CRITICISM OF BENACERRAF'S CRITICISM OF MODERN ELEATICS

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Abstract. I analyze here Benacerraf's criticism of Thomson arguments on the impossibility of ω-supertasks. Although Benacerraf's criticism is well founded, his analysis of Thomson's lamp is incomplete. In fact, it is possible to consider a new line of argument, which Benacerraf only incidentally considered, based on the functioning laws of the lamp. This argument leads to a contradictory result that compromises the formal consistency of the ω-ordering involved in all ω-supertasks.

1. Introduction

As is well known, to performs an ω-supertask means to perform an ω-ordered sequence of actions (tasks) in a finite interval of time [29], [7], [8], [24]. Supertasks are useful theoretical devices for the philosophy of mathematics [[5], [31], [24]], particularly for the formal discussion of certain problems related to infinity. Although their physical possibilities and implications have also been discussed ([19], [20], [24], [26], [13], [15], [14] ([20], [21], [22], [12], [23], [18], [2], [3], [25] [31], [16], [10], [11], [18], [9], [27]. Probably was Gregory the first in proposing how a supertask could be accomplished ([17], p. 53):

> If God can endlessly add a cubic foot to a stone -which He can- then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and son on *ad infinitum*. He would then have before Him an infinite stone at the end of the hour.

But the term "*supertask*" was introduced by J. F. Thomson in his seminal paper [29] of 1954 . Thomson's paper was motivated by Black's argument on the inconsistency of performing infinitely many actions [6], and the rejections of Black's argument by R. Taylor [28] and J. Watling [30]. In his paper Thomson tried to prove the impossibility of such supertasks. Thomson argument was, in turns, criticized in other seminal paper, in this case by P. Benacerraf [4]. Benacerraf's successful criticism finally motivated the foundation of a new infinitist theory: supertask theory.

The basic idea of Benacerraf's criticism against Thomson argument is the impossibility of deriving any formal consequence on the final state of a supermachine from the ω-ordered succession of states the machine traverses along a supertask. But, as we will see, Benacerraf's analysis of Thomson's lamp is incomplete. In fact, if supertasks do not change the nature of the world, Thomson's argument can be reoriented in order to consider the very nature of the theoretical devices that perform supertasks; a nature which is independent of the number of performed actions and that, consequently, remains before, during and after any supertask.

2. Thomson's Lamp

As Thomson did in 1954, in the following discussion we will deal with one of these ([29], p. 5):

> ... reading-lamps that have a button in the base. If the lamp is *off* and you press the button the lamp goes *on*, and if the lamp is *on* and you press the button the lump goes *off*.

It will be referred to as Thomson's Lamp (TL). Assume now that TL's button is pressed at each one of the countably many instants t_i of any ω-ordered sequence of instants $\langle t_i \rangle_{i\in\mathbb{N}}$ defined within any finite half-closed interval of time $[t_a, t_b)$, for instance the classical one defined in accordance with:

$$t_i = t_a + (t_b - t_a)\sum_{k=1}^{i}\frac{1}{2^k},\ \forall i \in \mathbb{N} \tag{1}$$

whose limit is t_b. In these conditions, at t_b TL will have completed an ω-ordered sequence of switchings $\langle s_i \rangle_{i\in\mathbb{N}}$, i.e. a supertask. Thomson tried to derive a contradiction from this supertask by speculating on the final state of the lamp at instant t_b in terms of the succession of the performed switchings along the supertask ([29], p. 5):

> [The lamp] cannot be *on*, because I did nor ever turn it *on* without at once turning it *off*. It cannot be *off*, because I did in the first place turn it *on*, and thereafter I never turned *off* without at once turning it *on*. But the lamp must be either *on* or *off*. This is a contradiction.

Benacerraf criticized this argument as follows: ([4], p. 768):

> The only reasons Thomson gives for supposing that his lamp will not be *off* at t_a are ones which hold only for

> times *before* t_b. The explanation is quite simply that Thomson's instructions do not cover the state of the lamp at t_b, although they do tell us what will be its state at every instant *between* t_a and t_b (including t_a). Certainly, the lamp must be *on* or *off* (provided that it hasn't gone up in a metaphysical puff of smoke in the interval), but nothing we are told implies which is to be. The arguments to the effect that it can't be either just have no bearing on the case. To suppose that they *do* is to suppose that a description of the physical state of the lamp at t_b (with respect to the property of being of *on* or *off*) is a *logical* consequence of a description of its state (with respect to the same property) at times prior to t_b.
>
> (t_a and t_b appears respectively as t_0 and t_1 in Benacerraf's paper)

In short, the problem possed by Thomson is not sufficiently described since no constraints have been placed on what happens at t_b [1].

