South American Journal of Logic
Vol. 3, n. 2, pp. 517–531, 2017
ISSN: 2446-6719

Oppositional Geometry in the Diagrammatic Calculus *CL*

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Abstract

The paper presents the diagrammatic calculus *CL*, which combines features of tree, Euler-type, Venn-type diagrams and squares of opposition. In its basic form, '*CL*' (= Cubus Logicus) organizes terms in the form of a square or cube. By applying the arrows of the square of opposition to *CL*, judgments and inferences can be displayed. Thus *CL* offers on the one hand an intuitive method to display ontologies and on the other hand a diagrammatic tool to check inferences. The paper focuses mainly on the adaptation of the square of opposition in *CL* and offers an algebraic notation, which corresponds to the diagrammatic representation.

Keywords: Calculus *CL*, square of opposition, oppositional geometry, universal logic, ontology engineering.

1 Introduction

The expression 'universal logic' can mean a universal logical system or a general theory of logic (Fu [3]). In the field of research which deals with the second meaning, logicians interested in diagrammatic reasoning and esp. oppositional geometry have for several years concentrated on the square of opposition (Beziau, Basti [1]). The fact that the logical square has a close relationship with other diagrams has been known at least since Thomson ([12], § 84) who had explained the categorical propositions (a, e, i, o) by focusing on Euler diagrams, and with the help of two further Euler diagrams he invented u- and y-propositions. With these two propositions, the square of opposition could be expanded into a logical hexagon and octagon (Moretti [7]).

A much older contribution to universal logic can be found in Johann Christian Lange's book *Inventum novum quadrati Logici Universalis*, published in the year 1714 [4]. Lange himself uses the term 'Logica Universalis' because he was searching for a universal diagram in order to unite the functions of Euler-type circular diagrams, squares of oppositions, porphyrian tree diagrams, and step diagrams in only one image. According to Lange, this universal diagram should express the metaphors of containment, of opposition, of subordination and of ascent and descent. This logic diagram is intended to illustrate not only a universal logic, but also a rational form of representationalism. That means that Lange wanted to represent and explain all functions of rationality (starting from conceptual semantics, extending to sentence analysis, up to inferentialism) by the intuition of only one universal diagram which was entitled 'cubus logicus'.

Because Lange thus builds an entire system of logic using one diagram, it is not possible to discuss the content and all aspects of the 170-page book in this article. Although many important logicians such as Leibniz ([5], p. 405), Ploucquet ([9], p. 43), DeMorgan [2], Venn ([13], p. 501), Peirce ([8], p. 298 (4.353)), and Risse ([10], p. 51) were interested in Lange's cubus logicus, there is still no intensive and continuous research on Lange's diagrams. This is partly due to the fact that throughout his entire book, Lange refers only to the illustration of one diagram, which can be interpreted in many different ways.

All of these are reasons which have led me to the decision not to discuss Lange's own book and the image of his universal diagram within it. Rather, I would like to present a calculus that is based on Lange's principles found in the *Inventum*. I will adopt Lange's basic form of the diagram, but I will go on specify a concrete state of the diagram for each conceptual relationship, for every logical judgment and all inferences which are discussed in what follows. To avoid giving the impression that this calculus was my invention, I have decided to name the calculus *CL* (in accordance with Lange's expression "cubus logicus"). And with the help of the references, historically interested readers will soon be able to recognize the parallels between the Calculus *CL* presented here and Lange's *Inventum*.

In Section 2, I will restrict my explanation of the universal diagram to some of its basic functions only. Sections 3 and 4 provide more details about the functions which *CL* takes from the square of opposition. This means that in the explanation of *CL*, I mainly limit myself to the aspect of opposition, which is represented here in geometric forms. Even if the form of the logical square is not always evident in *CL*, the diagrams can nevertheless be interpreted as contributions to oppositional geometry. Finally, Section 5 gives some syllogisms as examples in order to explain how *CL* is suitable for decision-making and proofs.

