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The size of $\tilde{T}$

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Abstract. Given a stationary subset T of ω_1, let $\tilde{T}$ be the set of ordinals in the interval (ω_1, ω_2) which are necessarily in the image of T by any embedding derived from the nonstationary ideal. We consider the question of the size of $\tilde{T}$, given T, and use Martin's Maximum and $\mathbb{P}_{max}$ to give some answers.

1. Introduction

In the context of elementary embeddings derived from the nonstationary ideal on ω_1, a canonical function for $\beta \in (\omega_1, \omega_2)$ is a function from ω_1 to ω_1 of the form $g(\alpha) = ordertype(f[\alpha])$, where $f : \omega_1 \to \beta$ is a bijection. The reason for this terminology is that in any elementary embedding derived from forcing with $\mathscr{P}(\omega_1)/I_{NS}$ (where I_{NS} is the nonstationary ideal), any such g represents β. This idea is implicit in the following definition.

Definition 1.1. *([20]) For* $T \subset \omega_1$, $\tilde{T} = \{\beta \in (\omega_1, \omega_2) \mid \exists f : \omega_1 \to \beta$, *one-to-one and onto, and* $C \subset \omega_1$, *club, s.t.* $\forall \alpha \in C$, $o.t.(f[\alpha]) \in T\}$.

Given a subset T of ω_1, $\tilde{T}$ is the set of ordinals in the interval (ω_1, ω_2) which are forced to be in the image of T after forcing with $\mathscr{P}/I_{NS}$ and taking the induced elementary embedding. This follows from the fact that $\tilde{T}$ is the set of $\beta \in (\omega_1, \omega_2)$ such that any canonical function for β maps a club into T.

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Canonical functions have been extensively studied in [1], [14] and elsewhere, but the tilde function seems to have attracted little attention until recently, when it has found application in [20], [13] and [11].

In this paper, we ask a very simple question about the tilde function: given a stationary set $T \subset \omega_1$, what do we know about the size of $\tilde{T}$? For instance, must it be nonempty, must it be stationary on $Cof(\omega)$, or must it be stationary on $Cof(\omega_1)$? It turns out that the answers to these questions are very sensitive to which extension of ZFC we are working in. In particular, we have the following ladder of facts, presented roughly in increasing order of strength of context.

1. If $\diamond$ holds, then there is a stationary, costationary subset T of ω_1 such that $\tilde{T}$ and $\tilde{\bar{T}}$ (the tilde of the compliment of T) are both empty, and a club $C \subset \omega_1$ such that $\tilde{C}$ is empty.
2. If the nonstationary ideal is saturated, then for all club $C \subset \omega_1$, $\tilde{C}$ contains a club in ω_2.
3. Assuming AD + $V = L(\mathbb{R})$ (or, assuming the existence of a Woodin cardinal), there is a two step forcing in whose extension ZFC and the saturation of the nonstationary ideal hold, along with the existence of a stationary, costationary subset T of ω_1 such that $\tilde{T}$ and $\tilde{\bar{T}}$ are both empty.
4. If SRP(ω_2) holds, then for every stationary $T \subset \omega_1$, $\tilde{T} \cap Cof(\omega_1)$ is stationary and $\tilde{T} \cap Cof(\omega)$ is unbounded in ω_2.
5. If $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass closed under continuous preimages and
$$L(\Gamma, \mathbb{R}) \models \text{“AD}_{\mathbb{R}} + \Theta \text{ regular”},$$
then there is a three step forcing over $L(\Gamma, \mathbb{R})$ which creates a model of ZFC + Martin's Maximum$^{++}(c)$ + $(*)$ (the $\mathbb{P}_{max}$ axiom) + “for all costationary $T \subset \omega_1$, $\tilde{T}$ is not stationary on $Cof(\omega)$.”
6. If Martin's Maximum holds, there is an (ω_1, ∞)-distributive forcing in whose extension we have Martin's Maximum for all forcings which don't change the cofinality of ω_2 to ω, but also that for every costationary subset $T \subset \omega_1$, $\tilde{T} \cap Cof(\omega)$ is nonstationary.
7. ([20]) If Martin's Maximum holds or $(*) + V = L(\mathscr{P}(\omega_1))$ holds, and $T \subset \omega_1$ is stationary, then $\tilde{T} \cap Cof(\omega)$ is stationary.

The first two facts listed here are very easy to prove. The third fact refers to two separate proofs, one using a $\mathbb{P}_{max}$ variation and the other using Shelah's forcing to make the nonstationary ideal saturated. The first of these proofs requires a significant amount of $\mathbb{P}_{max}$ machinery to prove, and since a proof is given in detail in [12], we give just a brief discussion of it here. The second proof is relatively simple if one takes for granted certain facts about revised countable support.

The last fact is proved in [20] so we do not reproduce the proof here. The proof of the fifth is similar to the proof of the sixth, but it follows from the $\mathbb{P}_{max}$ analysis. Here $(*)$ is the statement that $L(\mathbb{R}) \models$ AD and $L(\mathscr{P}(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.

Hidden in this list of facts are two results of independent interest. The last two facts together imply that Martin's Maximum cannot be decomposed into the conjunction of a statment about $L(\mathscr{P}(\omega_1))$ and a reflection principle into $H(\omega_2)$, at least as far as the currently used reflection principles are concerned. The proof of the fourth fact on the list concerns the question of whether certain subsets of $[\omega_2]^{<\omega_1}$ are projective stationary. We give a general lemma (Lemma 3.9.) which shows that the saturation of the nonstationary ideal implies that these sets are projective stationary. From this fact one gets immediately that SRP(ω_2) implies Woodin's axiom ψ_{AC} as well as the fourth fact above for $Cof(\omega_1)$.

Most of this paper is excerpted from the author's dissertation [10].

2. Preliminaries

2.1. Notation

We use the notation C^α_β to denote the set of ordinals less than β of cofinality α. We use Ψ to denote the class of partial orders such that forcing with them does not change the cofinality of ω_2 to ω.

2.2. The nonstationary ideal

For an ordinal γ, we say that $A \subset \gamma$ is club, or closed unbounded, in γ if A contains all its limit points below γ and is cofinal in γ. $A \subset \gamma$ is stationary in γ if it intersects every club subset of γ. Otherwise, A is called nonstationary. In this paper, I_{NS} is used to denote the collection of nonstationary subsets of ω_1.

Our interest in I_{NS} lies primarily in the following operation. Say M is a model of ZFC, or a strong enough fragment, such as ZFC* as defined in Sect. 4.1. Forcing over M with the boolean algebra $(\mathscr{P}(\omega_1)/I_{NS})^M$ gives us an M-ultrafilter on ω_1^M. Using functions from M, then, we can construct an ultrapower embedding j with domain M. We say that I_{NS} is precipitous if the range of the derived j is always well founded. The following two standard definitions describe stronger properties.

Definition 2.1. *I_{NS} is presaturated if for any $A \in \mathscr{P}(\omega_1) \setminus I_{NS}$ and for any sequence $\langle A_i : i < \omega \rangle$ of maximal antichains in $\mathscr{P}(\omega_1) \setminus I_{NS}$ there exists $B \subset A$ stationary such that for each $i < \omega$, $\{X \in A_i \mid X \cap B$ is stationary $\}$ has cardinality at most ω_1.*

The presaturation of I_{NS} implies that for the derived j, $j(\omega_1) = \omega_2$. The following property, saturation, is even stronger.

Definition 2.2. *I_{NS} is saturated if there are no antichains in $\mathscr{P}(\omega_1)/I_{NS}$ of cardinality ω_2.*

One particularly useful hypothesis is that the nonstationary ideal is presaturated plus $\utilde{\delta}^1_2 = \omega_2$. One example of this is the following standard fact.

Theorem 2.3. *Assume that I_{NS} is presaturated and $\utilde{\delta}^1_2 = \omega_2$. Then for every ordinal $\alpha \in \omega_2$, the value of $j(\alpha)$ is independent of the generic, for j the embedding derived from forcing with $\mathscr{P}(\omega_1)/I_{NS}$.*

The following consequence of this hypothesis is shown in [20] and used in [13] and [11]. The uniform indiscernibles are those ordinals which are indiscernibles for every real.

Theorem 2.4. *([20]) Assume that I_{NS} is presaturated and that $\utilde{\delta}^1_2 = \omega_2$. Then the critical sequence of any iteration of length γ by the nonstationary ideal is the set of the first γ uniform indiscernibles as computed in V.*

2.3. $\diamond$

Using a $\diamond$ sequence, we can construct subsets of ω_1 whose tildes are empty. We first note some elementary properties of $\diamond$ related to the tilde function. The following definition and lemma are standard.

Definition 2.5. *For $S \subset \omega_1$, $\diamond(S)$ is the statement that there exists a sequence $\langle \sigma_\alpha \mid \alpha \in S \rangle$ such that for all $E \subset \omega_1$ $\{\alpha \in S \mid \sigma_\alpha = E \cap \alpha\}$ is stationary.*

Lemma 2.6. *Say $\diamond(S)$ holds, for $S \subset \omega_1$. Then there is a partition*

$$\langle S_\beta \mid \beta < \omega_1 \rangle$$

of S into stationary sets such that for each β, $\diamond(S_\beta)$ holds.

Proof. Let $\langle x_\beta \mid \beta < \omega_1 \rangle$ enumerate the reals and let $\langle \sigma_\alpha : \alpha \in S \rangle$ be a $\diamond(S)$-sequence. For $a \subset \omega_1$, let

$$a^* = \{\gamma < \omega_1 \mid \omega + \gamma \in a\}.$$

Let $S_\beta = \{\alpha \mid \sigma_\alpha \cap \omega = x_\beta\}$, and let $\sigma^\beta = \langle \sigma^*_\alpha \mid \alpha \in S_\beta \rangle$. Then σ^β witnesses $\diamond(S_\beta)$. □

The following is the first fact from the list in the introduction.

