Axiomatics Without Foundations. On the Model-theoretical Viewpoint In Modern Axiomatics *

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Abstract: Two conflicting interpretations of modern axiomatics will be considered. The logico-analytical interpretation goes back to Pasch, while the model-theoretical approach stems from Hilbert. This perspective takes up the distinction between logic as calculus ratiocinator versus lingua characterica that Heijenoort and Hintikka placed emphasis on. It is argued that the Heijenoort-Hintikka distinction can be carried over from logic to mathematical axiomatics. In particular, the model-theoretical viewpoint is deeply connected to a philosophy of mathematics that is not committed to a foundational perspective, but oriented more at applications and at mathematical practice.

∗. My standpoint and arguments are based on common work with Michael Otte [Lenhard and Otte 2002]. I acknowledge the helpful comments given at PILM.

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1 Introduction

The received view in philosophy of mathematics treats the 'axiomatical standpoint' as a position that is more or less equivalent to the so-called 'Hilbertian formalism'. Moreover, it is interpreted in the coordinate system of the foundational crisis in mathematics that took place in the early 20^{th} century. In the following I shall argue that this gives a blurred, or at least incomplete, picture of modern axiomatics and of the standpoint that David Hilbert called "Axiomatisches Denken" (as explained in his lecture [Hilbert 1918] of the same name). There are significantly different interpretations of axiomatics. I shall distinguish between the *logicist* and the *model-theoretical* viewpoint concerning modern axiomatics. Both were held right from the beginning of modern axiomatics at the end of the 19th century.

This typology of axiomatics will be explained further in the second chapter. It takes up a famous differentiation that Jean van Heijenoort had pointed out concerning logic [Heijenoort 1967]. In his short note, he distinguished between logic as *calculus ratiocinator* and logic as *lingua characterica*, drawing upon Frege's statement in his debate with Schröder. The standpoint that sees logic as *lingua characterica* (held by Frege and Russell), insists upon logic being the universal language of the world. Heijenoort speaks also of 'logic as calculus' and 'logic as language'.

> "For Frege it cannot be a question of changing universes. One could not even say that he restricts himself to *one* universe. His universe is *the* universe." [1967, 235]

This view contains a strong ontological commitment. On the other side, the standpoint of logic as *calculus ratiocinator* (held by Boole, De Morgan, Grassmann, Peirce and Schröder) conceives logic as a means, a calculus that can be - and has to be - adapted to specific contexts. The particular interest of gaining knowledge, one can say, decides upon the semantics. In Heijenoort's words:

> "Boole has his universe class, and De Morgan his universe of discourse... But these have hardly any ontological import. They can be changed at will. The universe of discourse comprehends only what we agree to consider at a certain time, in a certain context." [1967, 235]

Warren Goldfarb has contributed a more detailed historical study much in the same vein (surprisingly not mentioning Heijenoort). He considers the discussions about the nature of the quantifier in the 1920ies and considers "the two major schools of logic prior to that period: the logicist, as represented by Frege and Russell, and the Schröderian algebraists of logic." [1979, 351] And Goldfarb names the same criterion of distinction as Heijenoort, saying about the logicist viewpoint:

> "The ranges of the quantifiers - as we would say - are fixed in advance once and for all. The universe of discourse is always the universe, appropriately striated." [1979, 352]

This distinction is well established in the literature (cf. *e.g.*, Ivor Grattan-Guiness about the classification of C. S. Peirce as an algebraist [1997]). And it has more than a mere historical significance. Jaakko Hintikka has picked up the categorization from Heijenoort and has transferred it from logic to the philosophy of language. This differentiation marks, so the point of Hintikka in a couple of nice articles, an "ultimate presupposition of Twentieth-Century Philosophy" [Hintikka 1998].

I shall argue for the applicability of this distinction in mathematical axiomatics. My main topic will be the standpoint of Hilbert, who was, as is well known, one of the decisive founders of modern axiomatics. In philosophical contexts, however, Hilbert usually is discussed from a foundational perspective and the Hilbertian formalism has become a fixed name for his position. I shall not follow this tradition. Admittedly, the common picture includes essential aspects of Hilbert's program, or at least of one of Hilbert's programs that he was after for a certain time span. My aim here is to argue for an application-oriented perspective on Hilbertian axiomatics for which the name "model-theoretical" seems to be rather appropriate. Already Hintikka has expressed himself in that direction:

> "There is no doubt that Hilbert's FG ((Foundations of Geometry)) was one of the main gateways of model-theoretical thinking into twentieth-century logic and philosophy." [1998, 109]

But Hintikka is somewhat sceptical whether Hilbert maintained his model-theoretical conception. I hope to show, by considering texts of the later Hilbert, that he can be seen to keep a model-theoretical position also then. At least, one can identify such a position, even if there may be different and not entirely compatible positions held by Hilbert.

