ON BIFURCATED SUPERTASKS AND RELATED QUESTIONS

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Abstract. Bifurcated supertasks entail the actual infinite division of time (accelerated system of reference) as well as the existence of half-curves of infinite length (supertask system of reference). This paper analyzes both issues from a critique perspective. It also analyzes a conflictive case of hypercomputation performed by means of a bifurcated supertask. The results of these analyzes suggest the convenience of reviewing certain foundational aspects of infinitist theories.

1. ω-Order and ω-Asymmetry

At the beginning of the second half of the XX century, the discussions on the possibilities to perform infinitely many actions in a finite interval of time (a supertask according to J. F. Thomson [50]) promoted new discussions on certain classical problems related to infinity [9], [54], [50], [51], [8]. The possibilities to perform an uncountable infinitude of actions were examined, and ruled out, by P. Clark and S. Read [12]. Supertasks have also been considered from the perspective of nonstandard analysis [34], [33], [1], [31], although the possibilities to perform an hypertask along an hyperreal interval of time have not been discussed, despite that finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals (hyperfinite partitions) [48], [21], [29], [27], etc. But most supertasks are ω-supertasks, i.e. ω-ordered sequences of actions performed (or perceived as performed) in a finite interval of time. Next paragraphs **1-1**/**1-5** resume the corresponding definitions.

1-1. The first transfinite ordinal[1] ω is the less ordinal greater than all finite ordinals. It defines a type of well order called ω-order: a set is ω-ordered if it has a first element and every element has an immediate successor[2]. In consequence there is not last element in an ω-ordered set. The set $\mathbb{N}$ of natural numbers in its natural order of precedence is a well known example of ω-ordered set.

[1]Transfinite ordinals are the ordinals of the second class according to Cantor classical terminology [11]. An ordinal of the second class is of the second kind if, as ω, it is the limit of an infinite sequence of ordinals; it is of the first kind if it is of the form $\alpha + n$, where α is an ordinal of the second class second kind and n a finite ordinal.

[2]Between an element and its immediate successor no other element of the sequence exists.

1-2. ω^*-Order is the symmetrical reflection of ω-order : a set or sequence is ω^*-ordered if it has a last element and each element has an immediate predecessor, in consequence there is not first element:

$$\overbrace{\ldots t_{3^*}, t_{2^*}, t_{1^*},}^{\omega^*-order} \Big| \overbrace{t_1, t_2, t_3, \ldots}^{\omega-order} \tag{I}$$

where $1^*, 2^*, 3^*, \ldots$ means last, last but one, last but two, etc.

1-3. As Cantor proved [11], ω-order is a formal consequence of the actual infinity, i.e. of assuming the existence of infinite sets as complete totalities. Notice that ω-ordered sequences are completed (as the actual infinity requires) and *uncompletable* (in the sense that no last element completes them).

1-4. Supertask theory[3] is founded on the assumption of the actual infinite division of time. Particularly on the existence of ω-ordered sequences of successive instants within any finite interval of time.

1-5. The arguments that follow will exclusively deal with ω-supertasks, i.e. with ω-ordered sequences of actions performed in a finite interval time, or observed as performed in a finite interval of time from a system of reference conveniently accelerated with respect to the supertask system of reference (bifurcated supertasks[4]). In this case supertasks could take an infinite amount of their proper time.

1-6. Next paragraphs **1-7**/**1-10** introduce ω-asymmetry.

1-7. In accordance with the definition of ω-order given in **1-1**, every element of an ω-ordered set has a finite number of predecessors and an infinite number of successors. This immense asymmetry in the number of predecessors and successors (ω-asymmetry) is a well known fact, although it is usually ignored in infinitist literature, particularly in supertask literature.