3. Criticism of Benacerraf criticism

In the discussion that follows, I will develop a new argument exclusively based on the theoretical definition of Thomson lamp: a reading-lamp that has "a button in the base such that if the lamp is *off* and you press the button the lamp goes *on*, and if the lamp is *on* and you press the button the lump goes *off*". According to Benacerraf ([4], p.770):

> "But [at t_b]the lamp must be either on or off" is striking by it obvious irrelevance.

and ([4], p. 768):

> ... Certainly, the lamp must be *on* or *off* [at t_b],...

which, in fact, has nothing to do with the number of performed switchings but on the nature of the lamp stated in its theoretical definition. The discussion that follows will also be exclusively based on that theoretical definition, particularly on the successiveness of its functioning, i.e. on the fact that TL can successively, but not simultaneously, be turned on and off.

Consider then a (theoretical) Thomson lamp TL. In order to avoid unnecessary discussions we will assume that it is permanently powered by its appropriate *theoretical fuel*. I will symbolize the functioning and states of TL in accordance with the following simple conventions:

- $ON[t]$ denotes that TL is *on* at instant t, while $ON(t_a,\, t_b)$ means that TL is *on* along the real interval $(t_a,\, t_b)$; the same applies to all closed and half closed real intervals.
- $OFF[t]$ denotes that TL is *off* at instant t, while $OFF(t_a,\, t_b)$ means that TL is *off* along the real interval $(t_a,\, t_b)$; the same applies to all closed and half closed real intervals.
- $P[t]^{ON}$ denotes that TL's button has been pressed down just at instant t so that the lamp is turned *on.* $P(t_a,\, t_b)^{ON}$ means that, at least one time in the real interval $(t_a,\, t_b)$, TL's button has been pressed down being the lamp turned *on*; the same applies to all closed and half closed real intervals.
- $P[t]^{OFF}$ and $P(t_a,\, t_b)^{OFF}$ means the same as $P[t]^{ON}$ and $P(t_a,\, t_b)^{ON}$ respectively, except the lump results *off*.

We can now express some elementary TL's laws. All of them exclusively derived from its functioning, which is explicitly stated in its definition, as well as from the strict successiveness of its two states (*on* and *off*). For instance:

- $P^{ON}[\mathrm{t}] \Rightarrow ON[t]$
- $P^{OFF}[\mathrm{t}] \Rightarrow OFF[t]$
- $\neg ON(\leftarrow,\ t) = OFF(\leftarrow,\, t)$[1]
- $ON[t,\, \rightarrow) \Rightarrow \neg P^{OFF}[t,\, \rightarrow)$
- $OFF[t,\, \rightarrow) \Rightarrow \neg P^{ON}[t,\, \rightarrow)$
- $OFF(\leftarrow,\, t) \wedge ON[t] \Rightarrow P^{ON}[t]$
- $P^{OFF}[t,\, \rightarrow)\ \Rightarrow\ \neg ON[t,\, \rightarrow)$
- $P^{OFF}(\leftarrow,\, t]\ \Rightarrow \exists t' \leq t : OFF[t']$
- etc.

Although it is so elementary as the above ones, we will prove now the following functioning law:

$$P^{OFF}(\leftarrow,\, t) \wedge ON[t,\, \rightarrow) \Rightarrow \exists t' \leq t :\ P^{ON}[t'] \wedge \neg P^{OFF}[t',\, \rightarrow) \tag{2}$$

denoting that if TL has been turned *off* at least one time before t, symbolically $P^{OFF}(\leftarrow,\, t)$, and it is *on* from t on, in symbols $ON[t,\, \rightarrow)$, then there must exist an instant t' equal or less than t at which the lamp is turned *on* being no longer turned off: $P^{ON}[t'] \wedge \neg P^{OFF}[t',\, \rightarrow)$).

Proof. Assume first that:

$$\neg \exists t' \leq t :\ P^{ON}[t'] \tag{3}$$

which means that TL has not been turned *on* at t', for any t' equal or less than t, that is to say:

$$\neg P^{ON}(\leftarrow,\, t] \tag{4}$$

[1]As usual, $(\leftarrow,\ t)$ is the interval of all real numbers less than t; similarly $[t,\, \rightarrow)$ is the interval of all reals equal or greater than t.