2 The basic diagram of the calculus *CL*

The basic diagram (Fig. 1) consists of a cube with 16 columns and 5 lines. The rows symbolize the extension, the columns the subordination. Terms, classes or concepts are identified by letters from A to t. The containment of a term is symbolized by solid lines.

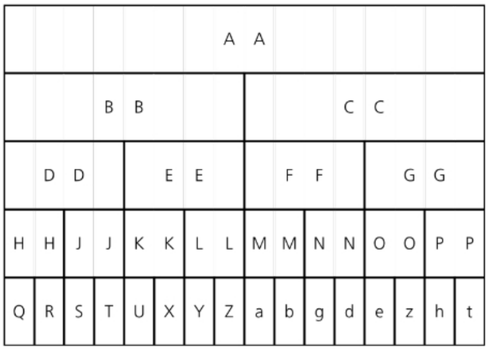


Figure 1:

The basic diagram can be interpreted in two ways: either as a porphyrian tree diagram describing the subordination or as an Euler-type diagram (cf. Shin, Moktefi [6]) representing the extension.

If the diagram is interpreted as a porphyrian tree diagram, one can say: A is the uppermost genus. This can be seen in the fact that A occupies the top row, above which there are no more rows. B to G are, on the one hand, genera in relation to the species below them, and on the other, species in relation to the genera which are superior to them. This is shown by the fact that there is at least one row above and below them. H to P are the lowest species, and Q to t are individuals. The individuals are recognizable by the fact that Q to t occupy the lowest row under which there are no more rows. As far as subordination is concerned, the higher the concept, the more concepts are subordinate to it. A is above B and C. B is above D and E. D is above H and J, etc.

With the help of dotted lines in the columns (Fig. 2) and the associated width of the terms in a row, however, Euler-type relationships of containment can also be depicted. The solid lines indicate what the term contains: A contains everything, Q to t contain nothing, i.e., no other term. The dotted lines indicate if and how one concept is contained in another: A is never contained in another concept, whereas Q to t are, in any case, each contained in another. One can imagine the dotted lines as auxiliary lines: The dotted columns are a reading aid to illustrate what is contained by any given term. For example, the fact that Q is completely contained in H is indicated by continuing the solid lines of the columns of Q as dotted lines in H. This makes the term H wider than the term Q. And so the dotted lines in H show that Q and R are contained completely in H and that H and J are contained entirely in D, and so on. A contains B and C. Since B now contains D and E and since C also contains F and G, A contains both D, E, and F, G. If we follow this rule, it becomes clear why A contains all the individuals. Furthermore, the rule explains why Q to t contain nothing, but all are contained by something else, and finally all are contained in A. In terms of extension, the rule can be kept: the further a term is, the more dotted columns it contains.

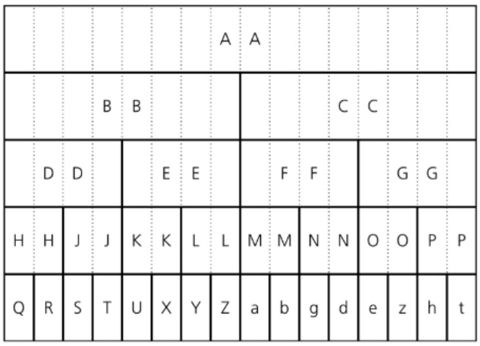


Figure 2:

By the eulerian extension, the porphyrian subordination can also be explained: The fact that Q to t are individuals is recognizable by the fact that they occupy only one column in the whole cube. The fact that A represents the top class is shown by the fact that it contains all the columns. The fact that B to P can function as both genera and species is apparent from the fact that they are both containing and contained.

3 Introducing Oppositional Geometry in *CL*

In Section 2 it was shown that porphyrian tree diagrams, as well as some Euler-type diagrams, can be depicted completely in *CL*. For the oppositional geometry so far, several relations of opposition have already been shown in Fig. 2. Following Lange ([4], § XLVII), these relations are called:

Contrary Opposition: Concepts in the same line are in a contrasting relationship. For example, the term B in relation to C, D in relation to E, F in relation to G, H in relation to J, and so on.