1-8. Let S be a supertask whose infinitely many actions $\langle a_i \rangle_{i\in\mathbb{N}}$ are performed at the infinitely many successive instants of the ω-ordered sequence of instants $\langle t_i \rangle_{i\in\mathbb{N}}$; each action a_i performed at the precise instant t_i. Being $\langle t_i \rangle_{i\in\mathbb{N}}$ strictly increasing and upper bounded by the finite duration of the supertask, the sequence $\langle t_i \rangle_{i\in\mathbb{N}}$ has a finite limit t_b.

[3] See for instance [10], [12], [40]
[4] See for instance [17], [16], [32]

1-9. The limit t_b is the first instant at which all actions $\langle a_i \rangle_{i\in\mathbb{N}}$ have already been performed. As a consequence of the ω-asymmetry, at any instant t before t_b and arbitrarily close to it, only a finite number of tasks will have been performed and infinitely many of them remain still to be performed.

1-10. To grasp the colossal magnitude of the above ω-asymmetry, assume the interval $[t_1,\ t_b]$ is trillions of times greater than the age of the universe and consider an interval of time $\tau = 0.000\ldots001$ seconds so small that we would need trillions and trillions of standard pages to write all its zeroes between the decimal point and the final digit 1, a number of pages so huge that the whole visible universe[5] would not have sufficient room for all of them; well, only a finite number of tasks will have been preformed during the trillions of years elapsed between t_1 and $t_b - \tau$ while infinitely many tasks, practically all of them, will have to be performed just in our unimaginably small interval of time τ. Thus, rather than anaesthetic, ω-asymmetry is repulsive.

1-11. In the following paragraphs **1-12**/**1-16** it will be proved that ω-asymmetry produces dichotomies of the *all or nothing* type.

1-12. Consider any finite interval of time $[t_a, t_b]$ and within it two sequences of instants: the ω^*-ordered sequence of Z^*-instants:

$$\langle t_i^* \rangle :\ t_i^* = t_a + \frac{1}{2^i},\ \forall i \in \mathbb{N} \tag{II}$$

and the ω-ordered sequence of Z-instants:

$$\langle t_i \rangle :\ t_i = t_a + \frac{2^i - 1}{2^i}(t_b - t_a),\ \forall i \in \mathbb{N} \tag{III}$$

1-13. We will examine the way the successive Z^*-instants $\langle t_i^* \rangle_{i\in\mathbb{N}}$ and Z-instants $\langle t_i \rangle_{i\in\mathbb{N}}$ elapse as time passes from t_a to t_b, for this we will make use of the two following functions:

$$f^*(t) = \text{number of } Z^*\text{-instants elapsed at } t, \forall t \in [t_a, t_b] \tag{IV}$$

$$f(t) = \text{number } Z\text{-instants not elapsed at } t, \forall t \in [t_a, t_b] \tag{V}$$

1-14. In accordance with the definitions of ω^*-order and ω-order we can write:

$$f^*(t) = \begin{cases} 0 \text{ if t} = t_a \\ \aleph_0 \text{ if } t > t_a \end{cases} \qquad f(t) = \begin{cases} \aleph_0 \text{ if } t < t_b \\ 0 \text{ if } t = t_b \end{cases} \tag{VI}$$

Otherwise, if being n any finite natural number, an instant t would exist such that $f^*(t) = n$ or $f(t) = n$, then there would also exist the impossible first n elements of an ω^*-ordered sequence or the impossible last n elements of an ω-ordered sequence.

[5] A sphere of 93000 billions light years.

1-15. According to **1-14** , functions f^* and f are well defined for every t in $[t_a, t_b]$; they map the interval $[t_a, t_b]$ to the set of two elements $\{0, \aleph_0\}$:

$$f^* : \quad [t_a, t_b] \mapsto \{0, \aleph_0\} \qquad \text{(VII)}$$

$$f \quad : \quad [t_a, t_b] \mapsto \{0, \aleph_0\} \qquad \text{(VIII)}$$

1-16. Function f^* defines, therefore, a dichotomy, the Z^*-dichotomy:

- Regarding the number of Z^*-instants elapsed when time passes from t_a to t_b only two values are possible: 0 and $\aleph_0$.