In short, I hope to make plausible two things. Firstly, the application of the Heijenoort-Hintikka distinction to mathematical axiomatics, especially as held by Hilbert, can count as an argument for the fruitfulness of the model-theoretical viewpoint. And secondly, this viewpoint is deeply connected to a philosophy of mathematics that is not committed to a foundational perspective, but oriented more at applications and at mathematical practice. Therefore, a broader consideration of the model-theoretical viewpoint promises to support such a philosophy of mathematics. Indeed, this is easily said, but still to be done. The title of my paper should hint in this direction and be reminiscent of Hilary Putnam's landmark "Mathematics Without Foundations" [1967] in which he argued in favor of dropping the philosophical ambitions for foundations while considering mathematics philosophically.

2 Two Interpretations of Modern Axiomatics

In this chapter two interpretations of mathematical axiomatics will be introduced and analyzed. At the very beginning of the modern axiomatical method, towards the end of the 19^{th} century, two conflicting interpretations were established:

(L) The logico-analytical interpretation of axiomatics can be traced back to *Moritz Pasch*, whose "Vorlesungen über neuere Geometrie" appeared in 1882. There he formulates axioms ("Kernsätze"), whose truth he considered as guaranteed. Starting from them geometry should be deduced "without any ingredient and purely deductive". This methodological goal aims at justification and leads to a reformulation of mathematics in logical terms. According to this standpoint logic is the basic science of epistemology.

(M) The model-theoretical interpretation of axiomatics has its origin in *David Hilbert's* "Grundlagen der Geometrie" from 1899. (Guiseppe Peano also played an important role.) Hilbert starts from a scientific theory — in this case from geometry — and axiomatically reconstructs its structure. An axiomatized theory has reached the highest level of development. In such a system the axioms are formulated in general terms which admit interpretations in different models. This viewpoint is based on an interest in application, that means the justification of a theory lies more in its successful applications. The evolution of the sciences is thought of as coupled with

> their mathematization (including mathematics itself) and consequently mathematics — not logic — counts as the basic science.

Both axiomatic approaches intend to make explicit the more or less implicit assumptions on which a theory can be reconstructed. The two interpretations, however, are based on radically different concepts about the nature of theories.

Pasch represents the traditional "aristotelic" [Beth 1968] conception of a theory, according to which a theory first identifies its basic truths and then combines them logically. Accordingly one can differentiate between two steps: first the axioms are anchored as true propositions, including only defined concepts. Sure enough, there exists considerable internal variability in opinions about the right way of justification. Pasch, for instance, saw a genuine correspondance between geometrical concepts and empirical objects. Especially he held that there is no need to reinterprete any formula. And the second step consists in the unfolding of the rest of the theory by the truth-conserving means of logic.

The other concept of theory, as represented by Hilbert, introduced the reference to objects and the concept of truth first by intended applications. That means, for Hilbert axioms in his sense were mere schemes of axioms in Pasch's sense, insofar they contain uninterpreted terms and no propositions which are thought of as true or false. The concept of truth, for Hilbert, requires at first the concept of a model. In short: (M) and (L), much in the same manner as the *calculus ratiocinator* and the *lingua characterica* standpoints, are based on a different conception of the relation between ontology and epistemology. And both standpoints differ in respect of their 'logic of justification': while (L) emphasizes the well-founded basis, *i.e.*, locates the justification in the 'past', (M) seeks for confirmation in future applications. According to (L), one can obtain the truths about the mathematical objects from the axioms, while according to (M), one can get only the propositions of the theory. The intensions of the concepts are determined only by the axiomatic system, this is what Moritz Schlick in [1918] called the method of 'implicit definition'. But the axioms do not deal with particular objects, rather with the relations between (indefinite) general objects. Thus, in general the extension of a concept is not determined uniquely. To achieve this, one needs the interpretation of an axiomatic system in a model - and interesting systems have several nonisomorphic models.

