HILBERT'S MACHINE AND THE AXIOM OF INFINITY

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Abstract. Hilbert's machine is a supertask machine inspired by Hilbert's Hotel whose functioning leads to a contradiction that compromises the Axiom of Infinity.

1. Hilbert's Machine

In the following conceptual discussion we will make use of a theoretical device that will be referred to as *Hilbert's machine*, composed of the following elements (see Figure 1):

(1) An infinite tape similar to those of Turing machines which is divided in two infinite parts, the left and the right side:
  (a) The right side is divided into an ω-ordered sequence of adjacent cells $\langle c_i \rangle_{i \in \mathbb{N}}$[1] which are indexed from left to right as c_1, c_2, c_3, ... These cells will be referred to as right cells.
  (b) The left side is also divided into an ω-ordered sequence of adjacent cells $\langle c'_i \rangle_{i \in \mathbb{N}}$ indexed now from right to left as c'_1, c'_2, c'_3, ..., being c'_1 adjacent to c_1. These cells will be referred to as left cells.

(2) An ω-ordered sequence of rings $\langle r_i \rangle_{i \in \mathbb{N}}$ being each ring r_i initially placed on the right cell c_i and permanently bound to its successor r_{i+1} by means of a rigid rod of the appropriate length. The rings r_i will be termed Hilbert's rings and the sequence $\langle r_i \rangle_{i \in \mathbb{N}}$ Hilbert's chain.

(3) A multidisplacement mechanism which moves simultaneously all Hilbert's rings one cell to the left, so that the ring placed on $c_{k,\, k>1}$ is placed on c_{k-1}, the one placed on c_1 is placed on c'_1, and the one placed on $c'_{k,\, k \geq 1}$ is placed on c'_{k+1}. This simultaneous displacement of all Hilbert's rings one cell to the left will be termed *multidisplacement*. Multidisplacements are the only actions performed by Hilbert's machine.

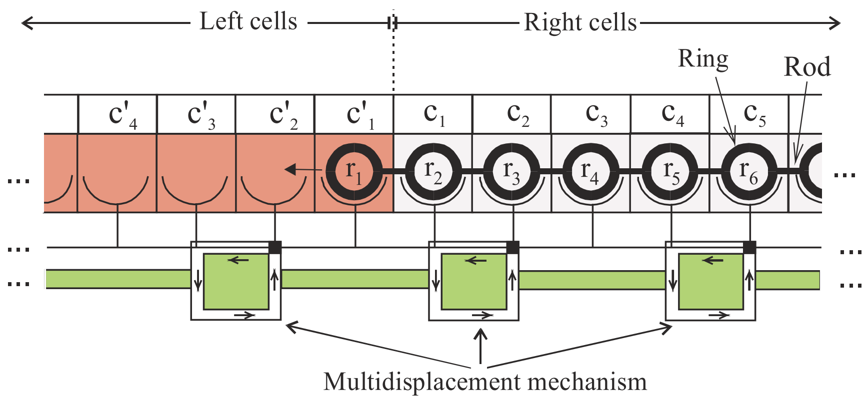

Figure 1. Hilbert's machine just before performing the second multidisplacement.

The functioning of Hilbert's machine is always subjected to the following Hilbert's restriction: the machine will perform a multidisplacement if, and only if, the multidisplacement

[1] As usual, $\mathbb{N}$ stands for the set of natural numbers $\{1, 2, 3, \dots\}$

does not remove any ring from the tape; otherwise the machine halts before performing the multidisplacement. Consequently no ring can be removed from the tape.