Privative Opposition: To each concept stand in a privative opposition those terms which are mediated completely by vertical columns with the term. For example, the terms Q, H, D, B, A stand in privative opposition.

Contradictory Opposition: Opposed to each concept, contradictory terms are those which are not completely mediated by vertical columns with the concept. For example, the terms A, D, E, H, J, K, L may be in contradictory opposition to the term B. However, the diagram (Fig. 2) has several disadvantages. Porphyrian tree diagrams can be translated satisfactorily in *CL*, but there are problems with Euler diagrams dealing with relative opposition. For example, a-judgments (affirmative universal) such as "All D are B" can be represented as shown in Fig. 3 by red columns. And if one can illustrate a-judgments, one can also represent i-judgments (affirmative particular), for example, "Some D are B", shown in Fig. 4 (orange colored). But negative e- and o-judgments are not apparent.

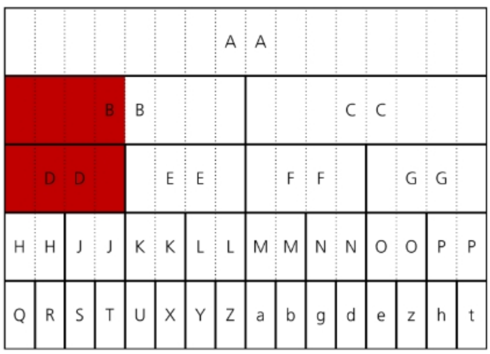


Figure 3:

At the suggestion of Johann Albrecht Bengel, Lange ([4], § XXVI) then came to the idea of combining the diagram with the arrows of the logical square. Yet with the help of the square of opposition, we are able to represent all four categorical propositions in *CL*. The fletching of the arrow indicates the subject, the arrowhead the predicate of a judgment (Fig. 5). The four arrow types can be used to display all the categorical propositions such as in Euler diagrams: 1. vertical bottom-up (↑), 2. vertical top-down (↓), 3. transversal (↘, ↖, ↙, ↗), and 4. horizontal (→, ←).

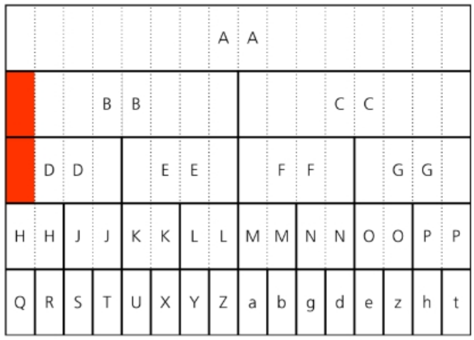


Figure 4:

Figure 5:

Furthermore, from the possible combinations of arrows, two affirmative and two negative quantifiers can be read which represent the four categorical propositions expressed in the traditional syllogism by the letters a, i, e, o:

<u>**affirmative**</u>

⇈ **(a-proposition) vertical I (completely subordinated):** The relationship between subject and predicate can be expressed completely by vertical bottom-up arrows. If vertical bottom-up arrows go up to the predicate from all columns of the subject, this means that the subject is completely contained in the predicate, or that the subject is completely subordinated to the predicate.

⇅ **(i-proposition) vertical II (partly superordinated or partly subordinated):** The relationship between subject and predicate is either partial expressed by vertical bottom-up arrows (↑) or wholly (⇊) or partially (↓) by vertical top-down arrows. If vertical bottom-up arrows from at least one column, but not from all columns, of the subject go to the predicate, this means that the subject is partially contained in the predicate, or the predicate is partly superordinate to the subject. One or more vertical top-down arrows means that the subject contains the predicate partially.

negative

⇄ **(e-proposition) horizontal (completely contrary):** The relationship between subject and predicate can be expressed completely by a horizontal arrow. If the horizontal arrow fills all the columns of a subject, it means that the subject is completely contrary to the predicate. If the subject and the predicate are on different rows, e-propositions can also expressed by transversal arrows. In this case, the transversal arrow fills all columns of the subject.