Lemma 2.7. *If $\diamondsuit$ holds then there exists a stationary, costationary $T \subset \omega_1$ such that $\tilde{T}, \tilde{\bar{T}}$ are empty. Also, there exists a club $C \subset \omega_1$ such that $\tilde{C} = \emptyset$.*

Proof. Let S_0, S_1 be disjoint subsets of S such that $\diamondsuit(S_0), \diamondsuit(S_1)$ hold. Fix a bijection h between $\omega_1 \times \omega_1$ and ω_1 and note that for any wellordering A of ω_1, $h[A \cap \lambda \times \lambda]$ is a subset of λ for a club set of λ. Define a function $f : S_0 \cup S_1 \to \omega_1$ such that if $\sigma_\alpha, \alpha \in S_0$, is on our $\diamondsuit(S_0)$-sequence and $h^{-1}[\sigma_\alpha]$ is a wellordering of α of ordertype $\beta > \alpha$, then $f(\alpha) > \beta$ (and similarly for S_1). Let $C \subset \omega_1$ be club such that for all $\eta \in C$, $\alpha < \eta \Rightarrow f(\alpha) < \eta$. Then $\tilde{C} = \emptyset$, since for any $\alpha < \omega_2$, if $g_\alpha : \omega_1 \to \alpha$ is a bijection, then by $\diamondsuit(S_0)$, for stationarily many $\beta < \omega_1$ $o.t.(g_\alpha[\beta]) \notin C$. Let T be stationary, costationary such that if $\gamma \in C \cap S_0$, then $(\gamma, \gamma^+) \cap T = \emptyset$, and if $\gamma \in C \cap S_1$, then $(\gamma, \gamma^+) \cap \bar{T} = \emptyset$, where γ^+ is the next element of C above γ. Then $\tilde{T}, \tilde{\bar{T}}$ are empty, for the same reason. □

Almost the same argument reproduces the well-known fact that a strengthening of $\diamondsuit$ known as $\diamondsuit^*$ is inconsistent with the presaturation of the nonstationary ideal.

Definition 2.8. *$\diamondsuit^*$ is the statement that there exists a sequence*

$$\langle A_\alpha \in \mathscr{P}_{\omega_1}(\alpha) : \alpha < \omega_1 \rangle$$

such that for all $A \subset \omega_1$, the set $\{\alpha < \omega_1 \mid A \cap \alpha \in A_\alpha\}$ contains a club.

Lemma 2.9. *If $\diamondsuit^*$ holds, then there is a club set $C \subset \omega_1$ such that $\tilde{\tilde{C}} = (\omega_1, \omega_2)$, and so the nonstationary ideal is not presaturated.*

Proof. This is basically the same argument as for Lemma 2.7.. Fix the sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ and let $C \subset \omega_1$ be a club set such that if $\beta \in C$, $\alpha < \beta$, and $\sigma \in A_\alpha$ codes a well ordering of α, then the ordertype of this wellordering is less than β. Then for any $\gamma \in (\omega_1, \omega_2)$, if E is a wellordering of ω_1 of ordertype γ, then a set coding $E \restriction \alpha$ is in A_α for club many α, showing that $\gamma \in \tilde{\tilde{C}}$. Since if j an elementary embedding $j(C)$ must be club in $j(\omega_1)$, it can't be then that $j(\omega_1) = \omega_2$, and so the nonstationary ideal isn't presaturated. □

Note that the following theorem shows that $\diamondsuit$ is consistent with the presaturation of the nonstationary ideal.

Theorem 2.10. *([20]) Suppose that δ is a Woodin cardinal and that*

$$G \subset Coll(\omega_1, < \delta)$$

is V-generic. Then in $V[G]$, I_{NS} is presaturated.

The proof of the following fact is a simple catch-up argument, using the fact that if the nonstationary ideal is saturated, and τ is name for an ordinal less than ω_2, then there is some $\alpha < \omega_2$ such that $1 \Vdash \tau < \check{\alpha}$.

Lemma 2.11. *Say I_{NS} is saturated. Then if $C \subset \omega_1$ is club, then $\tilde{C}$ contains a club in ω_2.*

3. Semi-proper forcing and the tilde function

3.1. SRP

Definition 3.1. *([18]) The strong reflection principle (SRP) is the following: Suppose that $\lambda \geq \omega_2$, $Z \subset \mathscr{P}_{\omega_1}(\lambda)$ and that M is a transitive set such that $M^\lambda \subset M$. Then there exists a continuous, increasing $\in$-chain*

$$\langle X_\alpha \prec M : \alpha < \omega_1 \rangle$$

of elements of $\mathscr{P}_{\omega_1}(M)$ such that for all $\alpha < \omega_1$, if there exists $X \in \mathscr{P}_{\omega_1}(M)$ such that

1. $X_\alpha \subset X$,
2. $X \prec M$,
3. $X \cap \omega_1 = X_\alpha \cap \omega_1$
4. $X \cap \lambda \in Z$,

then $X_\alpha \cap \lambda \in Z$.

SRP was formulated by Todorčević in [18]. It follows from MM and is equivalent to the simpler formulation below. The following defintion is a variation of the one in [6]

Definition 3.2. *([6]) Let A be a set such that $\omega_1 \subset A$. Say $X \subset [A]^\omega$. Then X is projective stationary if for all stationary $S \subset \omega_1$, the projection of X to S,*

$$\{x \in X \mid x \cap \omega_1 \in S\},$$

denoted $X \searrow S$, is stationary.

Definition 3.3. *([6]) Projective Stationary Reflection (PSR) is the statement that for every $\lambda \geq \omega_2$ every projective stationary subset of $[H(\lambda)]^\omega$ contains a continuous increasing length ω_1 $\in$-chain.*

Theorem 3.4. *([6]) SRP $\Leftrightarrow$ PSR.*

All of the known applications of SRP actually follow from the statement that for any $\lambda \geq \omega_2$ and any projective stationary $S \subset [\lambda]^{<\omega_1}$ and any $Y \subset \lambda$ of cardinality ω_1, S contains a continuous increasing sequence of length ω_1 whose union contains Y. This statement is an immediate consequence of SRP, but it is not known whether the two statements are actually equivalent. We denote by SRP(λ) the statement that any projective stationary subset of $[H(\lambda)]^\omega$ contains an increasing length ω_1 $\in$-chain. SRP(ω_2) is a straightforward consequence of MM(c). It is shown in [18] that SRP(ω_2) implies the saturation of the nonstationary ideal, and in [20] that it implies $c = \underset{\sim}{\delta}^1_2 = \omega_2$.

This section analyzes the tilde function in the context of SRP(ω_2). We note first that sequences given by SRP are the natural witnesses to the existence of elements of $\tilde{T}$.

Theorem 3.5. *Say that $T \subset \omega_1$, and $\beta \in [\omega_2, \omega_3)$ is such that the set*

$$A = \{x \in [\beta]^{<\omega_1} \mid o.t.(x) \in T\}$$

is projective stationary. Let $\mathbb{P}$ be the forcing whose conditions are countable continuous increasing sequences from A. Then forcing with $\mathbb{P}$ preserves stationary subsets of ω_1. Further, any sequence of length ω_1 whose initial segments are all elements of $\mathbb{P}$ defines a witness for $\tilde{T} \neq \emptyset$.

Proof. That $\mathbb{P}$ preserves stationary subsets of ω_1 follows from the fact that A is projective stationary, since if $X \prec H(\lambda)$ for sufficiently large λ with $X \cap \beta \in A$, then any X-generic for $\mathbb{P}$ is a condition.

For the second part, let $S = \langle x_\alpha : \alpha < \omega_1\rangle$ be such a sequence. Let $E = \bigcup S$, and for each $\eta \in E$, let $\gamma_\eta = o.t.(E \cap \eta)$. We claim $o.t.(E) \in \tilde{T}$. Let $f : \omega_1 \to E$ be a bijection such that if η appears in an earlier x_α than η', then $f^{-1}(\eta) < f^{-1}(\eta')$. Then the function $g : \omega_1 \to o.t.(E)$ such that $g(\alpha) = \gamma_{f(\alpha)}$ witnesses that $o.t.(E) \in \tilde{T}$, since for every α such that $f[\alpha] = x_\alpha$ (club many), $o.t.(g[\alpha]) = o.t.(x_\alpha) \in T$, by the definition of A. $\square$

Therefore, to keep elements out of $\tilde{T}$ in a model of MM$^{++}(c)$, there need to be protecting sets.

Definition 3.6. *$C \subset [\beta]^{<\omega_1}$, club, is a protecting set for (T, β) if there exists $S \subset \omega_1$ stationary, such that for all $x \in C$,*

$$x \cap \omega_1 \in S \Rightarrow o.t.(x) \notin T$$

We will need the following absoluteness lemma.

Lemma 3.7. *Say M is an inner or transitive set model of ZF, $\alpha < \beta$ are ordinals less than ω_1^M, γ is an ordinal in M, and $F : [\gamma]^{<\omega} \to \gamma$ is a function in M such that*

$$M \models \forall\sigma \in [\gamma]^\beta \text{ closed under } F,\ o.t.(\sigma \cap \omega_1) \neq \alpha$$

Then

$$\forall \sigma \in [\gamma]^{\beta} \textit{ closed under } F,\ o.t.(\sigma \cap \omega_1^M) \neq \alpha$$

Proof. We construct in M a tree of height ω such that any path through the tree would be a counterexample to the statement M satisfies, and such that any counterexample would define a path. Then, since M has no paths through the tree, in M there is a ranking function for the tree, which means that the absence of a path, and therefore a counterexample, is absolute.

Let $\pi : \omega \to \beta$ be 1-1 and onto, and let $\nu : \omega \to [\omega]^{<\omega}$ be an enumeration of the finite sequences from ω such that for all $n \in \omega$, $range(\nu(n)) \subset n$. The tree is the tree of attempts to build $\sigma \in [\gamma]^{\beta}$ closed under F with $o.t.(\sigma \cap \omega_1) = \alpha$. A node is of the form $\langle \eta_0, n_0, ...\eta_i, n_i \rangle$, where the ordinal ordering of $\langle \eta_0..\eta_i \rangle$ must match the ordinal ordering of $\pi(0)..\pi(i)$, $\eta_j < \omega_1 \Leftrightarrow \pi(j) < \alpha$ for each $j \leq i$, and for each $j \leq i$, if $n_j \leq i$ and $\nu(j) = \langle k_0..k_r \rangle$, then $\eta_{n_j} = F(\eta_{k_0}..\eta_{k_r})$. The order is by extension. Then for any path through the tree, the set $\{\eta_i : i \in \omega\}$ has the right ordertypes by its agreement with π, and is closed under F by the conditions on the n_i's. Our assumption on M means that the tree is well founded in M and thus has a ranking function, which means it is absolutely well founded. □

By the next theorem, if any forcing preserving stationary subsets of ω_1 can put an element into the tilde of a set, the forcing from Theorem $3.5.$ can.