Assume now that Hilbert's machine performs a multidisplacement m_i at each one of the countably many instants t_i of any ω-ordered sequence of instants $\langle t_i \rangle_{i\in\mathbb{N}}$ defined into any finite half-closed interval of time $[t_a, t_b)$, for instance the one defined in accordance with:

$$t_i = t_a + (t_b - t_a)\sum_{k=1}^{i} \frac{1}{2^k}, \ \forall i \in \mathbb{N} \tag{1}$$

whose limit is t_b. Accordingly, at t_b our machine will have completed the performance of an ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, i.e. a supertask. As is usual in supertask theory (see [5], [11], [12], [14], [15], [18], [10], etc.) we will assume that multidisplacements are instantaneous. Although it is irrelevant to our conceptual discussion, we could also assume that multidisplacements last a finite amount of time, for instance each m_i could take a time $1/(2^{i+1})$. It seems appropriate at this point to emphasize the conceptual nature of the discussion that follows. Here we are not interested in discussing the problems derived from the actual performance of supertasks in our physical universe, as would be the case of the relativistic restrictions on the speed of the multidisplacements and the like ([8], [9], [14], [15], [16], [10], [17], [13], [1], [2], [19], [4], [3], etc.). We will assume, therefore, that Hilbert's machine works in a conceptual universe in which no physical restriction applies to its functioning.

2. Performing the supertask

Consider the ω-ordered sequence of instants $\langle t_i \rangle_{i\in\mathbb{N}}$ defined according to (1), and a Hilbert's machine in the following initial conditions:

(1) At t_a the machine is at rest.
(2) At t_a each Hilbert's ring r_i is on the right cell c_i.
(3) At t_a each left cell c'_i is empty.

Assume that, if Hilbert's restriction allows it, this machine performs exactly one multidisplacement m_i at each one of the countably many instants t_i of $\langle t_i \rangle_{i\in\mathbb{N}}$, and only at them, being those successive multidisplacements the only performed actions. The objective of the following discussion is to analyze the performance of this ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$.

We begin by proving the following two basic propositions which are directly derived from the notion of ω-order, i.e. from assuming the existence, as *complete totalities*, of sequences in which there exist a first element and each element has a successor:

Proposition 1. *The ω-ordering makes it possible that all multidisplacements m_i of the ω-ordered sequence $\langle m_i \rangle_{i\in\mathbb{N}}$ observe Hilbert's restriction.*

Proof. It is evident that the first multidisplacement m_1 observes Hilbert's restriction (in fact, it places Hilbert's ring r_1 on the cell c'_1 and each $r_{i,\, i>1}$ on c_{i-1}). Assume the first n multidisplacements observe Hilbert's restriction. In these conditions, if m_{n+1} would not observe Hilbert's restriction, the ring r_1 would be removed from the tape by m_{n+1}. But this is impossible because, as a consequence of having performed the first n multidisplacements, and taking into account that each multidisplacement moves all Hilbert's rings one cell to the left, the ring r_1 has to be placed on the left cell c'_n, and in accordance with the

ω-order this cell has an adjacent cell to the left, the left cell c'_{n+1}, on which r_1 will be placed by m_{n+1}. Consequently m_{n+1} also observes Hilbert's restriction. We have just proved that m_1 observes Hilbert's restriction, and that if the first n multidisplacements observe Hilbert's restriction, then m_{n+1} also observes Hilbert's restriction. Therefore, all multidisplacements observe Hilbert's restriction.

□

Proposition 2. *At t_b the performance of the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$ has been completed.*

Proof. According to Proposition 1 all multidisplacements m_i observe Hilbert's restriction and therefore all of them can be performed by Hilbert's machine. Let us now prove that at t_b all of them have already been carried out. For this, consider the one to one correspondence f between $\langle t_i \rangle_{i\in\mathbb{N}}$ and $\langle m_i \rangle_{i\in\mathbb{N}}$ defined by:

$$f(t_i) = m_i, \ \forall i \in \mathbb{N} \tag{2}$$

Being t_b the limit of the ω-ordered sequence $\langle t_i \rangle_{i\in\mathbb{N}}$, and taking into account that by definition each multidisplacement m_i takes place just at the precise instant t_i, the above one to one correspondence f, together with the assumed completeness of the involved ω-ordered sequences, ensure that at t_b all multidisplacements m_i of $\langle m_i \rangle_{i\in\mathbb{N}}$ have already been carried out. Therefore at t_b the performance of the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$ has been completed.