On the other hand, and according to the first term of the antecedent of (2), TL has been turned *off* at least one time before t, accordingly we can write:

$$P^{OFF}(\leftarrow,\, t) \;\Rightarrow\; \exists t'' < t : OFF[t''] \tag{5}$$

and from (5) and (4):

$$OFF[t''] \;\wedge\; \neg P^{ON}(\leftarrow,\, t] \;\Rightarrow\; OFF[t] \tag{6}$$

But:

$$OFF[t] \;\Rightarrow\; \neg ON[t,\, \leftarrow) \tag{7}$$

and $\neg ON[t,\, \leftarrow)$ goes against the antecedent of (2). Assumption (3) is therefore impossible.
Now assume that:

$$\neg \exists t' \leq t : \neg P^{OFF}[t',\, \rightarrow) \tag{8}$$

This implies that:

$$\forall t' \leq t : P^{OFF}[t',\, \rightarrow) \tag{9}$$

Therefore:

$$P^{OFF}[t,\, \rightarrow) \tag{10}$$

And then:

$$\neg ON[t,\, \rightarrow) \tag{11}$$

which also goes against the antecedent of (2). We can therefore state that if:

$$P^{OFF}(\leftarrow,\, t) \wedge ON[t,\, \rightarrow) \tag{12}$$

then

$$\exists t' \leq t :\; P^{ON}[t'] \wedge \neg P^{OFF}[t',\, \rightarrow) \tag{13}$$

□

A symmetrical argument would prove that:

$$P^{ON}(\leftarrow,\, t) \wedge OFF[t,\, \rightarrow) \Rightarrow \exists t' \leq t :\; P^{OFF}[t'] \wedge \neg P^{ON}[t',\, \rightarrow) \tag{14}$$

We can now reconsider the possibility of performing a supertask with Thomson lamp TL. For this, assume that at each instant t_i of $\langle t_i \rangle_{i \in \mathbb{N}}$ defined according to (1), *and only an them*, TL's button is pressed down so that it is successively turned *on* and *off*. At t_b, the limit of the ω-ordered sequence $\langle t_i \rangle_{i \in \mathbb{N}}$, a supertask will have been performed. According to Benacerraf, at t_b the lamp will be either *on* or *off*. This conclusion does not depend on the number of performed switchings but on being a Thomson lamp with only two states, *on* and *off*, that can successively but non simultaneously be exhibited. Assume that, once completed the supertask, the lamp is *on* (a similar argument could be applied if it were *off*). If no other action is performed with the lamp, we can write:

$$P^{OFF}(\leftarrow,\, t_b) \wedge ON[t_b,\, \rightarrow) \tag{15}$$

denoting that TL has been turned off at least one time before t_b (first term of (15)) and that it remains *on* just from t_b on (second term of (15)). Now then, law (2) states[2]:

$$P^{OFF}(\leftarrow,\, t_b) \wedge ON[t_b,\, \rightarrow) \Rightarrow \exists t' \leq t_b:\ P^{ON}[t'] \wedge \neg P^{OFF}[t',\, \rightarrow)$$

As we have repeatedly said, this law does not depend on the (finite or infinite) number of performed switchings but on being a Thomson lamp (a theoretical object rightfully defined). Consequently, it must hold before, during and after any supertask. We will prove now, however, that it does not hold in the case of our supertask. In fact, let us prove that no t' in the real interval $[t_a,\, t_b]$ satisfies law (2). According to this law it must hold:

$$\exists t' \leq t_b:\ P^{ON}[t'] \wedge \neg P^{OFF}[t',\, \rightarrow) \tag{16}$$

On the one hand, t' cannot be equal to t_b because at t_b no switch is performed, otherwise we would have an $(\omega+1)$-ordered sequence of switchings (a super-duper-task in Benacerraf's words) rather than an ω-ordered one. On the other, it cannot be less than t_b because at any instant prior to t_b, whatsoever it be, only a finite number of switchings has been performed and infinitely many of them remains still to be performed[3]. Thus, no t' in the real interval $[t_a, t_b]$ satisfies $P^{ON}[t'] \wedge \neg P^{OFF}[t',\, \rightarrow)$ and then basic law (2). Notice this is not an indeterminacy but an impossibility: no real number within $[t_a, t_b]$ satisfies the required condition. The only way of performing our supertask with Thomson lamp is, therefore, by violating its own laws of functioning. Our supertask, and then that of Thomson, is in fact formally impossible.

It would be immediate to prove that if in the place of an ω-ordered sequence of switchings only a finite number n of them were performed, law (2) (or in its case law (14)) always holds. The reason is a simple one: in all finite cases there is always a last switching performed at the precise instant t_n (in the case of n switchings, each performed at the precise instant t_i of the finite sequence $\langle t_i \rangle_{1 \leq i \leq n}$) defining the instant t' of laws (2) and (14). We can therefore conclude that for any n in $\mathbb{N}$, to perform n switchings with TL is a consistent performance, while to perform an ω-ordered sequence of them is not. This seems to indicate that being complete and uncompletable[4], could be not only an abuse of language but a formal inconsistency derived from the first transfinite ordinal, which in turn derives from assuming the existence of infinite sets as completed totalities (Axiom of Infinity).

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[2]If TL were off at t_b we would have to consider law (14) instead of law (2).

[3]This unaesthetic and immense asymmetry is invariably forgotten in supertask literature.

[4]ω-ordered sequences have to be complete infinite totalities (as the actual infinity requires) although no last element completes them.

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