⤨ **(o-judgment) transversal (coordinated, partly contrary or contradictory):** The relationship between subject and predicate part can be expressed either by a short horizontal or by a transversal arrow. If a horizontal arrow fills at least one, but not all, columns of a subject, this means that the subject is partly contrary to the predicate. If a transversal arrow fills at least one column of the subject, that means that the subject is contradictory to the predicate.

The spelling of the judgments can be represented by the spelling using the arrows. In the following the old scholastic quantifiers (a, i, e, o) are replaced by the arrow quantifiers:

aFG	⇈ FG	Every F is G
iFG	⇅ FG	Some F is G
eFG	⇄ FG	No F is G
oFG	⤨ FG	Some F is not G

All categorical propositions can be represented in the diagram. For example: ⇈DB (red), ⇅DB (yellow), ⇄BC (blue), ⤨BD (green) (Fig. 6).

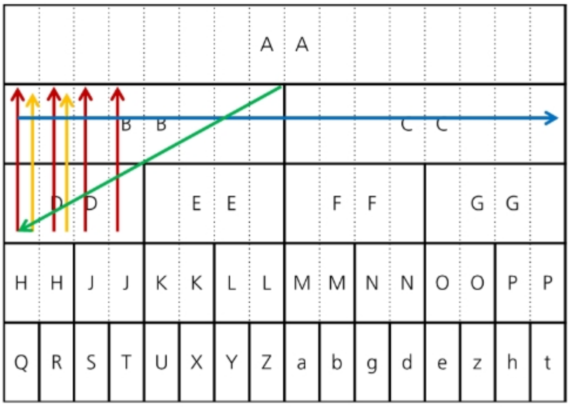


Figure 6:

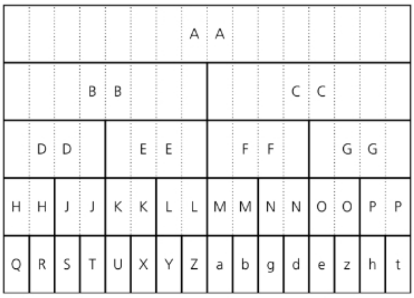
In contrast to Euler diagrams, however, only true judgments can be depicted in the *CL* diagram. Taking the above examples for the quantifiers, we can use Fig. 6 in order to show that ↑↑FG, ↑↓FG and ⤨FG are false, since they cannot be represented. On the other hand, ⇄FG can be represented and therefore the proposition is true. This has advantages, particularly in material areas of logic, in which work is already being done not with variables, but with concrete ontologies. Supporters of Euler diagrams should not have a disadvantage using the *CL* diagram, since all four types of judgments can be represented graphically. Moreover, a criterion for decision-making is immediately given by the fact that conceptual relations, judgments, and conclusions can be read directly from the diagram.

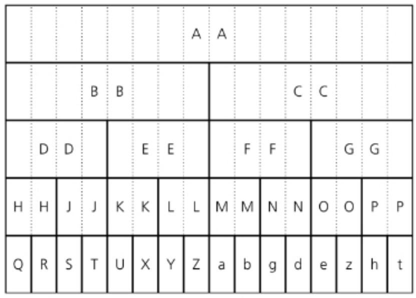
4 Oppositional Judgments in *CL*

In the following, I will present oppositional judgments with the help of some examples given by Lange ([4], § LXII). To do so I introduce the double turnstile, which can be interpreted as a semantic consequence (such as in model theory (cf. Smessaert and Demey [11])) or as a representationalistical symbol, meaning "representable" (⊨) or "non-representable" (⊭) in the diagrammatic sense: Thus, "*CL* ⊨ ↑↑BA" means "↑↑BA can be represented in *CL*"; in contrast, "*CL* ⊭ ⇄BA" means "⇄BA cannot be represented in *CL*". The question of how BA can be represented in *CL*is decided by the arrow which can be drawn between the subject B and the predicate A in *CL*.