In its turn, function f also defines a dichotomy, the Z-dichotomy:

- Regarding the number of Z-instants not elapsed when time passes from t_a to t_b only two values are possible: $\aleph_0$ and 0.

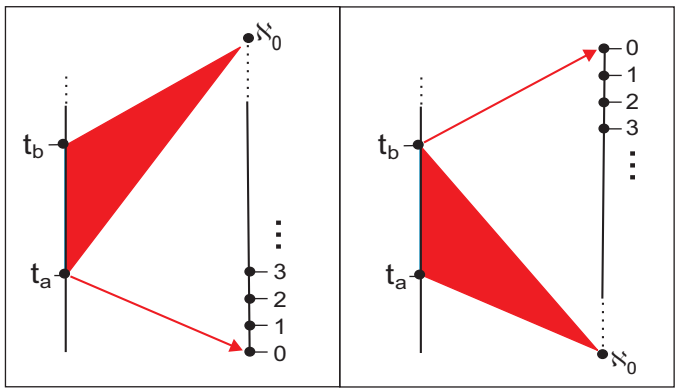

FIGURE 1. Z^*-Dichotomy (left) and Z-Dichotomy (right)

1-17. With respect to the number of Z^*-instants elapsed from t_a, the passing of time from t_a to t_b can only exhibit two states: the state $T^*(0)$ at which no Z^*-instant has elapsed, and the state $T^*(\aleph_0)$ at which infinitely many Z^*-instants have elapsed; without intermediate finite states $T^*(n)$ at which only a finite number n of Z^*-instants have elapsed; the passing of time becomes $T^*(\aleph_0)$ *directly* from $T^*(0)$. Similarly, with respect to the number of Z-instants not elapsed, the passing of time from t_a to t_b can only exhibit two states: $T(\aleph_0)$ and $T(0)$; without intermediate finite states $T(n)$ at which only a finite number n of Z-instants have to elapse; the passing of time becomes $T(0)$ *directly* from $T(\aleph_0)$.

1-18. Paragraphs **1-19**/**1-26** will finally prove that Z^*-dichotomy and Z-dichotomy lead to contradictions involving the assumption of the actual infinite division of time.

1-19. Let us examine the duration of the transitions:

$$T^*(0) \to T^*(\aleph_0) \tag{IX}$$

$$T(\aleph_0) \to T(0) \tag{X}$$

According to (VI) the number of Z^*-instants elapsed from t_a and the number of Z-instants not elapsed from t_a are well defined along the whole interval $[t_a, t_b]$. On the other hand, both transitions must take place within the same interval $[t_a, t_b]$.

1-20. Although the real interval $[t_a, t_b]$ is densely ordered the sequences $\langle t_i^* \rangle_{i\in\mathbb{N}}$ and $\langle t_i \rangle_{i\in\mathbb{N}}$ are not, these sequences are ω^*-ordered and ω-ordered respectively, which means that Z^*-instants and Z-instants are strictly successive, i.e. between any Z^*-instant and its immediate successor no other Z^*-instant exists, and the same applies to Z-instants. Thus, Z^* instants and Z-instants can only elapse successively, and in such a way that between any two of those successive instants a time greater than zero always passes. In consequence, the number of Z^*-instants elapsed from t_a can only increase *one by one*, from 0 to $\aleph_0$. The same applies to the way Z-instants decreases from $\aleph_0$ to 0. This way of elapsing will be capital in the subsequent discussion.

1-21. As a consequence of the Z^*-dichotomy the number of Z^*-instants elapsed from t_a must increase one by one from 0 to $\aleph_0$ without traversing the increasing sequence of natural numbers 1, 2, 3, ... Analogously, the number of Z-instants must decrease one by one from $\aleph_0$ to 0 without traversing the decreasing sequence of natural numbers ..., 3, 2, 1. This seems rather impossible, and in fact we will prove it is.

