



# Reopening the Hole Argument

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## ABSTRACT

This expository paper relates the Hole Argument in general relativity (GR) to the well-known theorem of Choquet-Bruhat and Geroch (1969) on the existence and uniqueness of globally hyperbolic solutions to the Einstein field equations. Like the Earman–Norton (1987) version of the Hole Argument (which is originally due to Einstein), this theorem exposes the tension between determinism and some version of spacetime substantivalism. But it seems less vulnerable to the campaign by Weatherall (2018) and followers to close the Hole Argument on the basis of “mathematical practice,” since the theorem only talks about isometries and hence does not make the pointwise identifications via diffeomorphisms that Weatherall objects to. Among other implications of the theorem for the philosophy of GR, we reconsider Butterfield’s (1987) influential definition of determinism. This should be amended if its goal is to express the idea that GR is deterministic in the absence of Cauchy horizons, although its original form does capture the way GR is indeterministic in their presence! Furthermore, in GR isometries come out as gauge symmetries, as do Poincaré transformations in special relativity.

Finally, I discuss some implications of the theorem for the philosophy of science: Accepting the determinism horn still requires a choice between Frege-style abstractionism and Hilbert-style structuralism; and, within the latter, between structural realism and empiricist structuralism (which I favor).



RESEARCH



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Initially, the Hole Argument (*Lochbetrachtung*) was an episode in Einstein’s struggle, between 1913–1915, to find the gravitational field equations of general relativity. At a time when he was already unable to find generally covariant equations for the gravitational field (i.e., the metric) that had the correct Newtonian limit and satisfied energy-momentum conservation, the Hole Argument confirmed him in at least temporarily giving up the idea of general covariance (which he later recovered without ever mentioning the Hole Argument again, at least in print).<sup>1</sup> Einstein’s invention of the argument formed part of his analysis of the interplay between general relativity (of motion), general covariance (of physical equations under coordinate transformations), and determinism (here: of the field equations of general relativity).

Thus, Einstein felt he had to choose between determinism and general covariance; the recent emphasis on the tension between determinism (siding with relationalism) and substantivalism is due to Earman and Norton (1987). But, since for Einstein the opposition between substantivalism and relationalism was closely related to the problem of absolute versus relative motion, and hence to his putative “principle of general relativity” (Earman 1989), he would certainly have been interested in it.

In modernized form (using a global perspective and replacing Einstein’s coordinate transformations by diffeomorphisms), his reasoning was essentially as follows:<sup>2</sup>

- Let  $(M, g)$  be a spacetime.<sup>3</sup> The transformation behavior of the Einstein tensor  $\text{Ein}(g)$  under diffeomorphisms  $\psi$  of the underlying manifold  $M$  is

$$\psi^*(\text{Ein}(g)) = \text{Ein}(\psi^*g). \quad (1)$$

Similarly, for any healthy energy-momentum tensor  $T(g, F)$  constructed from the metric  $g$  and the matter fields  $F$  that matter we should have

$$\psi^*(T(g, F)) = T(\psi^*g, \psi^*F). \quad (2)$$

Consequently, if  $g$  satisfies the Einstein equations  $\text{Ein}(g) = 8\pi T(g, F)$ , then  $\psi^*g$  satisfies these equations for the transformed matter fields  $\psi^*F$ .

- Now consider an open connected vacuum region  $H$  in spacetime, possibly surrounded by matter (i.e.,  $F = 0$  in  $H$ );  $H$  is referred to as a “hole,” hence the name

1 See Janssen and Renn (2022) for the final reconstruction of Einstein’s struggle, with §4.1 devoted to the Hole Argument. The earliest known reference to the Hole Argument is in a memo by Einstein’s friend and colleague Besso dated August 1913, provided this dating is correct (Janssen 2007). Einstein subsequently presented his argument four times in print; I just cite Einstein (1914) as the paper containing his final version. Implicitly, his later point-coincidence argument (Einstein 1916) was his own reply to his Hole Argument (Norton 1993; Giovanelli 2021). See Stachel (2014), Norton (2019), Pooley (2022), and Gomes and Butterfield (2023a), and references therein for reviews of the Hole Argument in both a historical and a modern context.

2 We write the Einstein tensor as  $\text{Ein}(g)$ , where its dependence on the metric  $g$  is explicitly denoted; in coordinates we have  $\text{Ein}(g)_{\mu\nu} = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ .

3 A spacetime is a smooth four-dimensional connected Lorentzian manifold with time orientation (this nomenclature of course hides philosophical issues to be discussed later in this paper). More generally, my notations and conventions follow Landsman (2021) and are standard, e.g., spacetime indices are Greek whereas spatial ones are Latin, the metric has signature  $-+++$ , etc.

of the argument.<sup>4</sup> Furthermore, find a diffeomorphism  $\psi$  that is nontrivial inside  $H$  and equals the identity outside  $H$ , so that in particular,

$$T(\psi^*g, \psi^*F) = T(\psi^*g, F) = T(g, F), \quad (3)$$

both outside  $H$  (where  $\psi$  is the identity) and inside  $H$  (where  $T(g, F) = 0$ ).

- It follows from the previous points that if  $g$  satisfies the Einstein equations for some energy-momentum tensor  $T$ , then so does  $\psi^*g$ . Hence the spacetimes  $(M, g)$  and  $(M, \psi^*g)$  satisfy the Einstein equations for the same matter distribution and are identical outside  $H$ . But they differ inside the hole.

Einstein saw this as proof that the matter distribution fails to determine the metric uniquely, and regarded this as such a severe challenge to determinism that, supported by the other problems he had at the time, he retracted general covariance.

From a modern point of view, the energy-momentum tensor is a red herring in the argument,<sup>5</sup> which may just as well be carried out *in vacuo*, as will be done from now on; this also strengthens my subsequent reformulation of the argument, since the theorem on which this is based is less well developed in the presence of matter.

Earman and Norton (1987), then, revived the Hole Argument, as follows:<sup>6</sup>

1. Although, in the case considered by Einstein,  $(M, \psi^*g)$  and  $(M, g)$  are mathematically speaking different spacetimes (unless  $\psi^*g = g$ , in which case the Hole Argument is void), physicists—usually tacitly—circumvent this alleged lack of determinism of GR by simply “identifying” the two, i.e., by claiming that  $(M, \psi^*g)$  and  $(M, g)$  represent “the same physical situation.”
2. In this practice they are encouraged by the observation that  $(M, \psi^*g)$  and  $(M, g)$  are *isometric*; trivially, the pertinent isometry is  $\psi$ , and so the conclusion would be that isometric spacetimes represent the same physical situation.

4 Einstein’s arrangement looks unnatural compared to Hilbert’s (1917) reformulation as an initial-value problem in the PDE sense (see Proposition 1 below), but Einstein was inspired by Mach’s principle, where “fixed stars at infinity” determine the local inertia of matter; see Maudlin (1990), Hofer (1994), and Stachel (2014). An argument that actually favors Einstein’s curious setting for the Hole Argument is this: The smaller the hole, i.e., the larger the complement of the hole, the greater the challenge to determinism, for if even things almost everywhere except in a tiny hole fail to determine things inside that hole, then we should really worry (Butterfield 1989). This pull admittedly gets lost in the initial-value formulation of the argument below. See Muller (1995) for the explicit construction of a hole diffeomorphism (the only one I am aware of).

5 Continuing footnote 4: Janssen (2007, footnote 98) notes that Einstein formulated his requirement that the matter distribution fully determines the metric only in 1917; in 1913 Einstein still thought of Mach’s principle in the light of the relativity of inertia. Furthermore, Einstein (1914) explicitly introduced the final version of the hole argument in terms of a conflict between general covariance and the “law of causality” (“*Kausalgesetz*”), which was contemporary parlance for determinism. In sum, it seems safe to say, with Janssen (2007), that the ‘worries about determinism and causality that are behind Einstein’s hole argument have strong Machian overtones.’ See Norton (1993) for Einstein’s general struggle with general covariance, and its aftermath.

6 This (as well as Weatherall’s critique) relies on a precise understanding of the notion of an *isometry* between spacetimes  $(M', g')$  and  $(M, g)$ : This is a diffeomorphism  $\psi : M' \rightarrow M$  for which  $g' = \psi^*g$ , or, equivalently,  $g = \psi_*g'$ , where  $\psi_* = (\psi^{-1})^*$ . In particular, following, e.g., Hawking and Ellis (1973), we always take an isometry to be a diffeomorphism.