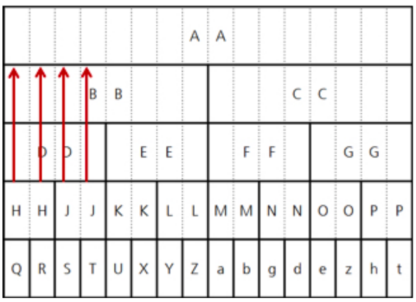
Perfect contradictions are those in which two judgments cannot be represented at the same time.
We speak of

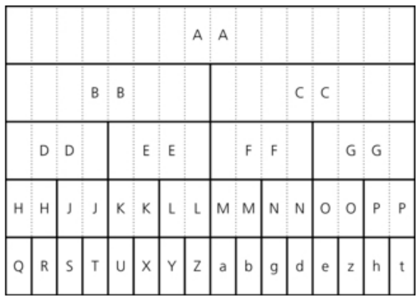
***CL*-contrary** if $CL \not\models \upuparrows$AB and $CL \not\models \rightleftarrows$AB;



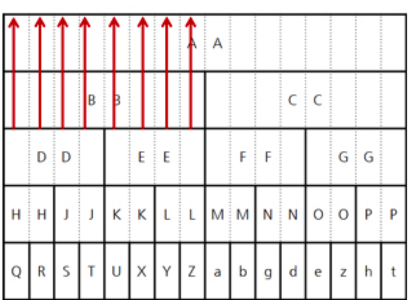


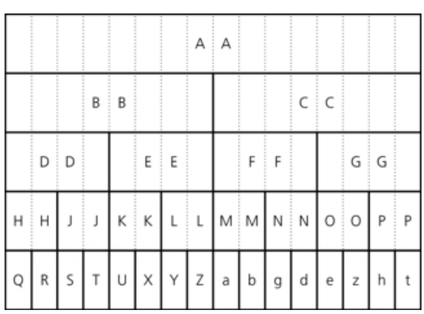
or if $CL \models \upuparrows$DB and $CL \not\models \rightleftarrows$DB.



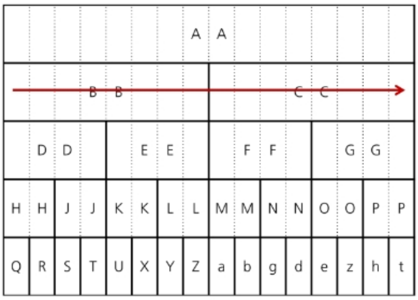


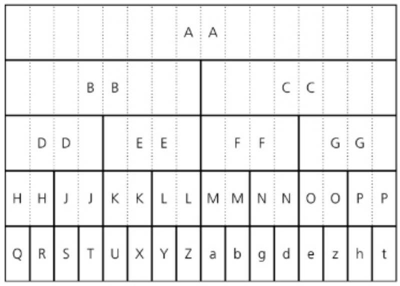
***CL*-contradictory** if $CL \models \upuparrows$BA and $CL \not\models \nwarrow\!\!\!\!\searrow$BA;





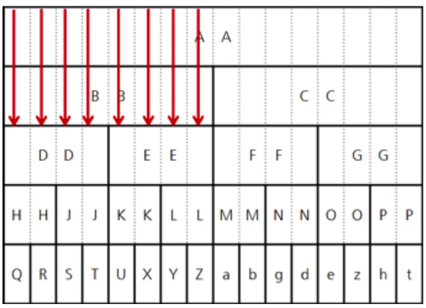
or if $CL \models \rightleftarrows \mathrm{BC}$ and $CL \not\models \updownarrows \mathrm{BC}$.

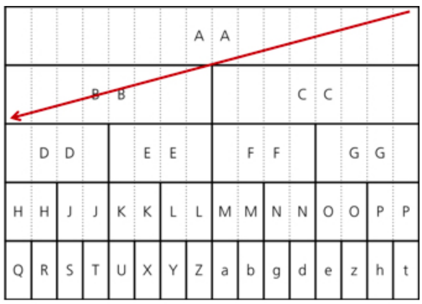




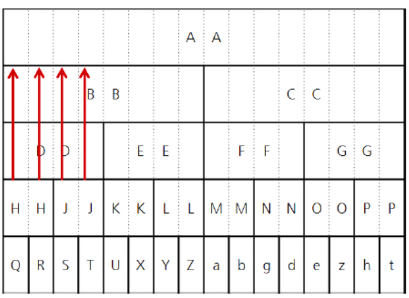
Imperfect contradictions are those in which two statements can be represented at the same time.