□

From the above propositions we now derive the following two auxiliary results:

Proposition 3. *Each multidisplacement m_i of $\langle m_i \rangle_{i\in\mathbb{N}}$ places the ring r_i on the first left cell c'_1 at the precise instant t_i.*

Proof. According to Proposition 2, Hilbert's machine completes the performance of the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$. In these conditions, let r_i be any Hilbert's ring. According to the initial conditions, r_i was initially placed on the right cell c_i. Taking into account that each multidisplacement moves each Hilbert's ring exactly one cell to the left, after the firsts $i-1$ multidisplacements the ring r_i will be placed just on the right cell $c_{i-(i-1)} = c_1$, and then the next multidisplacement m_i will place it on the first left cell c'_1. In consequence each multidisplacement m_i of $\langle m_i \rangle_{i\in\mathbb{N}}$ places the ring r_i on the first left cell c'_1 at the precise instant t_i.

□

Proposition 4. *Each Hilbert's ring r_i is placed on the left cell c'_k by the multidisplacement m_{i+k-1}, being c'_k any left cell.*

Proof. Let r_i be any Hilbert's ring and c'_k any left cell. According to Proposition 3 Hilbert's ring r_i is placed on the first left cell c'_1 by the multidisplacement m_i. Since each multidisplacement moves each ring one cell to the left, the next $k-1$ multidisplacements, which are all of them performed (Proposition 2), place the ring r_i on the left cell $c'_{1+(k-1)} = c'_k$. Therefore each ring r_i of $\langle r_i \rangle_{i\in\mathbb{N}}$ is placed on the left cell c'_k by the multidisplacement m_{i+k-1}, where c'_k is any left cell.

□

We can now derive the following two contradictory propositions:

Proposition 5. *Once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, and being those multidisplacements the only performed actions, all Hilbert's rings are in the left side of the tape.*

Proof 1. At t_b all multidisplacements have been performed (Proposition 2) and no ring have been removed from the tape (Hilbert's restriction). Consequently, we only have to prove that once completed $\langle m_i \rangle_{i\in\mathbb{N}}$, and being $\langle m_i \rangle_{i\in\mathbb{N}}$ the only performed actions, all Hilbert's rings $\langle r_i \rangle_{i\in\mathbb{N}}$ are placed in the left side of the tape. For this, consider any Hilbert's ring r_i. According to Proposition 3 the multidisplacement m_i places r_i on the first left cell c'_1 at t_i. Since all subsequent multidisplacements $m_{n,\, n>i}$ move r_i one cell to the left, r_i will remain on the left side of the tape from the performance of m_i. Consequently, the one to one correspondence f between $\langle m_i \rangle_{i\in\mathbb{N}}$ and $\langle r_i \rangle_{i\in\mathbb{N}}$ defined by:

$$f(m_i) = r_i,\ \forall i \in \mathbb{N} \tag{3}$$

proves that once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, and being those multidisplacements the only performed actions, all Hilbert's rings are in the left side of the tape.

□

Proof 2. According to Proposition 3 the multidisplacement m_1 places the ring r_1 on the first left cell c'_1 at t_1. Consequently, and taking into account that each multidisplacement moves all Hilbert's rings one cell to the left, the ring r_1 is in the left side of the tape from the performance of m_1. Thus, once completed $\langle m_i \rangle_{i\in\mathbb{N}}$, and being $\langle m_i \rangle_{i\in\mathbb{N}}$ the only performed actions, r_1 is in the left side of the tape. Assume now that, once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, and being those multidisplacements the only performed actions, the firsts n Hilbert's rings are in the left side of the tape. In these precise conditions, if r_{n+1} were not in the left side of the tape it would have to be on the first right cell c_1 and then none of the multidisplacements $m_{i,\, i\geq n+1}$ would have been carried out because m_{n+1} puts r_{n+1} on the first left cell c'_1 (Proposition 3) and all subsequent multidisplacements move r_{n+1} one cell to the left. But we know all $m_{i,\, i\geq n+1}$ have been carried out (Proposition 2), so it is impossible that in those conditions r_{n+1} be in the right side of the tape; it must also be in the left side of the tape.