Theorem 3.8. *Say $T \subset \omega_1$, and that $\tilde{T}$ is empty. Let $\beta \geq \omega_2$ be such that the set $\{x \in [\beta]^{<\omega_1} \mid o.t.(x) \in T\}$ is not projective stationary. Then there is no outer model preserving stationary subsets of ω_1 in which $\beta \in \tilde{T}$.*

Proof. Let $A = \{x \in [\beta]^{<\omega_1} \mid o.t.(x) \in T\}$, and let $B \subset \omega_1$ be stationary such that $\{x \in A \mid x \cap \omega_1 \in B\}$ is nonstationary. Let $F : [\beta]^{<\omega} \to \beta$ be a function such that for any $\sigma \in [\beta]^{<\omega_1}$ closed under F, if $\sigma \cap \omega_1 \in B$ then $o.t.(\sigma) \notin T$. Let $f : \omega_1 \to \beta$, $C \subset \omega_1$ club form a witness in any outer model that $\beta \in \tilde{T}$. Then there must be some $\alpha \in C \cap B$ such that $o.t.(f[\alpha]) \in T$, $\alpha = f[\alpha] \cap \omega_1$ and $f[\alpha]$ is closed under F. Then by Lemma $3.7.$ we have a contradiction. □

The following lemma shows that if the nonstationary ideal is presaturated and T is a stationary subset of ω_1, then there are no protecting sets for (T, ω_2).

Lemma 3.9. *If the nonstationary ideal is presaturated, then it cannot be that there exist S, T stationary subsets of ω_1, and $C \subset [\omega_2]^{<\omega_1}$ club such that*

$$\forall x \in C\ x \cap \omega_1 \in S \Rightarrow o.t.(x) \notin T.$$

Proof. Suppose the contrary. Let $F : [\omega_2]^{<\omega} \to \omega_2$ be such that all countable subsets of ω_2 closed under F are elements of C. Force with the nonstationary ideal, getting a generic G with $S \in G$, and over $V[G]$ with the nonstationary ideal of $j_G(V)$, getting generic H with $j_G(T) \in H$. Then

$$j_H(j_G(V)) \models \forall \sigma \in [\omega_2]^{\omega_2^V} \text{ closed under } j_H(j_G(F)),\ o.t.(\sigma) \cap \omega_1 \neq \omega_1^V,$$

since $\omega_1^V \in j_H(j_G(S))$ and $\omega_2^V \in j_H(j_G(T))$ (by presaturation). However, $\sigma = j_H\text{"}j_G\text{"}\omega_2^V$ is closed under $j_H(j_G(F))$, $\sigma \cap \omega_1^{j_H(j_G(V))} = \omega_1^V$ and $o.t.(\sigma) = \omega_2^V$, contradicting Lemma 3.7.. □

Another use of the tilde function, or a similar concept, is to give definable well-orderings of $\mathscr{P}(\omega_1)$. The following statement is the key means for proving AC in extensions by $\mathbb{P}_{max}$ variations.

Definition 3.10. *([20]) ψ_{AC} is the assertion : Suppose $A \subset \omega_1$ and $B \subset \omega_1$ are stationary, costationary sets. Then there exist $\alpha < \omega_2$, bijection $\pi : \omega_1 \to \alpha$, and a club set $C \subset \omega_1$ such that $\{\eta < \omega_1 \mid ordertype(\pi[\eta]) \in B\} \cap C = A \cap C$.*

Lemma 3.11. *([20]) Suppose that ψ_{AC} holds. Then*

$$2^\omega = 2^{\omega_1} = \omega_2.$$

The following corollary of Lemma 3.9. was pointed out by Hugh Woodin.

Theorem 3.12. *If SRP(ω_2) then ψ_{AC}.*

It is an instance of following corollary of Lemma 3.9..

Theorem 3.13. *(SRP(ω_2)) Let $\gamma \leq \omega_1$. Let $\langle A_\alpha : \alpha < \gamma\rangle$ be a collection of stationary subsets of ω_1, and let $\langle B_\alpha : \alpha < \gamma\rangle$ be a maximal disjoint antichain in $\mathscr{P}(\omega_1) \setminus I_{NS}$. Then $\{\eta \in C^{\omega_1}_{\omega_2} \mid \forall \alpha < \gamma\ B_\alpha \Vdash \eta \in j(A_\alpha)\}$ is stationary.*

Proof. Fix $E \subset \omega_2$ club. $\{x \in [\omega_2]^{<\omega_1} \mid sup(x) \in E\}$ is a club set. Since the intersection of a club set and a projective stationary set is projective stationary, by applying SRP(ω_2) we will be done if we see that

$$D = \{x \in [\omega_2]^{<\omega_1} \mid \forall \alpha < \gamma\ x \cap \omega_1 \in B_\alpha \Rightarrow o.t.(x) \in A_\alpha\}$$

is projective stationary. Let $S \subset \omega_1$ be stationary. Then for some $\alpha < \gamma$, $B_\alpha \cap S$ is stationary, and by Lemma 3.9.,

$$\{x \in [\omega_2]^{<\omega_1} \mid x \cap \omega_1 \in B_\alpha \cap S \wedge o.t.(x) \in A_\alpha\} \subset D \searrow S$$

is stationary. □

The following lemma shows that if I_{NS} is presaturated, ι is the ω-th uniform indiscernible and $\underset{\sim}{\delta}^1_2 = \omega_2$, then for no stationary $S, T \subset \omega_1$ is S a protecting set for (T, ι).

Lemma 3.14. [1] *Say that the nonstationary ideal is presaturated and $\underset{\sim}{\delta}^1_2 = \omega_2$, and let ι be the ω-th uniform indiscernible. Let $S, T \subset \omega_1$ be stationary. Then there is no $C \subset [\iota]^{<\omega_1}$ club, such that*

$$\forall x \in C \; x \cap \omega_1 \in S \Rightarrow o.t.(x) \notin T.$$

Proof. Suppose the contrary. Let $F : [\iota]^{<\omega} \to \iota$ be such that any countable subset of ι closed under F is in C. Force $\omega + 1$ times to get an iteration of V of length $\omega + 1$ by the nonstationary ideal, letting S be in the generic at the first stage, and T be in the generic at the last stage. Let j be the embedding induced by this iteration. We have then that $\omega_1^V \in j(S)$ and $\iota \in j(T)$, since by Lemma $2.4.$, ι is the critical point of the last step of the embedding. Further, $\sigma = j``\iota$ is closed under $j(F)$, contradicting Lemma $3.7.$. □

These give us the following facts.

Theorem 3.15. *Assume SRP(ω_2). Let $T \subset \omega_1$ be stationary. Then:*

1. *$\tilde{T} \cap C^{\omega_1}_{\omega_2}$ is stationary;*
2. *$\tilde{T} \cap C^{\omega}_{\omega_2}$ is unbounded in ω_2.*

Proof. Note that SRP(ω_2) implies $\underset{\sim}{\delta}^1_2 = \omega_2$ and I_{NS} saturated (by [20], [18]). Then the first conclusion is an instance of Theorem $3.13.$ (with $\gamma = 1$, $A_0 = T$ and $B_0 = \omega_1$). For the second, Theorem $3.5.$ and Lemma $3.14.$ show that $\tilde{T} \cap C^{\omega}_{\omega_2}$ is nonempty. But given any stationary $T \subset \omega_1$ and any $\gamma < \omega_2$, we can find stationary $T' \subset T$ such that $\tilde{T}' \cap \gamma$ is empty: let $h : \omega_1 \to \gamma$ be a bijection and let $T' \subset T$ be such that for a club of $\beta < \omega_1$, $T' \cap (\beta, o.t.(h[\beta]))$ is empty. Then since $\tilde{T}'$ is nonempty, $\tilde{T}$ must have an element above γ. □

3.2. MM

Definition 3.16. *([7]) Martin's Maximum (MM) is the following statement: Say $\mathbb{P}$ is a forcing which preserves stationary subsets of ω_1, and*

$$\langle D_\alpha : \alpha < \omega \rangle$$

is a collection of dense subsets of $\mathbb{P}$. Then there exists a filter $G \subset \mathbb{P}$ meeting all the D_α's.

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[1] Woodin presents a similar argument in Sect. 10.3 of [20], but without the assumption that $\underset{\sim}{\delta}^1_2 = \omega_2$. The point is that one needs to know in advance $\omega + 1$ many members of the critical sequence. Here we have used the uniform indiscernibles, but this can be done by cardinality considerations as well.

Martin's Maximum$^+$ *(MM*$^+$*) is MM with the added stipulation that if* τ *is a* $\mathbb{P}$*-name for a stationary subset of* ω_1*, then*

$$\{\alpha \in \omega_1 \mid \exists p \in G\ p \Vdash \check{\alpha} \in \tau\}$$

is stationary. MM^{++} *allows for an* ω_1*-sequence of names for stationary sets.*

MM(c*), MM*$^+$*(*c*) and MM*$^{++}$*(*c*) are the restrictions of MM, MM*$^+$ *and MM*$^{++}$ *to the case* $|\mathbb{P}| \leq c$*.*

MM and its consequences are presented in detail in [7]. In that paper a forcing construction is presented which when applied to a model with a supercompact cardinal creates a model of MM^{++}. Currently, this is the only known way of achieving MM. The relationship between MM and $(*)$ is one of the main open questions in the study of $\mathbb{P}_{max}$. It is shown in [20] that MM^{++}(c) and $(*)$ are independent, and that each implies that $\underset{\sim}{\delta}^1_2 = \omega_2$. It is shown in [11], by an application of the tilde function, that MM$^+$ does not imply $(*)$.

Theorem 3.17. *([20]) Assume Martin's Maximum. Suppose* $S \subset \omega_1$ *is stationary. Then*

$$\{\alpha \mid \alpha \in \tilde{S} \cap C^{\omega}_{\omega_2}\}$$

is stationary in ω_2*.*

We will contrast the above theorem with a couple of facts showing that its conclusion does not follow from MM(Ψ) (i.e., MM for the class of partial orderings which preserve stationary subsets of ω_1 and do not change the cofinality of ω_2 to ω). We use a strengthening of SRP which follows from MM(Ψ).