3. *However*—and this is their key point—this spells doom for spacetime substantialists (like Newton), who (allegedly) should be worried that if, in order to save determinism,  $x \in M$ , carrying the metric  $\psi^*g(x)$ , must be identified with  $\psi(x) \in M$ , carrying the same metric, then points have lost their “this-ness”: they cannot be identified *as such*, but only as carriers of metric information.
4. Thus, one seems forced to choose between determinism and substantialism.

For the purpose of this paper, it is sufficient (and considerably easier) to replace the concept of substantialism with what Gomes and Butterfield (2023a) call *Distinct*:<sup>7</sup>

*Though isometric,  $(M, \psi^*g)$  and  $(M, g)$  represent different physical possibilities.*

The tension exposed by the *modern* Hole Argument, then, is the one between *Determinism* (cf. §3) and *Distinct* (as opposed to general covariance, which is assumed).<sup>8</sup>

But this discussion would be pointless if the Hole Argument is a non-starter, as claimed by Weatherall (2018) and his followers (Bradley and Weatherall 2022; Fletcher 2020; Halvorson and Manchak 2022). Let me recall the main point:

This discussion may be summed up as follows: There is a sense in which  $(M, g_{ab})$  and  $(M, \tilde{g}_{ab})$  are the same, and there is a sense in which they are different. The sense in which they are the same—that they are isometric, or isomorphic, or agree on all invariant structure—is wholly and only captured by  $\tilde{\psi}$ . The (salient) sense in which they are different—that they assign different values of the metric to the same point—is given by an entirely different map, namely,  $1_M$ . But—and this is the central point—one cannot have it both ways. Insofar as one wants to claim that these Lorentzian manifolds are physically equivalent, or agree on all observable/physical structure, one has to use  $\tilde{\psi}$  to establish a standard of comparison between points. And relative to this standard, the two Lorentzian manifolds agree on the metric at every point—there is no ambiguity, and no indeterminism. (This is just what it means to say that they are isometric.) Meanwhile, insofar as one wants to claim that these Lorentzian manifolds assign different values of the metric to each point, one must use a different standard of comparison. And relative to this standard—that given by  $1_M$ —the two Lorentzian manifolds are not equivalent. One way or the other, the hole argument seems to be blocked (Weatherall 2018, 338–339).

Here,  $\tilde{g} = (\psi^{-1})^*g$ . Furthermore, the notation  $\tilde{\psi}$  stands for the promotion

$$(M, g) \xrightarrow{\tilde{\psi}} (M, \tilde{g})$$

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7 Gomes and Butterfield (2023a) argue that Earman and Norton (1987) assumed the implication *Substantivalism*  $\implies$  *Distinct*, which later literature questioned via attempts at “sophistication.”

8 Note that there is a kind of indeterminism in GR that is outside the scope of the Hole Argument (whatever its worth): In the language detailed in §2 below, this is the possibility that strong cosmic censorship (in the current, initial-value problem sense) fails; in other words, that the MGH (i.e., maximal Cauchy development) of some well-posed “generic” initial data for the Einstein equations is extendible in a suitable regularity class (of the metric). See, e.g., Dafermos (2019), Doboszewski (2017, 2020), Smeenk and Wüthrich (2021), Landsman (2021), Chapter 10, and references therein. For conceptual history, see also Earman (1995) and Landsman (2022). I return to this in §3.

of the diffeomorphism  $\psi : M \rightarrow M$  of (bare) manifolds to an isometry of *Lorentzian* manifolds, seen as the objects in a category **Lor** whose arrows are isometries.<sup>9</sup> There is no such extension  $\tilde{1}_M : (M, g) \rightarrow (M, \tilde{g})$  of the identity  $1_M$  of  $1_M$ , since  $\tilde{1}_M$  is not an isometry (unless  $\psi$  happens to be one);<sup>10</sup> in **Lor**, only  $\tilde{1}_M : (M, g) \rightarrow (M, g)$  is defined. And this is exactly Weatherall's point: One cannot meaningfully identify  $x \in M$  seen as a point in the spacetime  $(M, g)$  with  $x \in M$  seen as a point in a different spacetime  $(M, \tilde{g})$ , in order to be able to say that  $\tilde{g}_{ab}(x) \neq g_{ab}(x)$ , which would launch the Hole Argument. A similar point was made by Penrose:

The basic principles of general relativity—as encompassed in the term ‘the principle of general covariance’ (and also ‘principle of equivalence’)—tell us that there is no natural way to identify the points of one spacetime with corresponding spacetime points of another (Penrose 1996, 591).<sup>11</sup>

A simpler way to make the same *mathematical* point, in the spirit of Weatherall's own abelian group example but closer to the mathematical structure of GR, would be to take pairs  $(M, f)$  where  $M$  is a manifold (or just a set without further structure) and  $f : M \rightarrow \mathbb{R}$  is a smooth function (or just a function), perhaps interpreted as some physical scalar field. The allowed maps between pairs  $(M', f')$  and  $(M, f)$ , i.e., the analogues of isometries, are those diffeomorphisms (or just bijections)  $\psi : M' \rightarrow M$  for which  $f' = f \circ \psi$ . Taking  $M' = M$ , Weatherall would undoubtedly say:

- one can send a pair  $(x, f(x)) \in (M, f)$  to  $(\psi(x), f(x)) \in (M, \psi^* f)$ , since  $f(x) = (\psi^* f)(\psi(x))$
- but one cannot send  $(x, f(x))$  to  $(x, f(\psi^{-1}(x)))$ , although the latter is a point in  $(M, \psi^* f)$ , since neither  $1_M$  nor  $\psi$  can accomplish this.

Since, on this view, one cannot compare  $(x, f(x))$  with  $(x, f(\psi^{-1}(x)))$ , one cannot relate  $f(\psi^{-1}(x))$  to  $f(x)$  at  $x$  (which is deemed crucial for the Hole Argument).

There are also philosophical arguments against such “trans-world identifications,” see, e.g., Lewis (1986) and, in connection with the Hole Argument, Butterfield (1988, 1989) and Gomes and Butterfield (2023b). However, Weatherall explicitly tries to undermine the Hole Argument by appealing to *mathematical practice*:

In contemporary mathematics, the relevant standard of sameness for mathematical objects of a given kind is given by the mathematical theory of those objects. In most cases, the standard of sameness for mathematical objects is some form of isomorphism. (...) mathematical models of a physical theory are only defined up to isomorphism, where the standard of

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9 In view of Theorem 2 below, within such reasoning one should optimally work in the category **ST** of spacetimes (see footnote 3), whose isomorphism are isometries *preserving time orientation*.

10 The emphasis Halvorson and Manchak (2022) put in this context on their otherwise highly valuable Theorem 1 (see footnote 23) seems like flogging a dead horse. This theorem implies that a hole diffeomorphism of the kind envisaged by Einstein (1914) and Earman and Norton (1987), and explicitly constructed by Muller (1995), cannot be an isometry (which, or so it is suggested, would be the only remaining hope for the Hole Argument to work, accepting Weatherall's critique). But if it were, then  $\psi^* g = g$  all across  $M$  and the dilemma of having both  $(M, g)$  and  $(M, \psi^* g)$  as models with the same matter distribution or other initial data simply would not arise: both (naive) determinism and substantivalism would be safe in GR: The Hole Argument would be a dud.

11 Taken from the penultimate version of Gomes (2021a); omitted, alas, from the final version.

isomorphism is given by the mathematical theory of whatever mathematical objects the theory takes as its models. One consequence of this view is that isomorphic mathematical models in physics should be taken to have the same representational capacities. By this I mean that if a particular mathematical model may be used to represent a given physical situation, then any isomorphic model may be used to represent that situation equally well. Note that this does not commit me to the view that equivalence classes of isomorphic models are somehow in one-to-one correspondence with distinct physical situations. But it does imply that if two isomorphic models may be used to represent two distinct physical situations, then each of those models individually may be used to represent both situations (Weatherall 2018, 331–332).