1-22. The duration of the transition $T^*(0) \to T^*(\aleph_0)$ is, according to **1-20**, the interval of time within $[t_a, t_b]$ during which the number of Z^*-instants elapsed from t_a increases *one by one* from zero to $\aleph_0$. Similarly, the duration of the transition $T(\aleph_0) \to T(0)$ is the interval of time within $[t_a, t_b]$ during which the number of Z-instants not elapsed from t_a decreases *one by one* from $\aleph_0$ to zero.

1-23. Assume the transition $T^*(0) \to T^*(\aleph_0)$ lasts a time t, being t any positive real number. Let t' be any instant within $(0, t)$. If the number of Z^*-instants elapsed at $t_a + t'$ were 0 then the transition would not have begun at $t_a + t'$ and its duration would be less than t; if that number were $\aleph_0$ the transition would have finished at $t_a + t'$ and its duration would be less than t. But 0 and $\aleph_0$ are the only possible values for the number of Z^*-instants elapsed from t_a. In consequence the duration of $T^*(0) \to T^*(\aleph_0)$ is less than t. And being t any real number greater than 0, we must conclude the duration of $T^*(0) \to T^*(\aleph_0)$ is less than any real number greater than zero. Or in other words, it lasts a null time.

1-24. An argument similar to **1-23** proves the transition $T(\aleph_0) \rightarrow T(0)$ must also be instantaneous. It could be argued that the transition $T(\aleph_0) \rightarrow T(0)$ lasts a time t_b - t_a but this is impossible because being t any instant within $(0, t_b - t_a)$, at $t_a + t$ the number of Z-instants not elapsed from t_a is $\aleph_0$ and then the transition $T(\aleph_0) \rightarrow T(0)$ has not begun, in consequence it last an amount of time less than $t_b - t_a$. The same applies to any real number greater than zero.

1-25. According to **1-23** and **1-24**, infinitely many successive Z^*-instants and infinitely many successive Z-instants have to simultaneously elapse when time passes from t_a to t_b; but this is impossible because successive instants cannot elapse simultaneously: between any two of those successive instants a finite interval of time greater than zero always passes. Thus, indeterminable as they may be, the duration of the transitions $T^*(0) \rightarrow T^*(\aleph_0)$ and $T(\aleph_0) \rightarrow T(0)$ must be greater than zero, but they cannot be greater than zero (**1-23**/**1-24**). We have therefore two contradictions proving the impossibility of dividing any finite interval of time into an actual infinitude of ω^*-ordered parts and into an actual infinitude of ω-ordered parts (see Z-Clock in Figure 2).

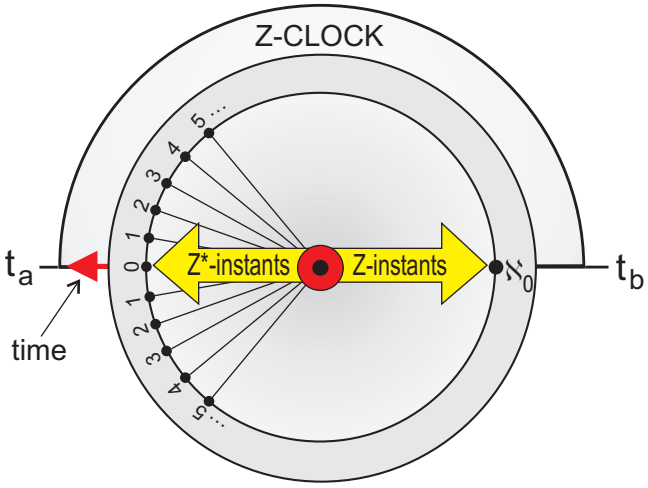

FIGURE 2. As time passes from t_a to t_b the arrow of Z^*-instants must turn clockwise from 0 to $\aleph_0$ without passing over the successive radius 1, 2, 3, ... At the same time the arrow of Z-instants must turn clockwise from $\aleph_0$ to 0 without passing over the successive radius ...3, 2, 1. This Z-Clock is so impossible as the actual infinite division of time.