Lemma 3.18. *Assume MM(*Ψ*). Let* $\lambda \geq \omega_2$ *and* $S \subset [\lambda]^{<\omega_1}$ *be projective stationary. Let* $\mathbb{P} \in H(\lambda)$ *be a forcing such that for any countable* $X \prec H(\lambda)$*, if* $X \cap \lambda \in S$*, then any* X*-generic for* $\mathbb{P}$ *can be extended to a condition in* $\mathbb{P}$*. Then for any set* $a \in H(\lambda)$ *there exist a continuous increasing* $\in$*-chain* $\langle X_\alpha \prec H(\lambda) : \alpha < \omega_1\rangle$ *such that* $a \in X_0$ *and each* $X_\alpha \cap \lambda \in S$*, and a sequence* $\langle p_\alpha \in \mathbb{P} : \alpha < \omega_1\rangle$ *such that for all* $\alpha < \beta < \omega_1$ $p_\beta < p_\alpha$*,* $p_\alpha \in X_{\alpha+1}$*, and* $X_\alpha \cap \{q \in \mathbb{P} : q \geq p_\alpha\}$ *is an* X_α*-generic filter for* $\mathbb{P}$*.*

Proof. We just need to see that the forcing to create such a pair of sequences preserves stationary subsets of ω_1. To do this, let τ be a name for a club subset of ω_1 under this forcing, and let $E \subset \omega_1$ be any stationary set. Then pick $X \prec H(\lambda)$ with $\tau, \mathbb{P} \in X$, and $X \cap \lambda \in S \searrow E$. Let $\beta = X \cap \omega_1$. We then note that any X-generic $\langle(X_\alpha, p_\alpha) : \alpha < \beta\rangle$ for our forcing can be extended to the condition $\langle(X_\alpha, p_\alpha) : \alpha < \beta + 1\rangle$ where $X_\beta = X$ and p is a condition in $\mathbb{P}$ extending the X-generic filter for $\mathbb{P}$ generated by the sequence of p_α's. That this filter is generic follows from the fact that each predense subset of $\mathbb{P}$ in X is in some X_α, and by the fact that each p_α generates an X_α-generic. □

Theorem 3.19. *Assume MM(Ψ). Then forcing to shoot an ω-club though a stationary subset of $C^\omega_{\omega_2}$ adds no subsets to ω_1 and preserves MM(Ψ).*

Proof. Let $\mathbb{D}$ denote the forcing to shoot an ω-club through a stationary set $A \subset C^\omega_{\omega_2}$. To see that $\mathbb{D}$ adds no subsets of ω_1, we apply Lemma 3.18.. That is, given a $\mathbb{D}$-name σ for a subset of ω_1, we can find a continuous increasing $\in$-chain $\langle X_\alpha : \alpha < \omega_1 \rangle$ of elementary submodels of some sufficiently large $H(\lambda)$ with $\sigma \in X_0$ and such that for each α, $sup(X_\alpha \cap \omega_2) \in A$, plus a sequence of generics for each X_α extending one another. The union of this sequence will be a condition in $\mathbb{D}$ deciding all of σ.

Let τ be a $\mathbb{D}$-name for a forcing which preserves stationary subsets of ω_1 and doesn't change the cofinality of ω_2 to ω. We may assume that forcing with $\mathbb{D} * \tau$ makes ω_2 have cofinality ω_1, since if necessary we can tack $Coll(\omega_1, \omega_2)$ onto τ. Then given names $\langle \rho_\alpha : \alpha < \omega_1 \rangle$ for dense sets in τ, we can find a filter in $\mathbb{D} * \tau$ which meets all the ρ_α (suitably generalized). Further, we can guarantee that the restriction of this filter to $\mathbb{D}$ has cofinality ω_1, since the forcing makes the cofinality of ω_2 equal ω_1. Therefore, the union of the restriction of the filter to $\mathbb{D}$ is a condition in $\mathbb{D}$ forcing the existence of the appropriate filter witnessing the desired instance of MM(Ψ) in the $\mathbb{D}$-extension. □

We have the following corollary. Qi Feng's Cofinal Branch Principle [4] is an immediate consequence of $MM(\Psi)$.

Corollary 3.20. *Let ϕ be a statement which is preserved by (ω_1, ∞)-distributive forcing. Then if ZFC + MM + ϕ is consistent, then so is ZFC + MM(Ψ) + ϕ + $\neg$MM.*

The forcing for which MM fails in this extension is Namba forcing followed by the forcing to put the old ω_2 into the tilde of the set the stationarity of whose tilde on $Cof(\omega)$ we have just killed. It seems unlikely that MM holds for Namba forcing itself in this extension, but this is not known.

The following forcing is designed to kill the stationarity of $\tilde{T} \cap C^\omega_{\omega_2}$ for every costationary $T \subset \omega_1$.

Definition 3.21. *$\mathbb{C}$ is the set of (c, A) such that*

1. *$c \subset \omega_2$ is closed and bounded,*
2. *A is a set of costationary subsets of ω_1,*
3. *$|A| \leq \omega_1$.*

The order on $\mathbb{C}$ *is as follows:*

$$(c, A) \leq (d, B) \Leftrightarrow B \subset A, c \text{ end-extends } d, \text{ and}$$
$$\forall S \in B\ \tilde{S} \cap (c \setminus d) \cap C^{\omega}_{\omega_2} = \emptyset.$$

The following theorems show that we can strengthen Theorem 3.19. using MM. The proof of the first part of the first theorem is just like the proof of Theorem 3.19. The proof of the second part is straight from the definition of $\mathbb{C}$.

Theorem 3.22. *Assume MM and that* $\mathbb{C}$ *adds no subsets of* ω_1*. Then forcing with* $\mathbb{C}$ *preserves MM(*Ψ*) and forces that for all costationary* $T \subset \omega_1$*,* $\tilde{T} \cap C^{\omega}_{\omega_2}$ *is nonstationary.*

Theorem 3.23. *Assume Martin's Maximum. Then the forcing* $\mathbb{C}$ *does not add any* ω_1*-sequences of ordinals.*

Proof. Fix τ, a $\mathbb{C}$-name for an ω_1^V-sequence of ordinals. The theorem follows from the following claim.

Suppose that MM holds and that $\langle T_\beta : \beta < \omega_2 \rangle$ *is an enumeration of the costationary subsets of* ω_1*. Then there exist stationarily many* $\beta \in C^{\omega}_{\omega_2}$ *such that for all* $\beta' < \beta$*,* $\beta \notin \tilde{T}_{\beta'}$*.*

Given this claim, we have that the set

$$\{X \prec H(\omega_2) \mid \text{ for all costationary } T \in \mathscr{P}(\omega_1)\ sup(X \cap \omega_2) \notin \tilde{T}\}$$

is projective stationary. Applying Lemma 3.18., let $\langle X_\alpha : \alpha < \omega_1 \rangle$ be a continuous increasing $\in$-chain contained in this set, with $\tau \in X_0$, and let $g \in \mathbb{C}$ such that for all $\alpha < \omega_1$, $g \restriction X_\alpha$ (abusing notation) is X_α-generic for $\mathbb{C}$. g then decides all of τ.

We now prove the claim. Let $\langle T_\beta : \beta < \omega_2 \rangle$ be as in the statement of the claim, and let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a disjoint maximal antichain in $\mathscr{P}(\omega_1) \setminus I_{NS}$.

We use the fact, shown in [20], that if S is a stationary subset of ω_1 and T is a costationary subset of ω_1, both in V, then after Namba forcing, the set

$$\{x \in [\omega_2^V]^{<\omega_1} \mid x \cap \omega_1 \in S \text{ and } o.t.(x) \notin T\}$$

is stationary.

Now consider the forcing $\mathbb{P}$ which is composed of first forcing with Namba forcing, and then, given a bijection $h : \omega_1 \to \omega_2^V$ added by Namba forcing, shooting a continuous increasing ω_1-sequence through the set

$$\{x \in [\omega_2^V]^{<\omega_1} \mid x \cap \omega_1 \in S_\alpha \Rightarrow o.t.(x) \notin T_{h(\alpha)}\}.$$

Since each stationary $S \subset \omega_1$ from V has stationary intersection with some S_α, this product preserves stationary subsets of ω_1.

Now for any club subset $D \subset \omega_2$, pick dense sets in $\mathbb{P}$ so that for any filter contained in $\mathbb{P}$ meeting all of our dense sets, the name for h is realized as a bijection between ω_1 and some $\beta \in D$ of cofinality ω, and the continous increasing ω_1 sequence through the above set (as a subset of $[\beta]^{<\omega_1}$) is defined. Then, given such a filter, the induced β is as desired. □

Note that this result requires MM and not just MM(Ψ). This can be seen by noting that the proof of Theorem 3.19. also allows us to shoot an ω-club through the tilde of some stationary set, preserving MM(Ψ). Forcing with $\mathbb{C}$ over this extension collapses ω_2.

3.3. Additional remarks

By contrast with the above remarks on Martin's Maximum and SRP, we should note that the Proper Forcing Axiom does not imply that $\tilde{T}$ is nonempty for each stationary $T \subset \omega_1$. To see this, note that if $T \subset \omega_1$ is added by forcing with initial segments, then for all ordinals $\alpha \geq \omega_1$, the set $\{x \in [\alpha]^{<\omega_1} \mid o.t.(\alpha) \notin T\}$ is stationary, and so $\tilde{T} = \emptyset$. Therefore, given a supercompact cardinal, one can first force the existence of such a T, then properly force PFA, leaving all these sets stationary, and thus $\tilde{T}$ empty.

Further, given a Woodin cardinal, if one forces by initial segments to create $T \subset \omega_1$ and then does Shelah's forcing [16] to make the nonstationary ideal saturated, $\tilde{T}$ remains empty in the final extension. This follows from the fact that Shelah's forcing is homogeneous and definable from an ordinal parameter (the Woodin cardinal), and the following lemma.