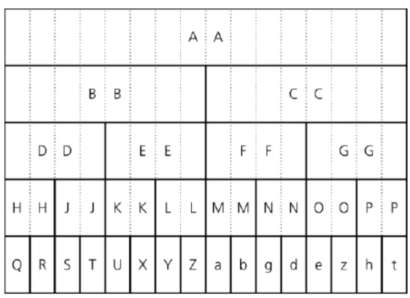
***CL*-subcontrary** if $CL \models \updownarrows \mathrm{AB}$ and $CL \models$ ⤨AB;



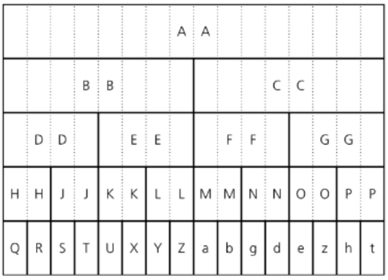


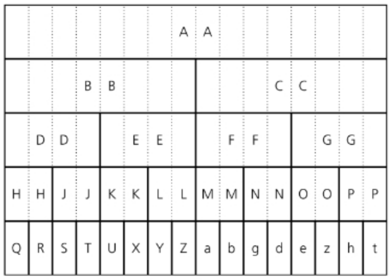
or if $CL \models \updownarrows \mathrm{DB}$ and $CL \not\models$ ⤨DB.



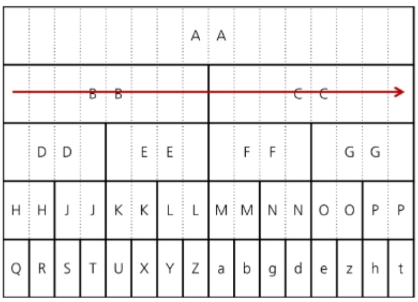


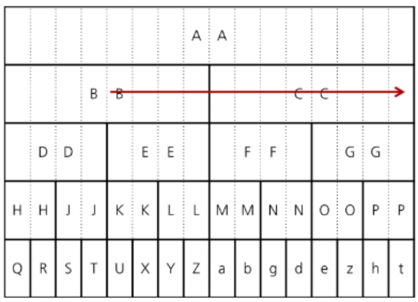
in *CL*-subalternation if $CL \not\models \upuparrows$BC and $CL \not\models \uparrow\!\downarrow$BC;





or if $CL \models \rightleftarrows$BC and $CL \models$ ⤨BC.





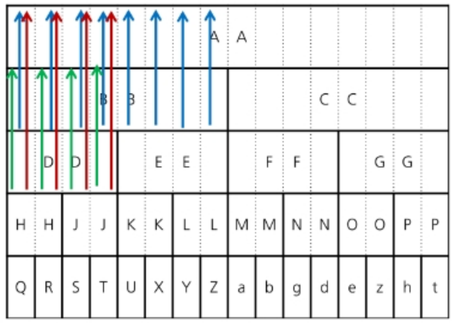
5 Some Examples of Syllogisms in *CL*

Finally, you can use the *CL*-diagram and the arrows to see whether a syllogism is valid or not. If all three judgments of the syllogism can be represented in the diagram, the inference is valid. If one of the premises or the conclusion cannot be represented in *CL*, the inference is invalid.

For this purpose, I use the judgments of some valid and some invalid inferences of the first figure, in order to illustrate the method of proof in *CL*. The first premise is always given in blue, the second in green, and the conclusion in red. False judgments are given in black, simply to indicate the conflict between what shall be represented and what is actually representable in *CL*. (As shown above, false judgments cannot be represented in *CL*.)