The structuralist approach suggested here implies that the *specific nature* of individual objects (in category theory) or models (in model theory) cannot be used.<sup>12</sup>

Weatherall’s arguments are controversial: see, e.g., Arledge and Rynasiewicz (2019), Roberts (2020), Pooley and Read (2021), Gomes (2021a,b), and Gomes and Butterfield (2023a) for criticism and discussion.<sup>13</sup> My own two pennies worth would be to say that Weatherall uses the notion of “contemporary mathematics” quite selectively: In many cases, the specific nature of mathematical objects—as opposed to just their isomorphism class—is used. Indeed, the very *definition* of an isometry rests on the ability to put  $\psi^*g$  at  $x$ , where originally there was  $g(x)$ , and no mathematician would have any qualms saying this is the same  $x$ . Subsequently, to call  $\psi$  an isometry one needs to ask whether or not  $\psi^*g(x)$  equals  $g(x)$  at  $x$ . More generally, defining the usual action of a diffeomorphism on a tensor (field)  $T$  (like the metric) puts  $(\psi^*T)(x)$  at  $x$  where previously there was  $T(x)$  at  $x$  (perhaps it is worth noting that in giving such definitions, mathematicians essentially stick to a Newtonian absolute spacetime, in which the points  $x$  are identifiable.) With it, the Lie derivative becomes questionable; see also Gomes (2021a), §2.4, and Gomes and Butterfield (2023a; 2023b). Furthermore, though indeed out of step with contemporary mathematics, all of the local coordinate-based definitions of tensors used in the past by Einstein (and even, both before and after the introduction of GR, by *mathematicians* like Ricci and Levi-Civita), while awkward as *definitions*, remain valid *theorems* in modern differential geometry. Do these definitions and theorems now become suspect? Finally, even if Weatherall (2018) were right, should mathematical practice really dictate the way physicists must interpret the mathematical objects they use? I would say the opposite: *Physical* practice should dictate the way mathematical objects are used, at least in the context of mathematical physics.<sup>14</sup>

## 2 THE CHOQUET-BRUHAT-GEROCH THEOREM

In any case, the apparently controversial Hole Argument is clarified by placing it in the context of an uncontroversial theorem due to Choquet-Bruhat–Geroch (1969) on the existence and uniqueness of maximal globally hyperbolic solutions to the Einstein field

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<sup>12</sup> This suggestion was subsequently somewhat weakened in Weatherall (2021) and is also challenged by, e.g., Roberts (2020), Gryb and Thébault (2022), and Pooley and Read (2022).

<sup>13</sup> See also Menon and Read (2023) for a valid critique of Halvorson and Manchak (2022).

<sup>14</sup> For example, Alexandre Grothendieck shaped current mathematical practice. But he famously refused to have anything to do with physics (because in his view physics led to nuclear weapons).

equations. This is contained in Theorem 2, below.<sup>15</sup> Together with Penrose’s work on the causal structure of spacetime, it is one of the pillars of mathematical relativity. In particular, all PDE-related work in GR is based on it, including current approaches to cosmic censorship (see footnote 8). In the theorem, Einstein’s somewhat obscure way of stating the initial value problem for his field equations (in which, inspired by Mach’s principle, he implicitly took initial data outside a hole, essentially at infinity) is replaced by a version first discussed by Hilbert (1917), who took initial data on a spacelike slice.<sup>16</sup> A Hilbert-style Hole Argument may then be based on Proposition 1 below), which comes out a special case of Theorem 2. The change from Einstein’s choice of initial data to Hilbert’s has little influence on any (relevant) philosophical discussion, which I will therefore base on Theorem 2.

Since the Hole Argument is closely related to the following central issues in the philosophy of GR, it is unsurprising that the Choquet-Bruhat–Geroch theorem clarifies those as well (in a way that can be separated from the Hole Argument):

1. Finding an appropriate notion of determinism for GR;
2. Interpreting isometries in GR as gauge symmetries.

Together with a reconsideration of Weatherall’s critique of the Hole Argument, the first issue will be taken up in §3, especially in the light of previous proposals by Butterfield (1987, 1988, 1989). The second will be discussed in §4.

First, I review the theorem in question. It is the culmination of the initial-value approach to GR, which is based on PDE-theory, and the following ideology:<sup>17</sup>

- All valid **assumptions** in GR are assumptions about initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ .

Such an initial data triple, assumed smooth, is obtained by equipping some 3d Riemannian manifold  $(\tilde{\Sigma}, \tilde{g})$  with a second symmetric tensor  $\tilde{k} \in \mathcal{X}^{(2,0)}(\tilde{\Sigma})$ , i.e., of the same “kind” as the 3-metric  $\tilde{g}$ , such that  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  satisfies the vacuum constraints

$$\tilde{R} - \text{Tr}(\tilde{k}^2) + \text{Tr}(\tilde{k})^2 = 0; \quad \tilde{\nabla}_j \tilde{k}_i^j - \tilde{\nabla}_i \text{Tr}(\tilde{k}) = 0. \quad (4)$$

Here  $\tilde{R}$  is the Ricci scalar on  $\tilde{\Sigma}$  for the Riemannian metric  $\tilde{g}$  and likewise  $\tilde{\nabla}$  is the unique Levi-Civita (i.e., metric) connection on  $\tilde{\Sigma}$  determined by  $\tilde{g}$  (so that  $\tilde{\nabla}\tilde{g} = 0$ ).

- All valid **questions** in GR are questions about “the” MGHD  $(M, g, \iota)$  thereof.

15 The original source is Choquet-Bruhat and Geroch (1969), who merely sketched a proof (based on Zorn’s lemma, which they even had to use twice). Even the 800-page textbook by Choquet-Bruhat (2009) does not contain a proof of the theorem (which is Theorem XII.12.2); the treatment in Hawking and Ellis (1973), §7.6, is slightly more detailed but far from complete, too. Ringström (2009) is a book-length exposition of the theorem, but ironically his proof of Theorem 16.6, i.e., Theorem 2 above, is wrong; it is corrected in Ringström (2013), §23. A constructive proof was given by Sbierski (2016), which is streamlined and summarized in Landsman (2021), §7.6.

16 Hilbert (1917) gave the first analysis of GR from a PDE point of view. He addresses the indeterminism of Einstein’s equations, and also refers to Einstein (1914), but does not explicitly relate his analysis to the *Lochbetrachtung*. See also Howard and Norton (1993) and Brading and Ryckman (2018). For some history of the PDE approach to GR see Stachel (1992), Choquet-Bruhat (2014), and Ringström (2015), summarized in Landsman (2021), §1.9. It is also possible to give initial data for the Einstein equations on a *null* hypersurface (Penrose 1963); see, e.g., Klainerman and Nicolò (2003) for a detailed treatment. That would also lead to a version of the Hole Argument.

17 Physicists would see this as the ADM approach to GR, as in Misner, Thorne, and Wheeler (1973). But the mathematical literature developed almost independently, led by Choquet-Bruhat.

Among these questions, the one relevant to the Hole Argument concerns the uniqueness of  $(M, g, \iota)$ , whence the scare quotes around “the.” Roughly speaking, a MGHD (for *maximal globally hyperbolic development*) of  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  is a maximal spacetime  $(M, g)$  “generated” by these initial data via the Einstein equations, in that

$$\iota : \tilde{\Sigma} \hookrightarrow M$$

injects  $\tilde{\Sigma}$  into  $M$  as a “time slice” on which the 4-metric  $g$  induces the given 3-metric  $\tilde{g}$  and extrinsic curvature  $\tilde{k}$ . In more detail,<sup>18</sup> a *Cauchy development* or *globally hyperbolic development* of given initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  satisfying the constraints (4) is a triple  $(M, g, \iota)$ , where  $(M, g)$  is a spacetime that solves the vacuum Einstein equations  $R_{\mu\nu} = 0$  and  $\iota$  is an injection making  $\iota(\tilde{\Sigma})$  a spacelike Cauchy (hyper)surface in  $M$  such that  $g$  induces these initial data on  $\iota(\tilde{\Sigma}) \cong \tilde{\Sigma}$ , i.e.,  $\tilde{g} = \iota^*g$  is the metric and  $\tilde{k}$  is the extrinsic curvature of  $\tilde{\Sigma}$ , induced by the embedding  $\iota$  and the 4-metric  $g$ .<sup>19</sup> It follows that  $(M, g)$  is globally hyperbolic, since it has a Cauchy surface.