Lemma 3.24. *Let $\mathbb{T}$ be the forcing which adds a subset T of ω_1 by initial segments, and let τ be a $\mathbb{T}$-name for a homogeneous forcing which is definable in $V[T]$ by the formula ϕ from a parameter X in V. Then in the extension of $V[T]$ by τ_T, $\tilde{T} = \emptyset$.*

Proof. Let α be an ordinal and $t \in \mathbb{T}$ be a condition such that

$$t \Vdash \text{“the forcing defined by } \phi \text{ with parameter } X \text{ puts } \alpha \text{ into } \tilde{T}\text{”}.$$

Let T_1 be V-generic extending t, and let $T_2 = (\bar{T}_1 \setminus sup(t)) \cup t$. Note that T_1 and T_2 are both V-generic for $\mathbb{T}$, that both extend t, and that $V[T_1] = V[T_2]$. Since τ is a name for the set defined from X by ϕ, $\tau_{T_1} = \tau_{T_2}$. But then

$$V[T_1] \models \tau_{T_1} \text{ puts } \alpha \text{ into } \tilde{T}_1 \cap \tilde{T}_2,$$

which is impossible since $T_1 \cap T_2$ is bounded, and so $\tilde{T}_1 \cap \tilde{T}_2$ must be empty. □

The same proof shows that $\tilde{\tilde{T}}$ be will empty in this extension also. Along with the last section of this paper, then, we see that models of ZFC in which I_{NS} is saturated and there exists a stationary, costationary $T \subset \omega_1$ with $\tilde{T}, \tilde{\tilde{T}}$ both empty can be created with $\mathbb{P}_{max}$ or by semi-proper forcing.

4. $\mathbb{P}_{max}$

In this section we present two $\mathbb{P}_{max}$ variations which reproduce results from the previous section. The first shows that MM$^{++}(c) + (*)$ is consistent with the statement that for all costationary $T \subset \omega_1$, $\tilde{T} \cap C^{\omega}_{\omega_2}$ is nonstationary. The second shows that ZFC + $c = \omega_2$ + "I_{NS} is saturated" is consistent with the existence of a stationary, costationary $T \subset \omega_1$ such that $\tilde{T}, \tilde{\tilde{T}}$ are both empty.

4.1. Iterable structures

The following weakening of ZFC is used in the definition of $\mathbb{P}_{max}$, though in the context of AD$^{L(\mathbb{R})}$ ZFC could be used instead. We include the definition for the sake of completeness.

Definition 4.1. *([20]) A transitive set M models ZFC* if the following hold.*

1. *M is closed under the Gödel operations.*
2. *If $R \subset M^{<\omega_1^M}$ is nonempty and definable in M with parameters from M such that for all $f \in R$ and $\alpha < dom(f)$ $f\restriction\alpha \in R$, then there exist $\alpha \leq \omega_1^M$ and a function*

$$f : \alpha \to M$$

   such that
   (a) $f \in M \setminus R$,
   (b) for all $\beta < \alpha$ $f\restriction\beta \in R$,
   (c) if $\alpha = \gamma + 1$, $g \in R$, and $f\restriction\gamma \subset g$, then $f\restriction\gamma = g$.

If M models ZFC*, then the appropriate version of Łos' Theorem holds and the embedding derived from forcing with the nonstationary ideal over M is elementary.

The following is the definition of iterability for sequences of models, the form which the $\mathbb{P}_{max}$ variations in this paper take.

Definition 4.2. *([20]) Suppose $\langle N_k : k < \omega \rangle$ is a countable sequence such that for each k, N_k is a countable transitive model of ZFC* and such that for all k, $N_k \in N_{k+1}$ and $\omega_1^{N_k} = \omega_1^{N_{k+1}}$. An iteration of $\langle N_k : k < \omega \rangle$ is a sequence*

$$\langle\langle N_k^\beta : k < \omega\rangle, G_\alpha, j_{\alpha,\beta} : \alpha < \beta < \gamma\rangle$$

such that for all $\alpha < \beta < \gamma$ the following hold.

1. *$j_{\alpha,\beta} : \cup\{N_k^\alpha \mid k < \omega\} \to \cup\{N_k^\beta \mid k < \omega\}$ is a commuting family of Σ_0 elementary embeddings.*
2. *For all $k < \omega$, $G_\beta \cap N_k^\beta$ is an N_k^β-normal ultrafilter on $(\mathscr{P}(\omega_1))^{N_k^\beta}$.*
3. *If $\beta + 1 < \gamma$ then $N_k^{\beta+1}$ is the $\cup\{N_k^\beta \mid k < \omega\}$-ultrapower of N_k^β by G_β and $j_{\beta,\beta+1} : \cup\{N_k^\beta \mid k < \omega\} \to \cup\{N_k^{\beta+1} \mid k < \omega\}$ is the induced Σ_0 elementary embedding.*
4. *For each $\beta < \gamma$ if β is a limit ordinal then for every $k < \omega$, N_k^β is the direct limit of $\{N_k^\alpha \mid \alpha < \beta\}$ and for all $\alpha < \beta$, $j_{\alpha,\beta}$ is the induced Σ_0 elementary embedding.*

If γ is a limit ordinal then γ is the length of the iteration, otherwise the length of the iteration is δ where $\delta + 1 = \gamma$.

A sequence $\langle N_k^* : k < \omega\rangle$ is an iterate of $\langle N_k : k < \omega\rangle$ if it occurs in an iteration of $\langle N_k : k < \omega\rangle$.

The sequence $\langle N_k : k < \omega\rangle$ is iterable if every iterate of it is well founded.

If $B \subset \mathbb{R}$, then $\langle N_k : k < \omega\rangle$ is B-iterable if it is iterable, and if for every iterate $\langle N_k^* : k < \omega\rangle$ of $\langle N_k : k < \omega\rangle$, $j(B \cap N_0) = B \cap N_0^*$, where j is the induced embedding and $j(B \cap N_0)$ is defined to be $\cup\{j(a) : a \in N_0$ and $a \subset B\}$.

The following lemma is the main tool for verifying that the sequences as above are iterable.

Lemma 4.3. *([20]) Suppose*

$$\langle N_k : k < \omega\rangle$$

is a countable sequence such that for each k, N_k is a countable transitive model of ZFC* and such that for all k,

$$N_k \in N_{k+1}$$

and

$$(\omega_1)^{N_k} = (\omega_1)^{N_{k+1}}.$$

Suppose that for all $k < \omega$

(i) *if $C \in N_k$ is closed and unbounded in $\omega_1^{N_0}$, then there exists $D \in N_{k+1}$ such that $D \subset C$, D is closed and unbounded in C, and*

$$D \in L[x]$$

for some $x \in \mathbb{R} \cap N_{k+1}$.

(ii) *for all* $x \in \mathbb{R} \cap N_k$, $x^\# \in N_{k+1}$.
(iii) *for all* $k < \omega$,

$$|N_k|^{N_{k+1}} = \omega_1^{N_0}.$$

Then the sequence $\langle N_k : k < \omega \rangle$ *is iterable.*

We quote a lemma from [20] showing that under certain circumstances the ultrafilter needed to iterate a given sequence exists.

Lemma 4.4. *([20]) Suppose that*

$$\langle N_k : k < \omega \rangle$$

is a sequence of countable transitive sets such that for all $k < \omega$, $N_k \in N_{k+1}$,

$$N_k \models ZFC^*,$$

and

$$N_k \cap (I_{NS})^{N_{k+1}} = N_{k+1} \cap (I_{NS})^{N_{k+2}}.$$

Suppose that $k \in \omega$ *and that*

$$a \in (\mathscr{P}(\omega_1))^{N_k} \setminus (I_{NS})^{N_{k+1}}.$$

Then there exists

$$G \subset \cup\{(\mathscr{P}(\omega_1))^{N_i} \mid i < \omega\}$$

such that $a \in G$ *and such that for all* $i < \omega$, $G \cap N_i$ *is a uniform* N_i*-normal ultrafilter.*

The sequences of models in $\mathbb{P}_{max}$ variations satisfy a variation of ψ_{AC}.

Definition 4.5. *([20])* ψ^*_{AC}*: Suppose that* $\langle S_\alpha : \alpha < \omega_1 \rangle$ *and* $\langle T_\alpha : \alpha < \omega_1 \rangle$ *are each sequences of stationary, costationary sets. Then there exists a sequence* $\langle \delta_\alpha : \alpha < \omega_1 \rangle$ *of ordinals less than* ω_2 *such that for each* $\alpha < \omega_1$ *there exists a bijection*

$$\pi : \omega_1 \to \delta_\alpha,$$

and a closed unbounded set $C \subset \omega_1$ *such that*

$$\{\eta < \omega_1 \mid\ o.t.(\pi[\eta]) \in T_\alpha\} \cap C = S_\alpha \cap C.$$

The reason for this variation is that our conditions are sequences of models, and iterates of sequences modeling ψ^*_{AC} model ψ^*_{AC}. This isn't so for ψ_{AC}.

4.2. The $\mathbb{P}^*_{max}$ extension of $L(\Gamma, \mathbb{R})$

In this section, we prove the following theorem, where $\mathbb{P}^*_{max}$ is a reformulation of $\mathbb{P}_{max}$ and $\mathbb{C}$ is the forcing defined in Definition 3.21..

Theorem 4.6. *Suppose that $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass closed under continuous preimages such that*

$$L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \text{“}\Theta \textit{ is regular.”}$$

Suppose $G \subset \mathbb{P}^*_{max}$ is $L(\Gamma, \mathbb{R})$-generic. Suppose $F \subset \mathbb{C}^{L(\Gamma,\mathbb{R})[G]}$ is $L(\Gamma, \mathbb{R})[G]$-generic. Suppose

$$H \subset Coll(\omega_3, H(\omega_3))^{L(\Gamma,\mathbb{R})[G][F]}$$

is $L(\Gamma, \mathbb{R})[G][F]$-generic. Then

$$L(\Gamma, \mathbb{R})[G][F][H] \models ZFC + MM^{++}(c) + (*).$$

Furthermore, in $L(\Gamma, \mathbb{R})[G][F][H]$, for every costationary $T \subset \omega_1$, $\tilde{T} \cap C^{\omega}_{\omega_2}$ is nonstationary.

