., Parson's [1998].

[34] See, for example, Hallett's [1990] reconstruction of Hilbert's consistency criterion in terms of what he calls the acceptability thesis: "If a body of sentences S is acceptable according to the canons of acceptability laid down, then there must be objects to which singular terms of the S-sentences purport to refer."

[35] See Marquis [2007] for a more detailed description of categorical logic and for an extensive list of references; see especially the section entitled "Research papers on various aspects of categorical logic".

Abstract

This paper considers the nature and role of axioms from the point of view of the current debates about the status of category theory and, in particular, its relation to the algebraic approach to mathematical structuralism. My aim is to show that category theory has as much to say about an algebraic structuralist consideration of meta-mathematical analyses of logical structure as it does about the conceptual analyses of mathematical structure, *without* requiring either an assertory mathematical or meta-mathematical background theory, and too *without* turning logical issues into philosophical ones. Thus, we *can* be algebraic structuralists all the way down.