This formulation of the (spatial) initial-value problem for the (vacuum) Einstein equations was an achievement in itself. In particular, it circumvents the vicious circle one is forced into if one tries to find initial data for an already given spacetime (solving the Einstein equations); for, it is part of the problem to find the latter from the given initial data, and hence one cannot give  $dg/dt$  ( $t = 0$ ) as initial data.

However, the main achievement concerns the existence and uniqueness of  $(M, g, \iota)$ , which depends on a suitable notion of *maximality* (as in the far simpler case of ODEs, where in order to guarantee uniqueness, the time interval on which the solution is defined should be maximal). This notion is also non-trivial, and tied to GR. Namely:

- A *maximal Cauchy development* or *maximal globally hyperbolic development*, acronym MGHD, of given smooth initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ , satisfying the constraints (4), is a Cauchy development  $(M, g, \iota)$  with the property that for any other Cauchy development = globally hyperbolic development  $(M', g', \iota')$  of these same data there exists an embedding  $\psi : M' \rightarrow M$  that preserves time orientation, metric, and Cauchy surface as defined by  $\iota$ , i.e., one has

$$g' = \psi^*g; \quad \iota' = \psi^{-1} \circ \iota. \quad (5)$$

A Hole Argument à la Hilbert (1917) then follows from a simple observation:<sup>20</sup>

**Proposition 1.** *Given some MGHD  $(M, g, \iota)$  of the initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ , seen as the spacetime under review, let  $U$  be an open neighborhood of  $\iota(\tilde{\Sigma})$  in  $M$ . Take a (time orientation preserving) diffeomorphism  $\psi$  of  $M$  that is the identity on  $U$ . Then the triple  $(M', g', \iota')$ , where  $M' = M$ ,  $g' = \psi^*g$  (so that  $g' = g$  within  $U$ ), and  $\iota' = \iota$ , with time orientation induced by  $\psi$ ,<sup>21</sup> is a MGHD of the same initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ .*

18 See also the references in footnote 23, or Landsman (2021), §7.6. Tildes adorn 3d objects.

19 Let  $N$  be the unique (necessarily timelike) future-directed normal vector field on  $\iota(\tilde{\Sigma})$  such that  $g_x(N_x, N_x) = -1$ . Then  $\tilde{k}(X, Y) = -g(\nabla_x N, Y)$  defines the extrinsic curvature of  $\iota(\tilde{\Sigma})$ .

20 This construction also works if  $U = J^-(\iota(\tilde{\Sigma}))$ , cf. Curiel (2018) and Pooley (2022). The ‘Gauge Theorem’ of Earman and Norton (1987, 520) is similar in spirit but lacks the connection to the initial-value problem that is central here. Both results of course follow from general covariance.

21 Defining time orientation by (the equivalence class of) a global timelike vector field  $T$  on  $M$ , so that some causal vector  $X$  is future-directed iff  $g(X, T) < 0$ , this means that  $g(T, \psi^{-1}T) < 0$ .



This supports a decent version of the Hole Argument. It is superior to Einstein’s and Earman and Norton’s formulation in that it has shaken off any implicit reference to Mach’s principle and is closer to the usual initial value problem for hyperbolic PDEs, with initial data on a spacelike hypersurface.<sup>22</sup> Note in this respect that the open set  $U$  is the analog of the *complement* of Einstein’s hole. The larger  $U$  is, the stronger the potential challenge to determinism (since the ensuing spacetimes differ in the *complement* of  $U$ , which for Einstein is inside the hole and for us is away from the initial data Cauchy surface  $\iota(\tilde{\Sigma})$ ), but although  $U$  can be made arbitrarily thin (as long as it contains  $\iota(\tilde{\Sigma})$ ), it may as well be arbitrary large (*idem dito*). Thus, the logical strength of both versions of the Hole Argument seems quite similar.

But! With respect to Weatherall’s (2018) critique, Proposition 1, seen as Hilbert’s version of the Hole Argument, is not really different from Einstein’s, since it equally well starts from a *diffeomorphism*  $\psi$  of  $M$  that only becomes an *isometry* from  $(M, \psi^*g)$  to  $(M, g)$  “with hindsight.” Although I disagree with this critique (see §1), I intend to weaken it even further via a slight reformulation—and corollary of—the celebrated theorem of Choquet-Bruhat and Geroch (1969):<sup>23</sup>

**Theorem 2.** *For each initial data triple  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  satisfying the constraints (4) there exists a MGHD  $(M, g, \iota)$ . Any triple  $(M', g', \iota')$  that arises from an isometry  $(M', g') \xrightarrow{\psi} (M, g)$  that preserves time orientation and satisfies  $\psi \circ \iota' = \iota$  (fixing the Cauchy surface) is an MGHD of the same initial data.<sup>24</sup> Conversely, all MGHDs of these data arise in this way, so  $(M, g, \iota)$  is unique up to these specific isometries.*

The easy first part incorporates Proposition 1 as a special case. The difficult second part, which is the real thrust of the theorem, is a nontrivial *converse* to the first.

### 3 RETHINKING THE HOLE ARGUMENT

Like the Earman-Norton Hole Argument,<sup>25</sup> Theorem 2 exposes the tension between:

1. *Determinism*, in the precise version that the Einstein equations for given initial data have a unique solution in the sense that triples  $(M, g, \iota)$  and  $(M', g', \iota')$ , as in the statement of Theorem 2, are seen as different mathematical representatives of the *same* physical situation (i.e., are “physically identified”).
2. *Distinct*, in the sense that triples  $(M, g, \iota)$  and  $(M', g', \iota')$  represent *different* physical possibilities (although they are observationally indistinguishable).

22 With a special GR twist, though: the Einstein equations are not hyperbolic, but the six spatial ones are hyperbolic in a suitable gauge, in which the remaining four are elliptic constraints.

23 Though rarely, if ever, mentioned, the isometry  $\psi$  in the converse is unique. This can be shown by Proposition 3.62 in O’Neill (1983) or the equivalent argument in footnote 639 of Landsman (2021), to the effect that an isometry  $\psi$  is determined at least locally (i.e., in a convex nbhd of  $x$ ) by its tangent map  $\psi'_x$  at some fixed  $x \in M'$ . Take  $x \in \iota'(\tilde{\Sigma})$ . Since  $\psi$  in Theorem 2 is fixed all along  $\iota'(\tilde{\Sigma})$  by the second condition in (5) and since it also fixes the (future-directed) normal  $N_x$  to  $\iota'(\tilde{\Sigma})$  by the first condition in (5), it is determined locally. Theorem 1 in Halvorson and Manchak (2022) then applies, which is a rigidity theorem for isometries going back at least to Geroch (1969), Appendix A (as Halvorson and Manchak acknowledge).

24 At first sight, only the second half of (5) appears in the above theorem. But the first half is part of the definition of an isometry.

25 Recall that Einstein’s Hole Argument was meant to enforce a choice between determinism and general covariance. Theorem 2 is based on standard (generally covariant) GR and hence this choice has already been made, leaving the dilemma highlighted by Earman and Norton (1987).

All discussions of this tension (e.g., Butterfield 1989; Curiel 2018; Gomes & Butterfield 2023a; Pooley 2022), which I will not review, remain relevant if we support the Hole Argument by Theorem 2 instead of Proposition 1 or Einstein’s construction. However, after this replacement, options 1 and 2 do differ a little from before:

- Option 1 is, in the context of Theorem 2, a larger move compared to the original Hole Argument, since far more spacetimes are now declared to be “physically equivalent”: namely all triples  $(M', g', t')$  in its statement. But in return, the thrust of choosing this option is strengthened: Regarding  $(M, \psi^*g)$  and  $(M, g)$  as physically equivalent for some specific Hole diffeomorphism  $\psi$ , as in Proposition 1, merely restores determinism in a special case, whereas (the second part of) Theorem 2 gives us complete assurance (barring indeterminism caused by violations of strong cosmic censorship, see §1 and below).
- Option 2, on the other hand, requires no more commitment than in the original Hole Argument: If we do not even identify  $(M, g)$  with  $(M, \psi^*g)$  in Proposition 1, where the underlying manifolds are the same, then certainly we will not identify any of the more general triples  $(M', g', t')$ , where they are different.