1-26. Any infinite division (or more correctly partition) of time has to be α-ordered, being α an ordinal of the second class (first or second kind). Thus, we will have:

$$\alpha = \omega \tag{XI}$$

or (XII)

$$\alpha = \omega + \beta \tag{XIII}$$

where β is an ordinal or the second class (first or second kind). In consequence, any transfinite partition of time has to contain at least an impossible ω-ordered partition. Time is not, therefore, infinitely divisible.

1-27. If in the place of the passage of time and the sequences of Z^*-instants and Z-instants we would have considered the uniform linear motion of a particle traversing the Z^* points $\langle z_i^* \rangle_{i \in \mathbb{N}}$ and Z-points $\langle z_i \rangle_{i \in \mathbb{N}}$ defined within the real interval $[0, 1]$ of the real line as:

$$\langle z_i^* \rangle : \ z_i^* = \frac{1}{2^i}, \ \forall i \in \mathbb{N} \tag{XIV}$$

$$\langle z_i \rangle : \ z_i = \frac{2^i - 1}{2^i}, \ \forall i \in \mathbb{N} \tag{XV}$$

We would have come to the same conclusion on the infinite divisibility of space.

1-28. One of the first philosophical consequences of the actual infinity was the believing in the infinite divisibility of everything that could be divided, and in the actual existence of the infinitely many resulting parts. That was the case for space, time, matter, and energy. The obstinacy of facts proved, however, that matter and energy were not infinitely divisible (elementary particles and quanta respectively). From different areas of physics it is now being suggested that space and time could also be of a quantum nature [22], [23], [52], [19], [46], [6], [47], [49], [7], [30], [7], [49]. The above conclusions on the impossible infinite division of both space and time support that suggestion.

2. Curves of infinite length

Supertask theory has recently turned its attention towards the discussion of the physical plausibility of supertasks as well as towards the implications of supertasks in the physical world including relativistic and quantum mechanics aspects.[6]. The actual performance of supertasks frequently implies the pathological behavior of the physical world, but in the place of questioning the formal consistency of the pathogene, infinitism prefers to accept all those unbelievable, and never observed, pathologies. In spite of those pathologies and in spite of the above conclusion **2-26** on the infinite divisibility of time, we will assume that supertasks are formally possible after all. In these conditions we will examine the formal consistency of infinite length curves, which are necessary theoretical devices for bifurcated supertasks.

[6]See [42], [36], [40], [43], [24], [26], [25], [37], [38], [18], [39], [35], [2], [3], [41], [53], [28], [16], [17], [15], [45], [5], [4], [32]

2-1. A Malament-Hogarth spacetime is a relativist model in which supertasks could be actually performed [28]. The model assumes the existence of future-directed timelike half-curves γ of infinite length:

$$\int_{\gamma} dt = \infty \tag{XVI}$$

Next paragraphs examine the possibilities for a curve to have an infinite length

2-2. Let $\mathcal{C}$ be a Jordan curve[7] and assume its length is infinite in any appropriate metric g. Let x_0 be any point on $\mathcal{C}$, and r any positive real number. Consider now the partition $\langle [x_{i-1},\ x_i)\rangle_{i\in\mathbb{I}}$ of $\mathcal{C}$ defined clockwise from x_0 in accordance with (Figure 3):

$$g(x_i, x_{i-1}) = r;\ \forall\ [x_{i-1},\ x_i) \tag{XVII}$$

except the last part[8] whose g-length could be less than r. The set of indexes $\mathbb{I}$ has to be α-ordered, being α a transfinite ordinal. Otherwise, if the ordinal of $\mathbb{I}$ were finite then, and taking into account that every part $[x_{i-1}, x_i)$ has a finite g-length r, the g-length of $\mathcal{C}$ would also be finite.