A nice illustration of the confrontation between the two standpoints gives the correspondence between Frege and Hilbert that developed in connection with the publication of the "Foundations of Geometry". Frege is right in insisting upon first defining the concepts to be able to separate definitions from conclusions. For him, the mentioned (M)-typical relational approach to objects constitutes a dazzling ambiguity, which should be eliminated. Frege compares Hilbert's approach to axiomatics with a system of equations with several unknown variables from which one does not know whether they have a solution, i.e. are not contradictory, nor whether this solution is unique. Hilbert, on the other side, conceived of the possibilities of interpretation as a decisive advantage, because they guarantee the universal applicability of axiomatics. In this controversy one is able to see clearly how the different orientations at justification (L) and application (M) stand one against the other.

Under the dominating perspective of foundations in philosophy of mathematics, (M) was read as a strategy of justification. More precisely, there is a very efficient variant of (M), which I call (LM). It ignores the pragmatic, application-oriented aspects and amalgamates (M) with (L) in taking the intensional characterization of objects from the one and the purely logical, formal setup of the other. Sometimes (LM) is called *Hilbertian formalism*, mainly as a standpoint in the so-called foundational crisis of mathematics at the beginning of the 20^{th} century. This name is a bit unlucky, as Hilbert was in much closer connection to (M) than to (LM). Or, as Ewald has put it, he is "persistently misconstrued as a 'formalist'." [1996, 1106]. (LM) presents a kind of theory-stew, or a 'popular eclecticism' — as Michael Otte likes to put it. I think that Heijenoort's distinction, as well as that between (M) and (L), are useful analytical tools to get different things separated. In particular, the logico-analytical viewpoint is indepted to a language-based approach to mathematics. Anyway, to give this argument in more detail would be an issue of its own. Let us return to Hilbert.

3 Hilbertian Axiomatics as Model-Theoretical Axiomatics

Admittedly, Hilbert had formulated a 'strong' project, an axiomatic foundation of pure mathematics. It contained the proof of consistency as well as the completeness of the axiomatic system. The description of this famous programme has yielded Hilbert the label of a formalist. After the results of mathematical logic in the 20ies and 30ies, especially those obtained by Gödel about incompleteness, this project can count as inexecutable.

But those results and the failure of the 'strong' program are not a severe strike against the axiomatic method as such. The pragmatic core of (M) consists in the mathematization of sciences (including mathematics itself). While the ambitions of (L) to justify mathematical theories are indeed questionable, (M) works with a more pluralistic and application-oriented attitude. This fits well to the observation that the failure of the strong project today doesn't matter in mathematical practice (except in the fields that analyze just that failure). Still the axiomatic method counts as modern.

In the remaining chapter, I will examine writings of the 'later' Hilbert, namely his famous lecture "Axiomatisches Denken" from 1917 (published 1918 and quoted according to the English translation of Ewald 1996) and his lectures "Natur und mathematisches Erkennen" from 1919/20 that has appeared not until 1992. I hope to make plausible that Hilbert didn't drop his model-theoretical attitude after the "Foundations of Geometry" (other than Hintikka seems to suggest in [1997, 28]).

In his lecture "Axiomatisches Denken" Hilbert addresses the issue, how mathematical axioms emerge and evolve in analogy to physics (totally independent of the strong project):

> "In the theory of real numbers it is shown that the axiom of measurement — the so-called Archimedean axiom — is independent of all the other arithmetical axioms. As everybody knows, this information is of great significance for geometry; but it seems to me to be of capital interest for physics as well [...] The validity of the Archimedean axiom in nature stands in just as much need of confirmation by experiment as does the familiar proposition about the sum of the angles of a triangle." [Hilbert 1918, 4, quoted according to Ewald 1996, 1110]

One could object easily that mathematics is not concerned with the validity of their axioms in nature. But the crucial point is: *which* axioms are formulated in mathematics and *which* are explored further, is determined by an interplay with intended applications. To avoid a possible physicalistic misinterpretation, I repeat that mathematics can be applied to mathematics itself very well. The point is, that in mathematics one is not exploring *arbitrary* axiomatic systems.

> "One can not speak of such an arbitrariness. Instead it appears that the building of concepts in mathematics is led constantly by intuition and experience ..." [Hilbert 1919/20, 5]

> "An immense reservoir of formal relations is available for cognition. The point is to find such systems of formal relations that are adaptable to the relations found in reality." [ibid., 17, my translations.]