Given its much better embedding in the mathematical (physics) literature, it seems considerably more difficult for Weatherall and his followers to redirect their critique of the (modern) Hole Argument to Theorem 2. Although I can’t speak for them, here are, prophylactically, some options they might still invoke, with a reply:

- (a)** The part of Theorem 2 that is actually relevant to the Hole Argument should be stated as follows, assuming the existence of a ‘reference’ MGHD  $(M, g, t)$ :

*Any diffeomorphism  $\psi : M' \rightarrow M$  gives rise to another MGHD  $(M', g', t')$  of the same initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ , where  $g' = \psi^*g$  and  $t' = \psi^{-1} \circ t$ .<sup>26</sup>*

Weatherall’s original arguments (from his 2018) then apply almost *verbatim* (and indeed, I would say they are even clearer in this more general context).

But I was very careful in stating Theorem 2 the way I did: all reference to “pure” diffeomorphisms has gone, and all spacetimes that occur in the theorem are related by isometries: It is either *assumed* (in the first half) or *concluded* (in the second half) that  $(M', g') \xrightarrow{\psi} (M, g)$  is an isometry. No other maps are mentioned and no controversial comparisons need to be made.

In so far as an appeal to “mathematical practice” is made, I would answer that few if any mathematicians would be sensitive to the difference between the above reformulation of the middle part of Theorem 2 and its earlier statement.

- (b)** One might accept Theorem 2 as it stands, but somehow object to Proposition 1 being a special case of it (for example, because its construction mixes up diffeomorphisms and isometries). Although I would again doubt any such arguments, especially if they appeal to “mathematical practice,” even if they were valid I would point out that Theorem 2 *as a whole* raises the same dilemma as the Hole Argument and indeed may be taken to *be* the Hole Argument (2.0).
- (c)** One could reject Theorem 2, for example because its *proof* (admittedly!) does use (even local) diffeomorphisms and pointwise comparisons of metrics. But, further to the discussion in the Introduction, if Weatherall *et al.* would object to

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<sup>26</sup>  $M'$  acquires a time orientation from  $M$  and  $\psi$ , which  $\psi$  trivially preserves, cf. footnote 21.

generally accepted proofs of theorems by top mathematicians, then their appeal to “mathematical practice” would once again be self-defeating.<sup>27</sup>

I now compare the notion of *Determinism* (which is standard in mathematical relativity) used above with an influential definition appearing in the philosophy of physics literature due to Butterfield (1987, 1989), which was specifically developed in the context of the Hole Argument. In order to clarify the connection of his definition with Theorem 2, I first state a somewhat awkward weakening of this theorem:

**Corollary 3.** *If two globally hyperbolic spacetimes  $(M, g)$  and  $(M', g')$  contain Cauchy surfaces  $\tilde{\Sigma} \subset M$  and  $\tilde{\Sigma}' \subset M'$ , respectively, which carry initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  and  $(\tilde{\Sigma}', \tilde{g}', \tilde{k}')$  induced by the 4-metrics  $g$  and  $g'$  on  $M$  and  $M'$ , respectively, where both  $(M, g)$  and  $(M', g')$  are maximal for these initial data, and there is a 3-diffeomorphism  $\alpha: \tilde{\Sigma}' \rightarrow \tilde{\Sigma}$  such that  $\tilde{g}' = \alpha^* \tilde{g}$  and  $\tilde{k}' = \alpha^* \tilde{k}$ , then there exists an isometry  $\beta: M' \rightarrow M$  that preserves time orientation and restricts to  $\alpha$  on  $\tilde{\Sigma}$ .*

This corollary is weaker than Theorem 2, for it lacks the existence claim of  $(M, g)$ . In comparison, Butterfield’s Definition **Dm2** of determinism is as follows:

A theory with models  $(M, O_i)$  is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, if:

given any two models  $(M, O_i)$  and  $(M', O'_i)$  containing regions  $S$  and  $S'$  of kind **S**, respectively, and any diffeomorphism  $\alpha$  from  $S'$  onto  $S$ :

If  $\alpha^*(O'_i) = O_i$  on  $\alpha(S') = S$ , then there is an isomorphism  $\beta$  from  $M'$  onto  $M$  that sends  $S'$  to  $S$ , i.e.,  $\beta^*O'_i = O_i$  throughout  $M'$  and  $\beta(S') = S$ .

(Butterfield 1987, 29; Butterfield 1989, 9)

Here it would clarify the situation to add the requirement that  $\beta$  extends  $\alpha$ .<sup>28</sup>

To start with the good news: Though not intended for that purpose, this definition is quite suitable for expressing the idea of determinism inherent in strong cosmic censorship (and its possible violation!), cf. footnote 8. Indeed, define models (of GR) to be spacetimes  $(M, g)$  satisfying the (vacuum) Einstein equations, and take the regions **S** to be MGHDS of

27 It may be interesting to point out that some of the most beautiful theorems in category theory, such as Gelfand duality, have very ugly proofs involving all kinds of constructions that the final result sweeps under the carpet; see Landsman (2017), Appendix C, for this specific example.

28 Butterfield (1987, 1989) emphasizes that  $\beta$  need not extend  $\alpha$  (his primed objects are our unprimed ones). However, his counterexamples are easily avoided by requiring that  $\alpha$  is only defined on  $S'$ . It is clear from Butterfield and Gomes (2023a), end of §2.2.3, that Butterfield endorses my discussion. Butterfield contrasts **Dm2** with a Laplacian kind of definition of determinism **Dm1** he attributes to Montague and Earman: “A theory with models  $(M, O_i)$  is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, if: given any two models  $(M, O_i)$  and  $(M', O'_i)$  and any diffeomorphism  $\beta$  from  $M'$  onto  $M$ , and any region  $S$  of  $M$  of kind **S**: if  $\beta(S)$  is of kind **S** and also  $\beta^*O'_i = O_i$  on  $S'$ , then:  $\beta^*O'_i = O_i$  throughout  $M'$ .” If we correct this similarly to **Dm2**, Butterfield’s point still stands: The Hole Argument (in any version) shows that GR violates **Dm1**. See also Belot (1995), Melia (1999), and Pooley (2022) for a detailed analysis of similar definitions. Pooley’s version of **Dm2** is a bit more general and also applies to GR: “Theory  $T$  is deterministic just in case, for any worlds  $W$  and  $W'$  that are possible according to  $T$ , if the past of  $W$  up to some timeslice in  $W$  is qualitatively identical to the past of  $W'$  up to some timeslice in  $W'$ , then  $W$  and  $W'$  are qualitatively identical.” Apart from my complaint that also this definition assumes the existence of  $W$  and  $W'$  (instead of proving it), a definition like this requires a sub-definition of what is meant by ‘qualitative’, which Theorem 2 also takes care of.

initial data  $(\tilde{g}, \tilde{k})$  posed on some partial Cauchy  $\tilde{\Sigma}$  surface in  $M$ . Then, GR is **S**-deterministic precisely if strong cosmic censorship holds. Indeed: If not, let  $(M, g, \iota)$  be a MGHD of initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  as in Theorem 2, with  $(M, g)$  the associated spacetime, and let  $(M', g')$  be a proper extension thereof (which, by definition, exists if strong cosmic censorship fails). Then **Dm2** fails.<sup>29</sup>

However, the “Laplacian” context in which **Dm2** was originally proposed suggests that the idea was to take  $S \subset M$  to be a time-slice in a spacetime  $(M, g)$ . In that case, for there to be any hope that GR is deterministic even if strong cosmic censorship holds, Butterfield’s definition should be amended in the following way:

1. The class of models should be restricted to maximal globally hyperbolic solutions to the vacuum Einstein equations with initial data as in Theorem 2.
2. Either **S** should be a neighborhood of a Cauchy surface in  $M$  (as in the Hole Argument), or the models  $(M, g)$  should be triples  $(M, g, \iota)$ , in which case one should add a condition on the extrinsic curvature to ‘ $\alpha^*(g) = g'$  on  $\alpha(S') = S'$ ’.

Granting strong cosmic censorship, GR is then **S**-deterministic by Theorem 2.