2-3. Consider a point y on $\mathcal{C}$ anticlockwise from x_0 and at a g-distance of $r/2$ from it. This point has to belong either to the last or to the last but one part of $\langle [x_{i-1},\ x_i)\rangle_{i\in\mathcal{I}}$. In consequence the ordinal of $\mathbb{I}$ has to be of the second class first kind; i.e. an ordinal of the form $\alpha + n$, where α is an ordinal of the second class second kind, and n a finite ordinal. Consider then the point z on $\mathcal{C}$ anticlockwise from x_α and at a g-distance from it of $r/2$. This point would have to belong to $[x_{\alpha-1},\ x_\alpha)$, but this part is simply impossible because α is an ordinal of the second class second kind, i.e. the limit of an infinite sequence of ordinals, and then one whose immediate predecessor $x_{\alpha-1}$ does not exist.

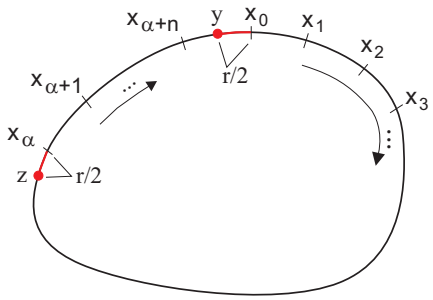

FIGURE 3. The impossible transfinite partition of a Jordan curve of infinite length.

[7] A Jordan curve is a plane closed curve which is topologically equivalent to a circle.

[8] Being C closed, x_0 is the start and the end point of the partition, which means it has a first and a last part.

2-4. According to **2-3** it is impossible to divide $\mathcal{C}$ into infinitely many finite parts of the same g-length r, being r any real number greater than zero and g any appropriate metric. Only finite partitions $\langle [x_{i-1},\ x_i)\rangle_{i=1,2,3,\ldots n}$ of the same g-length are possible, even if the g-length of the parts is arbitrarily small, which is absurd if the curve has an infinite g-length. It is therefore impossible that a Jordan curve has an infinite g-length.

2-5. Let now $\mathcal{C}$ be any open curve and assume its g-length is infinite. Let P be any point on $\mathcal{C}$ and r any positive real number. Consider the following partitions of the right and the left sides of $\mathcal{C}$ defined from P (Figure 4, a):

$$\text{Right side: } \langle [x_i,\ x_{i+1})\rangle_{i\in\mathbb{I}} :\ g(x_{i+1}, x_i) = r,\ \forall i \in \mathbb{I} \qquad \text{(XVIII)}$$

$$\text{Left side: } \langle [y_j,\ y_{j+1})\rangle_{j\in\mathbb{J}} :\ g(y_{j+1}, y_j) = r,\ \forall j \in \mathbb{J} \qquad \text{(XIX)}$$

where at least one of the sets of indexes $\mathbb{I}$ or $\mathbb{J}$ has to be α-ordered, being α a transfinite ordinal.

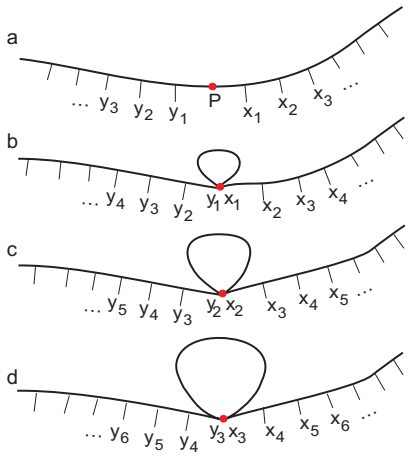

FIGURE 4. Transfinite partitions of both the right and the left side defined from an arbitrary point P on a curve of infinite g-length (a). The curve is then folded by successively joining the successive points of both partitions (b, c, and d).