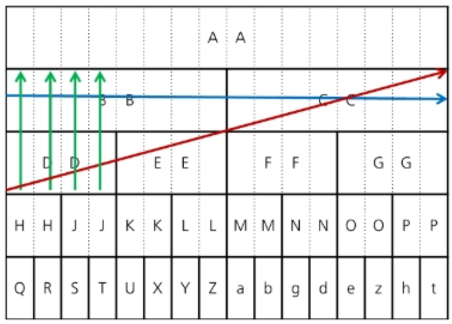
[a, a, a]-1 (Barbara)

All B is A	⇈BA	∀x (*B*x → *A*x)
All D is B	⇈DB	∀x (*D*x → *B*x)
All D is A	⇈DA	∀x (*D*x → *A*x)



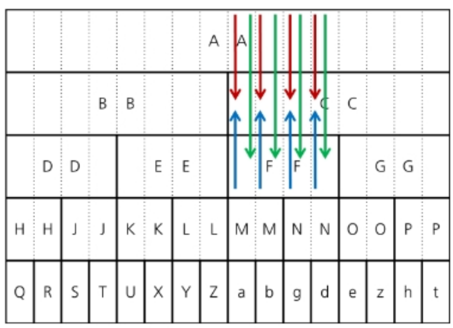
[e, a, e]-1 (Celarent)

No B is C	⇄BC	¬∃x (*B*x ∧ *C*x)
All D is B	⇈DB	∀x (*D*x → *B*x)
No D is C	⇄DC	¬∃x (*D*x ∧ *C*x)



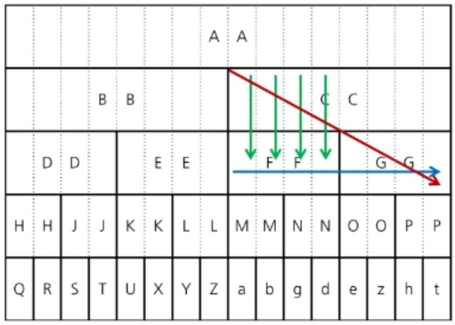
[a, i, i]-1 (Darii)

All F is C	⇈FC	∀x (*F*x ⟶ *C*x)
Some A is F	⇅AF	∃x (*A*x ∧ *F*x)
Some A is C	⇅AC	∃x (*A*x ∧ *C*x)



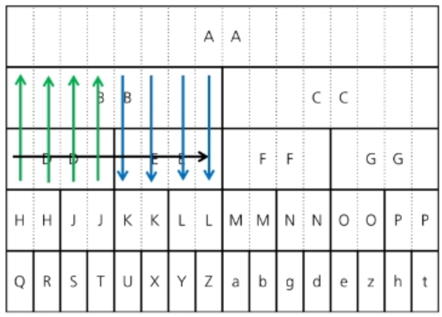
[e, i, o]-1 (Ferio)

No F is G	⇄FG	¬∃x (*F*x ∧ *G*x)
Some C is F	⇅CF	∃x (*C*x ∧ *F*x)
Some C is not G	⤨CG	∃x (*C*x ∧ ¬*G*x)



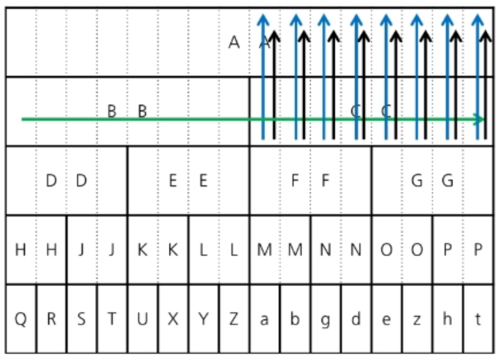
[i, a, a]-1 (irregular I)

Some B is E	⇅BE	∃x (*B*x ∧ *E*x)
All D is B	⇈DB	∀x (*D*x ∧ *B*x)
All D is E	**⇈DE**	**∀x (*D*x ∧ *E*x)**



[a, e, e]-1 (irregular II)

All C is A	⇈CA	∀x (*C*x → *A*x)
No B is C	⇄BC	¬∃x (*B*x ∧ *C*x)
No C is A	**⇄CA**	**¬∃x (*C*x ∧ *A*x)**



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