Of course, definitions like Butterfield’s **Dm2** or its close cousins are not sacrosanct. An anonymous referee very ingeniously suggested that *Determinism* and *Distinct* are compatible with each other *and with Theorem 2* in the following way:

*The initial data induced by the triple  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  on  $\iota(\tilde{\Sigma}) \subset M$  are taken to be distinct from those induced by the same triple on  $\iota'(\tilde{\Sigma}) \subset M'$  (as in the theorem).*

Consequently, this distinction should then also be made in the situation of Proposition 1 (which, after all, is the special case of the general situation just addressed), where the isometric space-times  $(M, g)$  and  $(M', g')$  both extend the “same” Cauchy surface  $\iota(\Sigma) = \iota'(\Sigma)$  carrying the “same” initial data  $(\tilde{g}, \tilde{k})$ . Indeed, if the initial data for  $(M, g)$  and  $(M', g')$  are different, no determinist would object to the ensuing dynamically evolved space-times being different; whereas, according to this new view, they were misled in believing that the initial data for the isometric but otherwise different space-times  $(M, g)$  and  $(M', g')$  were the same.

This kind of thinking would clearly be at odds with **Dm2** and similar (GR-adapted) “Laplacian” definitions of determinism. It would also affect the original Hole Argument: for, likewise, a way out of both Einstein’s and Earman and Norton’s Hole Argument would be to deny that the space-times outside the hole within  $(M, g)$  and  $(M, \psi^*g)$  are identical (whereas the Hole Argument is based on their identification). This would move the discussion from possible (mis)identifications of points and metrics *inside* the hole (or far *outside* the Cauchy surface), where the metrics are *different*, to (mis)identifications *outside* the hole (or *inside* the Cauchy surface), where the metrics are the *same*. Claiming that such identifications are wrong or undefined would be even more radical than Weatherall’s objection to the Hole Argument (which concerns the region where the metrics are *different*), but would equally well undermine it, in the sense that the tension or contradiction between *Determinism* and *Distinct* does not in fact arise.

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<sup>29</sup> The Kerr and Reissner-Nordström spacetimes are indeterministic in precisely this way, though strictly speaking, these spacetimes are usually not seen as indicating a violation of strong cosmic censorship in GR as a whole since their initial data are not “generic” in some suitable sense.

This idea is certainly worth further discussion; for the moment, my reply is:

- No mathematical relativist would ever consider making the said distinction between initial data, since all their practices are based on identifying them;
- Few philosophers would do so either, given the mismatch with **Dm2** and the like.

Finally, Theorem 2 (as well as Theorem 4 below) is reminiscent of the spontaneous breakdown of gauge symmetry through the Higgs mechanism.<sup>30</sup> Here, in order to settle into a minimum of the Higgs potential, the Higgs field  $\varphi$  must “choose” a point  $\varphi_c$  on a circle as its “frozen” vacuum value. The *global*  $U(1)$  symmetry involved in this choice is a *finite-dimensional* shadow of the original *infinite-dimensional local*  $U(1)$  symmetry of the theory (see also endnote xxxii, below). Different choices of  $\varphi_c$  yield phenomenologically indistinguishable worlds and hence the analogy is between moving the vacuum value  $\varphi_c$  around on a circle and moving a spacetime  $(M, g)$  around in its orbit under its isometry group.<sup>31</sup> Also, here we are talking about symmetries of the universe as a whole, which is what makes them unobservable; the situation changes completely if different domains in the universe have different values of  $\varphi_c$ .

Let us now turn to an important special (!) case of this situation of Theorem 2.

## 4 SPECIAL RELATIVITY: STATUS OF THE POINCARÉ GROUP

If one chooses the first option of *Determinism* from the binary menu opening the previous section, the isometries in Theorem 2 have to be interpreted accordingly as gauge symmetries. The way these symmetries reflect the original diffeomorphism invariance of the Einstein equations (which invariance launched the Hole Argument in the first place!) is clear from point **(a)** in the same section. In the special case  $M' = M$ , for some given MGHD  $(M, g, \iota)$  and any diffeomorphism  $\psi : M \rightarrow M$ , one obtains a new MGHD  $(M', g', \iota') = (M, \psi^*g, \psi^{-1} \circ \iota)$ , which (by the vote for determinism) is deemed physically equivalent to  $(M, g, \iota)$ . There is no reason, however, why, at least in the context of the initial-value formulation, the diffeomorphism invariance of GR should not include diffeomorphisms  $\psi : M' \rightarrow M$  for  $M' \neq M$  (and both mathematical and physical practice confirms this); the statement that the “gauge group” of GR is “ $\text{Diff}(M)$ ” seems too narrow (and even ill defined: which  $M$  is meant?), and Theorem 2 shows the full situation. The seeming lack of general covariance of the initial-value *problem* for GR, notably of its initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ , is then amply compensated for by the fact that (by Theorem 2) its *solution*, starting from some reference MGHD  $(M, g, \iota)$ , has covariance properties exceeding all expectations: The mathematical structure of the symmetry “thing” in the initial-value formulation of GR seems to be that of a *groupoid*, where for given initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  the base space consists of all manifolds diffeomorphic to the manifold  $M$  in some reference MGHD  $(M, g, \iota)$  of these data and the arrows are diffeomorphisms. The symmetry groupoid of GR therefore depends on the initial data and is not universal.

On the other hand, the special case where  $\psi$  is an isometry of  $(M, g)$  (as opposed to a diffeomorphism promoted to an isometry from  $(M, \psi^*g)$  to  $(M, g)$ , as in the Hole Argument)

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30 See Struyve 2011, Landsman (2017), §10.10, or any book on the Standard Model.

31 This analogy is admittedly weak, since Theorem 2 involves both the embedding maps  $\iota$  and the possibility that isometries move a given spacetime  $(M, g)$  to one  $(M', g')$  with a different underlying (but diffeomorphic) manifold  $M$ , neither of which have a counterpart in the Higgs mechanism.

is also clarified by Theorem 2 (except for the identity map  $1_M$ , such isometries exist only for exceptional spacetimes). Indeed, since these are merely a special case of the general isometries in Theorem 2, they have to be interpreted in exactly the same way, that is, as gauge transformations. This may be surprising, because the isometry group  $\text{Iso}(M, g)$  of a fixed spacetime  $(M, g)$  is finite-dimensional and, hence, is not given by freely specifiable functions on  $M$ , as one expects in gauge theories.<sup>32</sup> The explanation is simply that  $\text{Iso}(M, g)$  is a finite-dimensional subgroup of the infinite-dimensional gauge groupoid just defined (for given initial data).<sup>33</sup>

The simplest case where this occurs is in special relativity.<sup>34</sup> Minkowski spacetime  $\mathbb{M} = (\mathbb{R}^4, \eta)$  is a maximal globally hyperbolic solution to the vacuum Einstein equations,<sup>35</sup> which arises as “the” MGHD  $(\mathbb{R}^4, \eta, \iota_0)$  of the initial data

$$(\tilde{\Sigma} = \mathbb{R}^3, \tilde{g} = \delta, \tilde{k} = 0),$$

where  $\delta$  is the Euclidean metric on  $\mathbb{R}^3$  and

$$\iota(x^1, x^2, x^3) = (0, x^1, x^2, x^3). \quad (6)$$

By the general analysis above, the group of time-orientation-preserving Poincaré transformations is then contained in the gauge groupoid of these initial data and hence Poincaré transformations are, perhaps surprisingly, gauge transformations.

Another, more interesting way of reaching the same conclusion is to regard special relativity not as a specific solution to the vacuum Einstein equations, but as a generally covariant field theory by itself, formulated like GR but with the field equation

$$R_{\rho\sigma\mu\nu} = 0, \quad (7)$$

instead of  $R_{\mu\nu} = 0$ . The initial value problem is then almost the same as in general relativity,<sup>36</sup> except that the initial data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  now satisfy the (vacuum) constraints

$$\tilde{R}_{ijkl} - \tilde{k}_{il}\tilde{k}_{jk} + \tilde{k}_{ik}\tilde{k}_{jl} = 0; \quad \tilde{\nabla}_i\tilde{k}_{jk} - \tilde{\nabla}_j\tilde{k}_{ik} = 0. \quad (8)$$

The constraints (8) of generally covariant special relativity are stronger than their counterpart (4) in GR, which actually follows from (8) by contracting with  $\tilde{g}^{ik}\tilde{g}^{jl}$  and  $\tilde{g}^{ik}$ , respectively. The reason is that, in GR, one merely asks for an embedding of the initial data in a *Ricci-flat* Lorentzian manifold  $(M, g)$ , i.e.,  $R_{\mu\nu} = 0$ , whereas in special relativity one

32 If  $\dim(M) = n$ , then for any semi-Riemannian metric  $g$  the isometry group of  $(M, g)$  is at most  $\frac{1}{2}n(n+1)$ -dimensional. See O’Neill (1983), Lemma 9.28; Kobayashi and Nomizu (1963), Theorem VI.3.3, do the Riemannian case. Thus the Poincaré-group in  $n = 4$  has maximal dimension 10.