That is, according to Hilbert, the key for the success of axiomatics and there he locates the analogy between mathematics and physics:

> "[...] the contradictions that arise in physical theories are always eliminated by changing the selection of the axioms; the difficulty is to make the selection so that all the observed physical laws are logical consequences of the chosen axioms." [Hilbert 1917, 6, quoted according to Ewald 1996, 1112]

Admittedly, in mathematics no contradictions with physical laws occur and Hilbert considers as an example the paradoxes of set theory. In his view, they were resolved by Zermelo's axiomatization, whose success consisted in a paradigmatical application of the axiomatic method:

> "But in this precarious state of affairs as well, the axiomatic method came to a rescue. By setting up appropriate axioms which in a precise way restricted both the arbitrariness of the definitions of sets and the admissibility of statements about their elements, Zermelo succeeded in developing set theory in such a way that the contradictions disappear, but the scope and applicability of set theory remain the same." [ibid., my emphasize]

Hilbert is expressing here the decisive point, the mentioned pragmatical aspect of (M): systems of axioms emerge and develop further by the influence of intended applications. Insofar the model-theoretical interpretation of axiomatics fits well to what Hilbert has called "axiomatic thinking". By the way, it can be viewed as questionable, whether Hilbert was completely right in claiming that the contradictions of set theory have disappeared. But this does not diminish the significance of the mentioned criteria of successful axiomatization.

4 Conclusion

In my argumentation, the focus was on an application-oriented Hilbert, which is admittedly only one out of a couple of Hilberts, partly existing parallel in time and not necessarily compatible one with the other.

Surely it is not the case that 'Hilbertian formalism' is a philosophical standpoint completely erroneously ascribed to Hilbert. I shall not argue how much 'completely' should be emphasized. Instead, I was arguing for the philosophical significance of Hilbert's model-theoretical standpoint, that was strongly involved with applications of mathematics and the mathematization of the sciences. Foundational aspects are strikingly irrelevant to that kind of approach. My view coincides at this point with Ulrich Majer's, who argued for the independence of the axiomatic and the foundational projects of Hilbert [Majer 2001]. Maybe the availability of Hilbert's yet unpublished lectures will strengthen the claim of the 'model-theoretical' Hilbert in historical respect.

There is a rather fundamental objection against the distinction between logico-analytical and model-theoretical axiomatics. It goes like: a detailed historical study of any author would bring forward that he shows up features from both sides of the distinction. This is a general objection against distinctions of that kind, but it doesn't rule it out as invalid. The model-theoretical viewpoint of axiomatics has high heuristic and systematic value.

As our reading has shown, Hilbert takes applied theories as a *starting point* for the axiomatical method. What is used in practice, shall be provided with a common, and that means more general, basis or foundation. Consequently, he has pointed at the *scope* and *applicability* as criteria of a successful axiomatization, as we have seen in the last quotation.

Finally, I like to push this interpretation a bit further. One of the best-known of Hilbert's metaphors about the axiomatic method is that of "deepening the foundations", also given in his lecture "Axiomatisches Denken":

> "The procedure of the axiomatic method, as is expressed here, amounts to a deepening of the foundations of the individual domains of knowledge — a deepening that is necessary for every edifice that one wishes to expand and to build higher while preserving its stability." [Quote acc. to Ewald 1996, 1109]

Doesn't this confirm the axiomatic method as a foundational program? V. Peckhaus has ascribed Hilbert a "foundational pragmatism" [1999, 5]. In the light of the distinction between a model-theoretical and a logico-analytical standpoint, however, one could switch off the "fundamentalizing filter" [Corfield 2003] and ask whether Hilbert wasn't interested more in questions of development than in foundational aspects.

We saw that Hilbert was starting with applied theories that are to be axiomatically reconstructed, but not validated by being deduced from axioms. I.e., the model-theoretical viewpoint of axiomatics acts on the assumption of a pragmatic pluralism, whose "scope" and "applicability" have to be preserved.

Therefore, deepening the foundations does not aim primarily at a *justification* of the established practice. Just the opposite way round, Hilberts methodology acts on the *assumption* of a plurality of scientific theories, whose justification proceeds according to their own criteria. Taking into account the autonomous character of the theories which are to be axiomatized, one could perhaps speak of a "naturalized" approach to justification.

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