Recall that axiom $(*)$ is the statement that $L(\mathbb{R}) \models$ AD and $L(\mathscr{P}(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$. It is shown in [20] that if $(*) + V = L(\mathscr{P}(\omega_1))$ holds, then for every stationary $T \subset \omega_1$, $\tilde{T} \cap C^{\omega}_{\omega_2}$ is stationary. A corollary of the arguments in this section is that if $(*) + V = L(\mathscr{P}(\omega_1))$ holds, then the forcing $\mathbb{C}$ adds no subsets of ω_1. We note that the assumption $V = L(\mathscr{P}(\omega_1))$ is essential, since it is possible to have models of $(*)$ in which there exists a stationary subset of ω_1 whose tilde contains an ω-club (just force such a club over a model of $(*) + V = L(\mathscr{P}(\omega_1))$).

The following is established in [20].

Theorem 4.7. *([20]) Suppose $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass closed under continuous preimages such that*

$$L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \text{“}\Theta \textit{ is regular.”}$$

Suppose $G \subset \mathbb{P}^*_{max}$ is $L(\Gamma, \mathbb{R})$-generic. Then

$$L(\Gamma, \mathbb{R})[G] \models \omega_2\text{-}DC + MM^{++}(c) + (*).$$

What remains to be shown is that forcing with $\mathbb{C}$ over $L(\Gamma, \mathbb{R})[G]$ as above adds no new ω_1-sequences, preserves ω_2-DC and MM$^{++}(c)$, and destroys the stationarity of $\tilde{T} \cap C^{\omega}_{\omega_2}$ for every costationary $T \subset \omega_1$. However, forcing always preserves ω_2-DC, and $\mathbb{C}$ clearly destroys stationarities as desired, provided that is doesn't collapse ω_2, so our task is even easier.

We need the following theorem.

Theorem 4.8. *([14]) If κ is a successor cardinal and $\mathbb{P}$ is a partial order preserving κ^+, then for all generic $G \subset \mathbb{P}$,*

$$V[G] \models cof(|\kappa|) = cof(\kappa).$$

Therefore, no forcing of cardinality ω_2 preserving ω_1 can change the cofinality of ω_2 to ω. This gives us the following fact. The proof is just like the proof of Theorem 3.19., with an extra step to account for the names for stationary subsets.

Theorem 4.9. *Assume that $\mathbb{C}$ adds no subsets of ω_1. Then forcing with $\mathbb{C}$ preserves MM$^{++}(c)$.*

Proof. Note first that MM$^{++}(c)$ implies that $c = |\mathscr{P}(\omega_1)| = \omega_2$. Let τ be a $\mathbb{C}$-name for a forcing of cardinality ω_2 or less which preserves stationary subsets of ω_1. We may assume that forcing with $\mathbb{C} * \tau$ makes ω_2 have cofinality ω_1, since by Theorem 4.8. it cannot make it have cofinality ω, and since if necessary we can tack $Coll(\omega_1, \omega_2)$ onto τ. Then given names $\langle \rho_\alpha : \alpha < \omega_1 \rangle$ for dense sets in τ and $\mathbb{C}$-names $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$ for τ-names for stationary subsets of ω_1, we can find by MM$^{++}(c)$ a filter $G \subset \mathbb{C} * \tau$ which meets all the ρ_α (generalized to subsets of $\mathbb{C} * \tau$) and realizes each σ_α as a stationary set. Further, we can guarantee that the restriction of G to $\mathbb{C}$ has cofinality ω_1, since the forcing makes the cofinality of ω_2 equal ω_1. Since $\mathbb{C}$ is ω_1-closed, the union of the restriction of G to $\mathbb{C}$ is a condition in $\mathbb{C}$ forcing the existence of the appropriate filter (the restriction of G to τ) witnessing the desired instance of MM$^{++}(c)$ in the $\mathbb{C}$-extension. □

What is left, then, is to show that forcing with $\mathbb{C}$ over $L(\Gamma, \mathbb{R})[G]$ as above adds no ω_1-sequences. Showing this requires some analysis of $\mathbb{P}^*_{max}$.

Definition 4.10. *([20]) $\mathbb{P}^*_{max}$ is the set of pairs $(\langle M_k : k < \omega \rangle, a)$ such that the following hold.*

1. *$a \in M_0, a \subset \omega_1^{M_0}$, and $\omega_1^{M_0} = \omega_1^{L[a,x]}$ for some $x \in \mathbb{R} \cap M_0$.*
2. *Each M_k is a countable transitive model of ZFC*.*
3. *$M_k \in M_{k+1}, \omega_1^{M_k} = \omega_1^{M_{k+1}}$.*
4. *$(I_{NS})^{M_{k+1}} \cap M_k = (I_{NS})^{M_{k+2}} \cap M_k$.*
5. *$\cup \{M_k : k < \omega\} \models \psi^*_{AC}$.*
6. *$\langle M_k \mid k < \omega \rangle$ is iterable.*
7. *$\exists X \in M_0$ such that $X \subset \mathscr{P}(\omega_1)^{M_0} \setminus I_{NS}^{M_1}$, such that $M_0 \models$ "$|X| = \omega_1$," and such that for all $A, B \in X$, if $A \neq B$ then $A \cap B \in I_{NS}^{M_0}$.*

The ordering on $\mathbb{P}^*_{max}$ is as follows.

$$(\langle N_k : k < \omega \rangle, b) < (\langle M_k : k < \omega \rangle, a)$$

if $\langle M_k : k < \omega \rangle \in N_0, \langle M_k : k < \omega \rangle$ *is hereditarily countable in* N_0 *and there exists an iteration*

$$j : \langle M_k : k < \omega \rangle \to \langle M_k^* : k < \omega \rangle$$

such that:

1. $j(a) = b$,
2. $\langle M_k^* : k < \omega \rangle \in N_0$ *and* $j \in N_0$,
3. $j(I_{NS}^{M_{k+1}}) \cap M_k^* = (I_{NS})^{N_1} \cap M_k^*$ *for all* $k < \omega$,

The following lemma follows from the fact that $\mathbb{P}^*_{max}$ conditions model ψ^*_{AC}. The analogous lemma holds in the other $\mathbb{P}_{max}$ variations whose conditions are sequences of models.

Lemma 4.11. *([20]) Suppose that* $(\langle M_k \mid k < \omega \rangle, a) \in \mathbb{P}^*_{max}$. *Suppose that*

$$j_1 : \langle M_k \mid k < \omega \rangle \to \langle M_k^1 \mid k < \omega \rangle$$

and

$$j_2 : \langle M_k \mid k < \omega \rangle \to \langle M_k^2 \mid k < \omega \rangle$$

are well founded iterations such that $j_1(a) = j_2(a)$.

Then

$$\langle M_k^1 \mid k < \omega \rangle = \langle M_k^2 \mid k < \omega \rangle$$

and $j_1 = j_2$.

Since the order on $\mathbb{P}^*_{max}$ is determined by the existence of elementary embeddings, each condition $(\langle M_k : k < \omega \rangle, a)$ in the generic is iterated ω_1 times through the conditions below it in the generic. In fact, by Lemma 4.11., each $(\langle M_k : k < \omega \rangle, a)$ in the generic is uniquely iterated into the extension to a structure $\langle \langle M_k^* : k < \omega \rangle, a_G \rangle$, where

$$a_G = \cup \{a \mid \exists (\langle M_k : k < \omega \rangle, a) \in G\},$$

for generic G. The following definitions (all but the last from [20]) apply to all $\mathbb{P}_{max}$ variations.

Definition 4.12. *A filter* $G \subset \mathbb{P}^*_{max}$ *is semi-generic if for all* $\alpha < \omega_1$ *there exists a condition* $\langle M_k : k < \omega_1 \rangle \in G$ *such that* $\alpha < \omega_1^{M_0}$.

$A_G = \cup \{a \mid \exists (\langle M_k : k < \omega \rangle, a) \in G\}$.

$$\mathscr{P}(\omega_1)_G = \bigcup \{\mathscr{P}(\omega_1)^{M_0^*} \mid (\langle M_k : k < \omega \rangle, a) \in G\},$$

and

$$I_G = \cup \{I_{NS}^{M_1^*} \cap M_0^* \mid (\langle M_k : k < \omega \rangle, a) \in G\},$$

where for $(\langle M_k : k < \omega \rangle, a) \in G$, $\langle M_k^* : k < \omega \rangle$ *is the iterate of* $\langle M_k : k < \omega \rangle$ *by the unique iteration of* $\langle M_k : k < \omega \rangle$ *that sends* a *to* A_G.

Also, for $(\langle M_k : k < \omega \rangle, a) \in \mathbb{P}^*_{max}$ *and* $x \in M_k$, x^* *is the image of* x *under the unique iteration of* $\langle M_k : k < \omega \rangle$ *sending* a *to* A_G.

We will need to use another form of determinacy, AD^+. AD^+ implies AD trivially, but the reverse implication is open. The only fact about AD^+ that we will be using is that it follows from $\mathrm{AD}_{\mathbb{R}}$.

Definition 4.13. *([20]) Suppose $A \subset \mathbb{R}$. The set A is $^\infty$-Borel if there exists a set S of ordinals and a Σ_1 formula $\phi(x_0, x_1)$ such that*

$$A = \{y \in \mathbb{R} \mid L[S, y] \models \phi[S, y]\}.$$

Definition 4.14. *([20]) (ZF + $DC_{\mathbb{R}}$) AD^+ abbreviates the following assumptions.*

1. *Suppose $A \subset \mathbb{R}$. Then A is $^\infty$-Borel.*
2. *Suppose $\lambda < \Theta$ and*

$$\pi : \lambda^\omega \to \omega^\omega$$

is a continuous function. Then for each $A \subset \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.

The following theorem gives the basic analysis of $\mathbb{P}^*_{max}$ in the presence of AD^+.

Theorem 4.15. *([20]) Suppose $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass which is closed under continuous preimages such that*

$$L(\Gamma, \mathbb{R}) \models AD^+.$$

Then $\mathbb{P}^*_{max}$ is ω-closed and homogeneous.