We have proved that, once completed $\langle m_i \rangle_{i\in\mathbb{N}}$, and being $\langle m_i \rangle_{i\in\mathbb{N}}$ the only performed actions, the first Hilbert's ring r_1 is in the left side of the tape, and that if the first n Hilbert's rings are in the left side of the tape, then r_{n+1} is also in the left side of the tape. Therefore, once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, and being those multidisplacements the only performed actions, all Hilbert's rings are in the left side of the tape.

□

Proposition 6. *Once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, and being those multidisplacements the only performed actions, no Hilbert's ring is in the left side of the tape.*

Proof 1. Once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i\in\mathbb{N}}$, and being those multidisplacements the only performed actions, all Hilbert's rings are in the left side of the tape (Proposition 5). In these conditions and being c'_1 the first left cell, if c'_1 would contain a ring this ring would have to be the last ring of Hilbert's chain. But evidently there is not a last ring in the ω-ordered sequence $\langle r_i \rangle_{i\in\mathbb{N}}$, so c'_1 must be empty. Let us now assume that, once completed $\langle m_i \rangle_{i\in\mathbb{N}}$, and being $\langle m_i \rangle_{i\in\mathbb{N}}$ the only performed actions, the firsts n left cells are empty. In these precise conditions, if the next left cell c'_{n+1} were not empty, then it would have to contain the impossible last Hilbert's ring, so it has also to be empty.

We have proved that, once completed $\langle m_i \rangle_{i \in \mathbb{N}}$, and being $\langle m_i \rangle_{i \in \mathbb{N}}$ the only performed actions, the first left cell c'_1 is empty, and that if the first n left cells are empty then the next left cell c'_{n+1} is also empty. Therefore, once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$, and being those multidisplacements the only performed actions, all left cells $\langle c'_i \rangle_{i \in \mathbb{N}}$ are empty, and then no Hilbert's ring is in the left side of the tape.

□

Proof 2. Let c'_k be any left cell. Assume that once completed $\langle m_i \rangle_{i \in \mathbb{N}}$, and being $\langle m_i \rangle_{i \in \mathbb{N}}$ the only performed actions, c'_k contains any Hilbert's ring r_i. According to Proposition 4 the ring r_i was placed on the left cell c'_k by the multidisplacement m_{i+k-1}. Therefore, and taking into account that each multidisplacement moves r_i exactly one cell to the left, if once completed $\langle m_i \rangle_{i \in \mathbb{N}}$ and being $\langle m_i \rangle_{i \in \mathbb{N}}$ the only performed actions r_i were on c_k none of the multidisplacements $m_{n,\, n \geq (i+k)}$ would have been carried out, which contradicts the fact that all of them have been carried out (and Proposition 2 confirms). Therefore, once completed $\langle m_i \rangle_{i \in \mathbb{N}}$, and being $\langle m_i \rangle_{i \in \mathbb{N}}$ the only performed actions, the left cell c'_k contains no Hilbert's ring. Consequently, and taking into account that c'_k is any left cell, once completed $\langle m_i \rangle_{i \in \mathbb{N}}$, and being $\langle m_i \rangle_{i \in \mathbb{N}}$ the only performed actions, no Hilbert's ring is in the left side of the tape.