33 Subgroups of groupoids, seen as (small) categories in which each arrow is invertible (i.e., an isomorphism), are contained in the group of arrows from some base object to itself.

34 The following analysis was inspired by correspondence with Henrique Gomes and Hans Halvorson, who proposed to look at special relativity in this context. See also Iftime and Stachel (2006).

35 Maximality of Minkowski spacetime follows from its inextendibility; see, e.g., Corollary 13.37 in O’Neill (1983) for the smooth case and Sbierski (2018) for inextendibility even in  $C^0$ .

36 This upsets the idea that special relativity uses only linear subspaces of spacetime as hypersurfaces of simultaneity whereas general relativity uses general curved surfaces, but already Schwinger (1948) employed arbitrary initial data surfaces in relativistic quantum field theory.

seeks an embedding in a *flat* Lorentzian manifold, as follows from (7), and the so-called fundamental theorem of (semi) Riemannian geometry.<sup>37</sup>

To avoid global topological issues, I assume that  $\tilde{\Sigma}$  is diffeomorphic to  $\mathbb{R}^3$ , in which case the role of a (reference) MGHD  $(M, g, \iota)$  in Theorem 2 is simply played by Minkowski spacetime  $(\mathbb{R}^4, \eta)$ , with  $\iota$  to be found (see Theorem 4 below).<sup>38</sup> One could now state and prove a counterpart of Theorem 2 for generally covariant special relativity, but instead I rely on a Minkowskian version of the *fundamental theorem for hypersurfaces*, whose original version studied embeddings of two-dimensional surfaces  $\Sigma$  in  $\mathbb{R}^3$  with Euclidean metric (here lies the origin of the Gauss–Codazzi equations, which also play a key role in deriving the constraints (4) in GR).<sup>39</sup>

**Theorem 4.** *For each initial data triple  $(\mathbb{R}^3, \tilde{g}, \tilde{k})$  satisfying the constraints (8) there exists an isometric embedding  $\iota: \mathbb{R}^3 \rightarrow \mathbb{R}^4$  carrying the Minkowski metric  $\eta$ , whose extrinsic curvature is the given tensor  $\tilde{k}$ . Any triple  $(\mathbb{R}^4, \eta, \iota')$  that arises from an isometry  $\psi$  of  $\mathbb{M}$  (i.e., a Poincaré transformation), preserves time-orientation, and satisfies  $\psi \circ \iota' = \iota$  has the same properties (that is,  $\iota': \mathbb{R}^3 \rightarrow \mathbb{R}^4$  is an isometric embedding and the extrinsic curvature induced on  $\iota'(\mathbb{R}^3) \subset \mathbb{R}^4$  by the metric  $\eta$  is  $\tilde{k}$ ).*

*Conversely, all triples  $(\mathbb{R}^4, \eta, \iota')$  with these properties arise in this way from some given triple  $(\mathbb{R}^4, \eta, \iota)$ , which is therefore unique up to Poincaré transformations.*

There is a clear conceptual analogy between Theorems 2 and 4, except that unlike the former, the latter does not take into account the spacetimes  $(M, \eta')$  that are isometric to Minkowski spacetime  $\mathbb{M}$  (where  $M$  could even be  $\mathbb{R}^4$ ). However, the corresponding more general version of Theorem 4 would not affect my conclusion about Poincaré transformations, it would just assign a similar interpretation to even more transformations. And, exactly as in my discussion of special relativity as a special (vacuum) solution of GR, this interpretation is that Poincaré transformations in generally covariant special relativity play the same role as the isometries in general relativity that appear in Theorem 2. On the option of determinism as a way out of the Hole Argument, *Poincaré transformations are therefore physically inert!*

Now, whereas most physicists would be happy to regard isometries in general relativity as gauge symmetries, few would regard Poincaré transformations as such. Fortunately, Gomes (2021b), partly reflecting on Belot (2018), makes the right point:

37 In Lorentzian signature this theorem states that  $(M, g)$  is locally flat (in that its metric is locally Minkowski) if its Riemann tensor vanishes. See, e.g., Landsman (2021), Theorem 4.1.

38 By the splitting theorem of Geroch (1970) as improved by Bernal and Sánchez (2003), global hyperbolicity of  $(M, g)$  gives  $M \cong \mathbb{R} \times \tilde{\Sigma} = \mathbb{R}^4$ , diffeomorphically. Hence we may actually take  $M = \mathbb{R}^4$ , due to (7) necessarily with the Minkowski metric. Finally,  $\mathbb{M}$  is maximal, cf. footnote 35.

39 See Kobayashi and Nomizu (1969), Theorem VII.7.2 or Landsman (2021), Theorem 4.18. This theorem is concerned with embeddings of curved surfaces with prescribed second fundamental form into Euclidean space and goes back to the nineteenth century. The proof of the Minkowskian case is the same, up to some sign changes: In the Euclidean case, the first constraint in (8) is  $\tilde{R}_{jkl} + \tilde{k}_i \tilde{k}_{jk} - \tilde{k}_{ik} \tilde{k}_{jl}$ , the sign changes going back to the different signs in the Gauss–Codazzi equations in Euclidean and Lorentzian signature, see e.g., eqs. (4.147)–(4.148) in §4.7 in Landsman (2021). These sign changes do affect the outcome. For example, Hilbert (1901) proved that it is impossible to isometrically embed two-dimensional hyperbolic space  $(H^2, g_H)$  in Euclidean  $\mathbb{R}^3$ . But hyperbolic space *can* be isometrically embedded in  $\mathbb{R}^3$  with Minkowski metric, cf., e.g., Landsman (2021), §4.4. Hence, given  $(H^2, g_H)$ , a symmetric tensor  $\tilde{k}$  such that  $(g_H, \tilde{k})$  satisfy the Euclidean constraint does not exist, but such a  $\tilde{k}$  can be found satisfying the Minkowski constraints.

But some familiar symmetries of the whole Universe, such as velocity boosts in classical or relativistic mechanics (Galilean or Lorentz transformations), have a *direct* empirical significance when applied solely to subsystems. Thus Galileo's famous thought-experiment about the ship—that a process involving some set of relevant physical quantities in the cabin below decks proceeds in exactly the same way whether or not the ship is moving uniformly relative to the shore—shows that sub-system boosts have a direct, albeit relational, empirical significance. For though the inertial state of motion of the ship is undetectable to experimenters confined to the cabin, yet the entire system, composed of ship and sea registers the difference between two such motions, namely in the different relative velocities of the ship to the water.

(Gomes 2021b, 2)

In other words, in thinking about Poincaré transformations as bringing physical change, as for example in boosts of Galileo's ship or Einstein's train, we apply such transformations to *subsystems* of the universe. But Theorem 4 concerns the action of Poincaré transformations on spacetime *as a whole*. See also Wallace (2022). Similarly, since time translations are Poincaré transformations, even special relativity seems a "timeless" theory in the sense that time translation is a gauge transformation. But once again, this only applies to empty spacetime, where it seems correct.

Summarizing: In the substantialism versus relationalism debate (Earman 1989; Pooley 2013), I see general relativity and special relativity as qualitatively similar. Whatever differences there are seem technical rather than conceptual, just reflecting the underlying difference between the field equations  $R_{\mu\nu} = 0$  and  $R_{\rho\sigma\mu\nu} = 0$ .

## 5 THE HOLE ARGUMENT IN THE PHILOSOPHY OF SCIENCE

Despite their denial of the Hole Argument, Weatherall (2018) and Halvorson and Manchak (2022) make some of the most pertinent comments towards its resolution:

Mathematical models of a physical theory are only defined up to isomorphism, where the standard of isomorphism is given by the mathematical theory of whatever mathematical objects the theory takes as its models. One consequence of this view is that isomorphic mathematical models in physics should be taken to have the same representational capacities. By this I mean that if a particular mathematical model may be used to represent a given physical situation, then any isomorphic model may be used to represent that situation equally well. Note that this does not commit me to the view that equivalence classes of isomorphic models are somehow in one-to-one correspondence with distinct physical situations. But it does imply that if two isomorphic models may be used to represent two distinct physical situations, then each of those models individually may be used to represent both situations.