Suppose $G \subset \mathbb{P}^*_{max}$ is $L(\mathbb{R})$-generic. Then in $L(\Gamma, \mathbb{R})[G]$:

1. *$\mathscr{P}(\omega_1)_G = \mathscr{P}(\omega_1)$;*
2. *$\mathscr{P}(\omega_1) \subset L(\mathbb{R})[G]$;*
3. *I_G is the nonstationary ideal;*
4. *I_G is a normal saturated ideal;*
5. *for every $A \in \Gamma$, $B \in \mathscr{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ the set*

$$\{X \prec \langle H(\omega_2, A, \in\rangle \mid M_X \text{ is } B\text{-iterable and } X \text{ is countable}\}$$

contains a club, where M_X is the transitive collapse of X.

Note that the set

$$\{((\langle M_k : k < \omega\rangle, a), \sigma) \mid \exists c \in M_0\ (\langle M_k : k < \omega\rangle, a) \Vdash \sigma_G \\ = (c^*, (\mathscr{P}(\omega_1)^{M_0})^*)\},$$

is dense in $\mathbb{P}^*_{max} * \mathbb{C}$, where c^* and $(\mathscr{P}(\omega_1)^{M_0})^*$ as usual are the images of c and $\mathscr{P}(\omega_1)^{M_0}$ under the embeddings through the generic. That this set is dense follows from the fact that $\mathscr{P}(\omega_1)_G = \mathscr{P}(\omega_1)$ in the $\mathbb{P}^*_{max}$ extension, and the fact that $\mathbb{C}$ is closed under expansion of the second coordinates of its conditions.

First we will show that $\mathbb{P}^*_{max} * \mathbb{C}$ is ω-closed on this set and so adds no reals. Then we will show that all subsets of ω_1 added by $\mathbb{P}^*_{max} * \mathbb{C}$ are in fact added by $\mathbb{P}^*_{max}$. Then we will be done.

The following theorem is implicit in [20].

Theorem 4.16. *([20])Assume AD^+ holds in $L(\Gamma, \mathbb{R})$, where $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass closed under continuous preimages. Suppose $B \subset \mathbb{R}$ and $B \in L(\Gamma, \mathbb{R})$. Then there exists a $\mathbb{P}^*_{max}$ condition*

$$(\langle M_k : k < \omega\rangle, a)$$

such that the following hold.

1. *For all $k < \omega$, $I_{NS}^{M_{k+1}} \cap M_k = I_{NS}^{M_k}$.*
2. *For all $k < \omega$, $B \cap M_k \in M_k$*
3. *For all $k < \omega$, $\langle H(\omega_1)^{M_k}, \in, B \cap M_k\rangle \prec \langle H(\omega_1), \in, B\rangle$.*
4. *$\langle M_k : k < \omega\rangle$ is B-iterable.*

The following lemma shows that we can appropriately iterate $\mathbb{P}^*_{max}$ conditions.

Lemma 4.17. *(ZFC*) Say $\langle\langle M_k : k < \omega\rangle, a\rangle$ is a $\mathbb{P}^*_{max}$ condition. Then there exists an iterate $\langle M^*_k : k < \omega\rangle$ of $\langle M_k : k < \omega\rangle$ such that*

1. *$\omega_1^{M_0^*} = \omega_1$,*
2. *$\forall k < \omega$, $I_{NS}^{M^*_{k+1}} \cap M^*_k = I_{NS} \cap M^*_k$,*
3. *$\forall k < \omega, \forall T \in \mathscr{P}(\omega_1)^{M^*_k} \setminus (I_{NS})^{M^*_{k+1}} \cup\{M^*_k : k < \omega\} \cap Ord \notin \tilde{\tilde{T}}$.*

Proof. Build an iteration

$$\langle\langle M_k^\beta : k < \omega\rangle, G_\alpha, j_{\alpha,\beta} \mid \alpha < \beta \leq \omega_1\rangle,$$

with $\langle M^*_k : k < \omega\rangle = \langle M_k^{\omega_1} : k < \omega\rangle$. The first condition is satisfied by the length of the iteration. The second condition is routine: we take a partition $\langle S_\alpha : \alpha < \omega_1\rangle$ of ω_1 into ω_1 many stationary sets, and while we build the iteration we tag each set A in some $\mathscr{P}(\omega_1)^{M_k^\beta} \setminus (I_{NS})^{M_{k+1}^\beta}$ to some S_{α_A}, and from that point on construct so that if $\omega_1^{M_0^\gamma} \in S_{\alpha_A}$, then $j_{\beta,\gamma}(A) \in G_\gamma$.

For the last condition, we use the fact (see [20], [13]) that for any iteration of a $\mathbb{P}^*_{max}$ condition, for all $\beta + 1 < \alpha$,

$$\omega_1^{M_0^{\beta+1}} = j_{\beta,\beta+1}(\omega_1^{M_0^\beta}) = \cup\{M_k^\beta \cap \mathrm{Ord} \mid k < \omega\}.$$

Then we can assure the last condition by doing the same argument as for the second condition, just at successor stages. That is, for each A in some

$\mathscr{P}(\omega_1)^{M_k^\beta} \setminus (I_{NS})^{M_{k+1}^\beta}$, construct so that $j_{\beta\gamma+1}(A) \in G_{\gamma+1}$ for stationarily many γ. Let $f : \omega_1 \to \cup\{M_k^* : k < \omega\} \cap Ord$ be a bijection such that on a club of $\alpha < \omega_1$, $f[\alpha] = j_{\alpha,\omega_1}[\cup\{M_k^\alpha : k < \omega\} \cap Ord]$. Then the third condition will be satisfied, since for any T in any $\mathscr{P}(\omega_1)^{M_k^*} \setminus (I_{NS})^{M_{k+1}^*}$, $o.t.(f[\alpha]) \notin \bar{T}$ for stationarily many α. □

First we show that $\mathbb{P}^*_{max} * \mathbb{C}$ is ω-closed, and so adds no reals.

Theorem 4.18. *Assume AD$^+$ holds in $L(\Gamma, \mathbb{R})$, where $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass closed under continuous images. Then $\mathbb{P}^*_{max} * \mathbb{C}$ is ω-closed on the set $D =$*

$$\{((\langle M_k : k < \omega\rangle, a), \sigma) \mid \exists c \in M_0\ (\langle M_k : k < \omega\rangle, a) \Vdash \sigma_G = (c^*, (\mathscr{P}(\omega_1)^{M_0})^*)\}.$$

Proof. Suppose that

$$\langle(((\langle M_k^i : k < \omega\rangle, a^i), \sigma_i) \mid i < \omega\rangle$$

is a descending sequence in $\mathbb{P}^*_{max} * \mathbb{C} \cap D$. Let x be a real which codes the sequence

$$\langle(((\langle M_k^i : k < \omega\rangle, a^i), c_i)\rangle \mid i < \omega\rangle,$$

where c_i witnesses that $((\langle M_k^i : k < \omega\rangle, a^i), \sigma_i) \in D$.

Then let $(\langle M_i : i < \omega\rangle, b)$ be a $\mathbb{P}^*_{max}$ condition with $x \in M_0$, as given by Theorem 4.16..

We work in M_0.

There are embeddings $j_{i,i+1}$ witnessing the descent of our sequence of $\mathbb{P}^*_{max}$ conditions. Let $j_{i\omega}$ denote the embedding of $\langle M_k^i : k < \omega\rangle$ into the direct limit of this system, and consider the structure

$$(\langle N_k : k < \omega\rangle, a)$$

where $N_k = j_{k\omega}(M_k)$ and $a = \cup j_{0\omega}(a^0)$.

By Lemma 4.3., this sequence is iterable, and thus $(\langle N_k : k < \omega\rangle, a)$ is a $\mathbb{P}^*_{max}$ condition. By iterating it by an embedding j as in Lemma 4.17. we see that

$$(((\langle M_i : i < \omega\rangle, j(a)), \sigma)$$

is a lower bound in D for our descending sequence, where c is the closure of $\cup\{j(j_{k\omega}(c_k)) : k \in \omega\}$, and σ is a $\mathbb{P}^*_{max}$-name such that

$$(\langle M_i : i < \omega\rangle, j(a)) \Vdash \sigma_G = (c^*, (\mathscr{P}(\omega_1)^{M_0})^*)).$$

□

Lastly, we show that $\mathbb{C}$ adds no subsets of ω_1 over the the $\mathbb{P}^*_{max}$ extension in question.

Theorem 4.19. *Assume AD^+ holds in $L(\Gamma,\mathbb{R})$, where $\Gamma \subset \mathscr{P}(\mathbb{R})$ is a pointclass closed under continuous images. Let $G \subset \mathbb{P}^*_{max}$ be $L(\Gamma,\mathbb{R})$-generic, and let $F \subset \mathbb{C}^{L(\Gamma,\mathbb{R})[G]}$ be $L(\Gamma,\mathbb{R})[G]$-generic. Then if $A \subset \omega_1$ and $A \in L(\Gamma,\mathbb{R})[G][F]$, then $A \in L(\Gamma,\mathbb{R})[G]$.*

Proof. Again, let $D =$

$$\{((\langle M_k : k < \omega\rangle, a), \sigma) \mid \exists c \in M_0\ (\langle M_k : k < \omega\rangle, a) \Vdash \sigma_G \\ = (c^*, (\mathscr{P}(\omega_1)^{M_0})^*)\}.$$

Let τ be a D-name in $L(\Gamma,\mathbb{R})$ for a subset of ω_1 added by $\mathbb{C}$, and, noting that conditions in D can be coded by reals, let B be a set of reals coding τ. Then let $(\langle M_i : i < \omega\rangle, b)$ be a B-iterable $\mathbb{P}^*_{max}$ condition as in Theorem 4.16..

We work in M_0.

Using the ω-closure proved above as well as the fact that

$$\langle H(\omega_1)^{M_0}, \in, B \cap M_0\rangle \prec \langle H(\omega_1), \in, B\rangle,$$

we can build a descending ω_1 sequence $S = \langle (p_\alpha, \sigma_\alpha) : \alpha < \omega_1\rangle$ from D of length ω_1 such that for every $\alpha < \omega_1$ (of M_0) there exists a $(p,\sigma) \in S$ such that (p,σ) decides "$\check{\alpha} \in \tau$."