□

3. Consequences

We have just proved that once completed the ω-ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ and being $\langle m_i \rangle_{i \in \mathbb{N}}$ the only performed actions, Hilbert's chain is and is not in the left side of the tape[2]. Obviously, Hilbert's machine is a conceptual device whose theoretical existence and functioning is only possible under the consideration of the ω-order that legitimates the ω-ordered sequences $\langle c_i \rangle_{i \in \mathbb{N}}$, $\langle c'_i \rangle_{i \in \mathbb{N}}$, $\langle r_i \rangle_{i \in \mathbb{N}}$, $\langle m_i \rangle_{i \in \mathbb{N}}$ and $\langle t_i \rangle_{i \in \mathbb{N}}$ as complete totalities. Furthermore, the contradictory Propositions 5 and 6 are formal consequences of Proposition 1, which in turn is a formal consequence of the ω-order. It is, therefore, the ω-order the cause of the contradiction between Propositions 5 and 6.

We will come to the same conclusion on the inconsistency of the ω-order by comparing the consequences of the functioning of the above infinite Hilbert's machine (symbolically H_ω) with the functioning of any finite Hilbert machine H_n with a finite number n of both right and left cells; being, as in the case of H_ω, a Hilbert's chain of n rings placed on the right side of the tape, each ring r_i on the cell c_i (see Figure 2). In effect, it is immediate to prove that, according to Hilbert's restriction, H_n can only perform n multidisplacements because the $(n+1)$-th multidisplacement would remove from the tape the ring r_1 initially placed on the first right cell c_1 and placed on the last left cell c'_n by the multidisplacement m_n. Thus m_{n+1} does not observe Hilbert restriction. In these conditions it is impossible to derive Proposition 6 because once performed the n-th multidisplacement, and due to Hilbert's restriction, the machine H_n halts with each left cell c'_i occupied by the ring r_{n-i+1} and all right cells empty. Thus for any natural number n, H_n is consistent. Only the infinite Hilbert's machine H_ω is inconsistent. Consequently, and taking into account

[2]Although we will not do it here, it is possible to derive other contradictory results from the functioning of Hilbert's machine

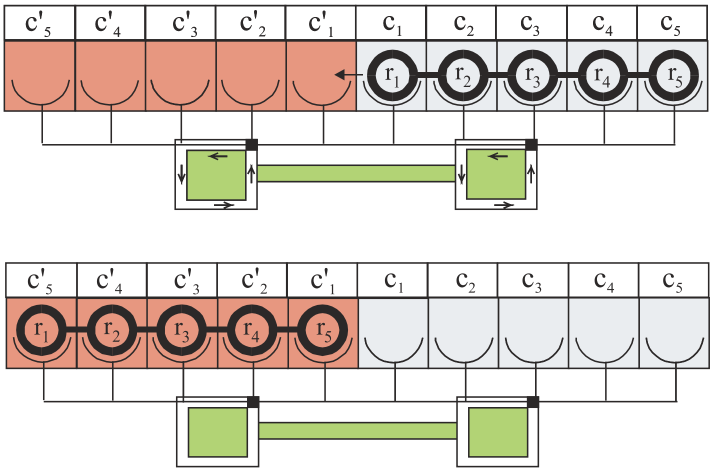

FIGURE 2. The finite Hilbert's machine H_5 can only performs five multidisplacements as a consequence of which the five rings of Hilbert's chain will be moved from the right to the left side of the tape and then machine will halt due to Hilbert's restriction.

that the ω-order is the only difference between H_ω and $H_{n,\,\forall n \in \mathbb{N}}$, only the ω-order can be the cause of the inconsistency of H_ω.

We should not be surprised by this conclusion on the inconsistency of the ω-order. After all, an ω-ordered sequence is one which is both complete (as the actual infinity requires) and uncompletable (because there is not a last element which completes it). To be simultaneously complete and uncompletable seems not to be an acceptable formal status. On the other hand, and as Cantor proved [6], [7], the ω-order is an inevitable consequence of assuming the existence of denumerable complete totalities. An existence which is solemnly stated in our days by the Axiom of Infinity, in both ZFC and BNG axiomatic set theories. It is therefore that axiom the ultimate cause of the contradiction between Propositions 5 and 6.

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