(Weatherall 2018, 331–332)

Why is it, then, that there has been, and will surely continue to be, a feeling that there is some remaining open question about whether general relativity is fully deterministic? Our conjecture is that the worry here arises from the fact that general relativity, just like any other theory of contemporary mathematical



physics, allows its user a degree of representational freedom, and consequently displays a kind of *trivial semantic indeterminism*: how things are represented at one time does not constrain how things must be represented at later times.

(Halvorson and Manchak 2022, 19)

These comments could just as well have been made about Theorem 2, which by itself already makes it worth delving into the idea of “representational freedom.”<sup>40</sup>

I suggest that the Hole Argument and/or Theorem 2 prompt us to choose not only between *Determinism* and *Distinct*, but, having chosen the first option, to also refine the consequences of this option—seeing isometries as gauge symmetries—through a further choice in this garden of forking paths. This second choice is between two positions in the philosophy of mathematics that are traditionally seen as opposites, namely a Hilbert-style structuralism and a Frege-style abstractionism.<sup>41</sup>

- *Structuralism*: Spacetimes (with fixed initial data) are mathematical structures, which by their very nature can only be studied up to isomorphism. Since isometry is the pertinent notion of isomorphism, the identification of isometric spacetimes called for by the Hole Argument or Theorem 2 was to be expected.
- *Abstractionism*: The relevant mathematical object is the equivalence class of all spacetimes (with fixed initial data) up to isometry. Quoting Wilson (2010):

Appeals to equivalence classes will seem quite natural if one regards the novel elements as formed by *conceptual abstraction* in a traditional philosophical mode: one first surveys a range of concrete objects and then *abstracts* their salient commonalities (Wilson 2010, 395).

In the case at hand, the ‘salient commonalities’ seem to be the property that all members of a given equivalence class satisfy the vacuum Einstein equations with identical initial data. In the spirit of the abstractionist program, this commonality may be expressed by the function  $f$  from the class of all triples  $(M, g, \iota)$  to the class of all triples  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  that maps  $(M, g, \iota)$  to the initial data it induces on  $\iota(\Sigma) \subset M$ , where it is assumed that each  $(M, g, \iota)$  is a maximal globally hyperbolic spacetimes with given Cauchy surface  $\iota(\tilde{\Sigma})$ .

These two options are put in perspective by the following quote from Martin, which Benaceraff chose as the opening quote of his famous 1965 work:

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40 See also Belot (2018), Fletcher (2020), Gomes (2022a,b), Luc (2022), and Pooley (2022).

41 See, e.g., Hallett (2010), Ebert and Rossberg (2016), Mancosu (2016), Blanchette (2018), Hellman and Shapiro (2019), and Reck and Schiemer (2020). Historically, Frege’s abstractionism served his higher goal of logicism, but the former stands on its own and can be separated from the latter. It may be objected that the heart of the Frege-Hilbert opposition does not lie in abstractionism versus structuralism but in differences about the nature of mathematical axioms, definitions, elucidations, and existence, and in particular about Frege’s insistence that every mathematical concept (such as “point” or “line”) be defined on its own through reference, against Hilbert’s revolutionary idea of implicit and “holistic” definition of concepts through an entire axiom system in which they occur. But these issues are closely related. For example, Hilbert’s contextual and relational way of defining concepts naturally implies that whatever makes them concrete is given only up to isomorphism. Abstractionism of the kind considered here arguably goes back to Aristotle, since the kind of equivalence relation lying at the basis of Frege’s abstraction principle is typically obtained by Aristotle’s procedure of *abstraction by deletion* (Mendell 2019). For example, a mathematician sees a bronze sphere as a sphere, deleting its bronzeness. Also in so far as Hilbert famously claimed that mathematical objects exist as soon as the axioms through which they are implicitly defined are consistent (leaving their precise manner of existence in the dark, like Plato), the Frege-Hilbert opposition has its roots in the Aristotle–Plato one (Bostock 2009).

THE attention of the mathematician focuses primarily upon mathematical structure, and his intellectual delight arises (in part) from seeing that a given theory exhibits such and such a structure, from seeing how one structure is “modelled” in another, or in exhibiting some new structure and showing how it relates to previously studied ones ... But ... the mathematician is satisfied so long as he has some “entities” or “objects” (or “sets” or “numbers” or “functions” or “spaces” or “points”) to work with, and he does not inquire into their inner character or ontological status.

The philosophical logician, on the other hand, is more sensitive to matters of ontology and will be especially interested in the kind or kinds of entities there are actually ... He will not be satisfied with being told merely that such and such entities exhibit such and such a mathematical structure. He will wish to inquire more deeply into what these entities are, how they relate to other entities ... Also he will wish to ask whether the entity dealt with is *sui generis* or whether it is in some sense *reducible* to (or *constructible* in terms of) other, perhaps more fundamental entities.

—R.M. MARTIN, *Intension and Decision*

Against abstractionism (both in the context of the Hole Argument and in Frege’s original application to the definition of Number), one may claim extravagance by noting that an equivalence class  $[x]$  with respect to any equivalence relation  $\sim$  on some given set  $X$  is typically huge;<sup>42</sup> no theoretical or mathematical physicist ever works with such equivalence classes of spacetimes, or even a tiny fraction of it.<sup>43</sup>

In practice, one picks some representative  $(M, g, \iota)$ , from which one may switch to an equivalent triple  $(M', g', \iota')$  now and then, but one never uses the entire equivalence class. And yet it is, strictly speaking, the entire equivalence class that Frege would invoke in order to obtain a proper definition or reference of the word “spacetime” (provided the analogy with his definition of natural numbers is valid). See also Benaceraff (1965). To resolve this, one might try to work with the single object  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ , i.e., the initial data that give rise to all of these isometric spacetimes, but no one does this either; all actual work in GR is done in terms of just a few of the triples  $(M, g, \iota)$ , whose choice (within its isometry class) is made for convenience.

Within mathematical structuralism, the Hole Argument seems compatible with both structural realism (Ladyman 2020) and empiricist structuralism (van Fraassen 2008); in the former, the structures in question are so to speak parts of reality whereas in the latter they model empirical phenomena. Let me quote van Fraassen:

1. Science represents the empirical phenomena as embeddable in certain *abstract structures* (theoretical models).
2. Those abstract structures are describable only up to structural isomorphism.

(...) How can we answer the question of how a theory or model relates to the phenomena by pointing to a relation between theoretical and data models, both of them abstract entities? The answer has to be that the data model represent the phenomena; but why does that not just push the problem [namely: *what*

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42 Recall that an equivalence class  $[x] \subset X$  consists of all  $y \in X$  such that  $y \sim x$ .

43 See Gomes (2021a) for an analysis of physical practice, which in the context of gauge theories and GR amounts to the choice of cross-sections of the canonical projection from  $X$  to  $X/\sim$ .

is the relation between the data and the phenomena it models] one step back?  
The short answer is this: construction of a data model is precisely the selective relevant depiction of the phenomena *by the user of the theory* required for the possibility of representation of the phenomenon.

(van Fraassen 2008, 238, 253)

This last comment seems to describe the practice of physicists and mathematicians working in GR: Some *user of the theory* chooses a member  $(M, g, \iota)$  of its equivalence class, whilst some other *user* (or even the same one) may pick another member.<sup>44</sup>

In conclusion, empiricist structuralism seems to have strong cards in confronting the Hole Argument (in both its original versions or rephrased as Theorem 2): It does not suffer from the calculational intractability and ontological extravagance of Frege-style abstractionism; and it seems to be warranted by actual scientific practice.


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The author has no competing interests to declare.

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<sup>44</sup> Van Fraassen's emphasis on the user also explains why say Kerr spacetime, even with fixed parameters  $m$  and  $a$ , can be used to describe different black holes, despite the mathematical identity of the two models. Indeed, one user models the phenomena produced by one black hole, whilst another user uses (!) the "spacetime" in question to model the phenomena produced by another.

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