By a bookkeeping argument as in Lemma $4.17.$ to ensure that the stationary sets in each p_α are mapped to stationary sets in M_1 (using the set X from the definition of $\mathbb{P}^*_{max}$), we can construct this sequence so that each

$$(p_\alpha, \sigma_\alpha) > ((\langle M_i : i < \omega\rangle, a), \sigma)$$

in D, for some fixed $a, \sigma, c \in M_0$ such that

$$(\langle M_i : i < \omega\rangle, a) \Vdash \sigma_G = (c^*, (\mathscr{P}(\omega_1)^{M_0})^*).$$

Here, if we let j_α be the embedding witnessing $p_\alpha > (\langle M_i : i < \omega\rangle, a)$ and let c_α be the set such that p_α forces that the first coordinate of $\sigma_{\alpha G}$ will be c^*_α, then $c = \cup\{j_\alpha(c_\alpha) : \alpha < \omega_1^{M_0}\}$.

Furthermore, by the B-iterability of our condition (working now in $L(\Gamma,\mathbb{R})$), if $(\langle M_i : i < \omega\rangle, j(a_0)) \in G$, then the value of τ is determined by S, A_G and σ_G, contradicting that fact that τ is a name for a subset of ω_1 added by $\mathbb{C}$. □

4.3. $\mathbb{P}_{max}$ variations for $\tilde{T} = \emptyset$

In [10], two $\mathbb{P}_{max}$ variations are presented in whose extensions the nonstationary ideal is saturated and there exists a stationary, costationary subset T of ω_1 such that $\tilde{T}$ and $\tilde{\tilde{T}}$ are both empty. The first of these variations, called $\mathbb{P}^g_{max}$, will be presented in detail in [12]. The relevant theorem regarding $\mathbb{P}^g_{max}$ for this context is a follows.

Theorem 4.20. *Assume AD + $V = L(\mathbb{R})$. Then there is a forcing $\mathbb{P}^g_{max}$ such that in the extension by $\mathbb{P}^g_{max}$ followed by adding a subset of ω_2 by initial segments the following hold.*

1. *ZFC + $c = \underset{\sim}{\delta}^1_2 = \omega_2$.*
2. *I_{NS} is saturated.*
3. *There exists a stationary, costationary $T \subset \omega_1$ such that for all finite sets $a \subset \omega_2$ there exist stationary subsets $S_0, S_1 \subset \omega_1$ such that*

$$S_0 \Vdash a \cap j(T) = \emptyset \text{ and } S_1 \Vdash a \cap j(\bar{T}) = \emptyset$$

where S_0 and S_1 are considered as conditions in $\mathscr{P}(\omega_1)/I_{NS}$ and j is the embedding derived from this forcing.

So, not only are $\tilde{T}$ and $\tilde{\tilde{T}}$ empty in the statement of this theorem, but T witnesses a failure of ψ_{AC} also.

Leaving the discussion of $\mathbb{P}^g_{max}$ to [12], in this section we briefly present a $\mathbb{P}_{max}$ variation which tries to maximize Π_2 sentences for the structure

$$\langle H(\omega_2), \in, I_{NS} \rangle$$

relative to the existence of a stationary, costationary set $T \subset \omega_1$ such that $\tilde{T}, \tilde{\tilde{T}}$ are empty. This is a straightforward variation of $\mathbb{P}^*_{max}$.

Definition 4.21. *$\mathbb{T}^0_{max}$ is the set of sequences*

$$\langle \langle M_k : k < \omega \rangle, T, X \rangle$$

such that the following hold.

1. *Each M_k is a countable transitive model of ZFC*.*
2. *$M_k \in M_{k+1}$, $\omega_1^{M_k} = \omega_1^{M_{k+1}}$.*
3. *$(I_{NS})^{M_{k+1}} \cap M_k = (I_{NS})^{M_{k+2}} \cap M_k$*
4. *$\langle M_k \mid k < \omega \rangle$ is iterable.*
5. *For each $k < \omega$,*

$$M_k \models \text{“}T \subset \omega_1 \text{ is stationary, costationary, } \tilde{T} = \emptyset, \text{ and } \tilde{\tilde{T}} = \emptyset.\text{”}$$

6. *$\exists Y \in M_0$ such that $Y \subset \mathscr{P}(\omega_1)^{M_0} \backslash I^{M_1}_{NS}$, such that $M_0 \models$ “$|Y| = \omega_1$,” and such that for all $A, B \in Y$, if $A \neq B$ then $A \cap B \in I^{M_0}_{NS}$.*

7. $X \in M_0$ *and* X *is a set, possibly empty, of pairs* $(\langle\langle N_k : k < \omega\rangle, S, Z\rangle, j)$ *such that the following hold:*

a) $\langle N_k : k < \omega\rangle$ *is countable in* M_0*;*
b) $\langle\langle(N_k : k < \omega\rangle S, Z\rangle \in \mathbb{T}^0_{max}$*;*
c) $j : \langle N_k : k < \omega\rangle \to \langle N^*_k : k < \omega\rangle$ *is an iteration such that*

$$j(I_{NS}^{N_{k+1}}) \cap N^*_k = (I_{NS})^{M_1} \cap N^*_k$$

for all $k < \omega$*;*
d) $j(Z) \subset X$*;*
e) $j(S) = T$*;*
f) if $(\langle\langle N_k : k < \omega\rangle, S, Z\rangle, j') \in X$ *then* $j = j'$*.*

Suppose

$$\langle\langle N'_k : k < \omega\rangle, T, X\rangle, \langle\langle N_k : k < \omega\rangle, S, Z\rangle$$

are conditions in $\mathbb{T}^0_{max}$*. Then*

$$\langle\langle N'_k : k < \omega\rangle, T, X\rangle < \langle\langle N_k : k < \omega\rangle, S, Z\rangle$$

if there exists an iteration

$$j : \langle N_k : k < \omega\rangle \to \langle N^*_k : k < \omega\rangle$$

such that $(\langle\langle N_k : k < \omega\rangle, S, Z\rangle, j) \in X$*.*

The basic analysis of $\mathbb{T}^0_{max}$ is roughly the same as that of $\mathbb{P}_{max}$. The basic properties of the extension are given below. Note that we do not know whether $\mathbb{T}^0_{max}$ is homogeneous. As we discuss below, the issue of homogeneity is related to the question of whether the $\mathbb{T}^0_{max}$ extension is indeed Π_2 maximal relative to the existence of a stationary subset of ω_1 whose tilde is empty. The additional forcing over the $\mathbb{T}^0_{max}$ extension is required to obtain AC, since we also don't know if ψ_{AC} holds after forcing with $\mathbb{T}^0_{max}$.

Definition 4.22. *For* $p = \langle\langle M_k : k < \omega\rangle, T, X\rangle \in \mathbb{T}^0_{max}$, $T_p = T$*. For* $g \subset \mathbb{T}^0_{max}$ *a filter,* T_g *is the union of* $\{T_p \mid p \in g\}$*.*

Theorem 4.23. *Assume AD*$^{L(\mathbb{R})}$*. Then* $\mathbb{T}^0_{max}$ *is* ω*-closed.*
Suppose $G \subset \mathbb{T}^0_{max}$ *is* $L(\mathbb{R})$*-generic. Then*

$$L(\mathbb{R})[G] \models \omega_1 - DC$$

and in $L(\mathbb{R})[G]$*:*

1. $\mathscr{P}(\omega_1)_G = \mathscr{P}(\omega_1)$.
2. $\tilde{T}_G, \tilde{\tilde{T}}_G$ *are empty.*
3. I_G *is the nonstationary ideal.*

Further, say that H is $L(\mathbb{R})[G]$-generic for adding a subset of ω_2 by the initial segment forcing. Then in $L(\mathbb{R})[G][H]$ the nonstationary ideal is saturated.

The key point in the basic analysis of $\mathbb{T}^0_{max}$ is showing that any $\mathbb{T}^0_{max}$ condition can be iterated in a way that preserves the emptiness of the tilde of its T (and $\bar{T}$). More precisely, it is the following lemma.

Lemma 4.24. *($\diamond$) Say $\langle\langle M_k : k < \omega\rangle, T, X\rangle$ is a $\mathbb{T}^0_{max}$ condition.*

Then there is an iteration by the nonstationary ideal, $j : \langle M_k : k < \omega\rangle \to \langle M^*_k : k < \omega\rangle$, such that (for $j(T) = P$) the following hold.

1. $\omega_1^{M^*_0} = \omega_1$.
2. *For all $k \in \omega$, $I_{NS} \cap M^*_k = (I_{NS})^{M^*_{k+1}} \cap M^*_k$.*
3. *$\tilde{P}, \tilde{\tilde{P}}$ are both empty.*

Proof. We construct an ω_1-length iteration, using the usual trick to ensure that $I_{NS} \cap M^*_k = (I_{NS})^{M^*_{k+1}} \cap M^*_k$. That is, we take a partition of ω_1 into stationary sets $\langle S_\alpha : \alpha < \omega_1\rangle$ and, enumerating the stationary subsets which appear during the iteration, we make sure each such set is in the generic at all stages in some S_α.

To get $\tilde{P}, \tilde{\tilde{P}}$ empty, we use the same construction as in Lemma 2.7.. The key idea is that since $M \models \tilde{T}, \tilde{\tilde{T}} = \emptyset$, if at some stage λ σ_λ of our $\diamond$ sequence codes a well ordering of λ of ordertype $\gamma > \lambda$, we can extend the iteration to keep γ out of P or to put it in P as desired. □

The key issue here is whether the assumptions on this lemma can be improved to "there exists a stationary, costationary set $S \subset \omega_1$ such that $\tilde{S}, \tilde{\tilde{S}}$ are empty," instead of $\diamond$. In the terminology of [17], we are asking if there exists an optimal iteration lemma for this statement. If there does, then $\mathbb{T}^0_{max}$ maximizes Π_2 sentences in $H(\omega_2)$ relative to this sentence. If there doesn't, then there should be incompatible Π_2 sentences relative to this one, and therefore no such maximal model.

5. Questions

The following issues remain unresolved.

1. Does MM(Namba forcing) hold in the forcing extension in Corollary 3.20.?
2. Does Shelah's forcing to make the nonstationary ideal saturated from a Woodin cardinal ever put an ordinal α into $\tilde{T}$ if in the ground model $\{x \in [\alpha]^{<\omega_1} \mid o.t.(x) \in \bar{T}\}$ is stationary?
3. Is there an optimal iteration lemma for "There exists a stationary, costationary $T \subset \omega_1$ such that $\tilde{T}$ and $\tilde{\tilde{T}}$ are empty"?

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