2-6. $\mathcal{C}$ can be folded by successively joining the corresponding successive points of both partitions, as Figure 4 (b, c, d) shows:

$$y_1x_1,\ y_2x_2,\ y_3x_3,\ \ldots \qquad \text{(XX)}$$

If one of the partitions, for instance $\langle [y_j,\ y_{j+1})\rangle_{j\in\mathbb{J}}$, were finite there would be a last part $[y_k - 1,\ y_k)$ (whose g-length could be less than r) in that partition, in whose case the folding would continue by joining y_k with the successive points of the other partition:

$$y_kx_k,\ y_kx_{k+1},\ y_kx_{k+2}\ \ldots \qquad \text{(XXI)}$$

2- 7. Once folded, $\mathcal{C}$ becomes a Jordan type curve $\mathcal{C}'$. Since the increasing g-length of the successive loops is 2r, 4r, 6r, ... the g-length of $\mathcal{C}'$ has to be infinite, otherwise only a finite number of pairs of points y_1x_1, $y_2, x_2, \ldots, y_nx_n$ would have be successively joined. Now then, according to **2-2**, Jordan curves of infinite g-length are impossible, in consequence the initial hypothesis on the infinite g-length of $\mathcal{C}$ from which the infinite g-length of $\mathcal{C}'$ is derived, has to be false.

2-8. We must therefore conclude that, in accordance with **2-4-2-7** curves of infinite length are inconsistent objects.

3. A conflictive case of hypercomputation

Despite the above inconveniences on the divisibility of time and on the existence of curves of infinite length, let us assume bifurcated supertasks are possible and then that it is also possible to perform hypercomputations, i.e. computations of infinitely many steps. We will now define and analyze the consequences of an elementary, although conflictive, case of hypercomputation.

3-1. Let $\mathbb{Q}^+$ be the set of all positive rational numbers and f a one to one correspondence between the set $\mathbb{N}$ of natural numbers and $\mathbb{Q}^+$ that induces the following ω-order in $\mathbb{Q}^+$:

$$\mathbb{Q}^+ = \{f(1), f(2), f(3), \ldots\} \tag{XXII}$$

Being $\mathbb{N}'$ the set $\mathbb{N} - \{1\}$, let $\langle d_i\rangle_{i\in\mathbb{N}'}$ be an ω-ordered sequence of rational numbers, and r a rational variable whose initial value is 1, both defined in accordance with:

$$\left.\begin{array}{r} d_i = \mid f(i) - f(1) \mid \\ d_i < r \Rightarrow r = d_i \end{array}\right\} \; i = 2, 3, 4, \ldots \tag{XXIII}$$

where $\mid f(i+1) - f(1) \mid$ is the absolute value of $f(i+1) - f(1)$, and '$<$' stands for 'less than' in the usual ordering of $\mathbb{Q}$; i.e $d_i < r$ means $d_i - r < 0$

3-2. Hypercomputation (XXIII) defines the sequence $\langle d_i\rangle_{i\in\mathbb{N}'}$ as a complete infinite totality. It also redefines the rational variable r a finite or infinite number of times. We know neither the number of times r is redefined nor its current value once completed the ω-ordered sequence of computations (XXIII). Notwithstanding, and whatsoever be its current value, r will continue to be a rational variable[9]. And this is all we need to prove the two following contradictory results.

[9]The completion of a finite or infinite sequence of definitions does not change the nature of things, otherwise no demonstration would be possible

3-3. In the usual ordering of $\mathbb{Q}$ and whatsoever be the current value of r once completed (XXIII), the rational $f(1)+r$ is less than any rational greater than $f(1)$.

Assume it is not. There would be a rational $f(k)$ greater than $f(1)$ and less than $f(1)+r$:

$$f(1) < f(k) < f(1) + r \tag{XXIV}$$

and then:

$$0 < f(k) - f(1) < f(1) + r - f(1) = r \tag{XXV}$$

which is impossible because r would have been redefined as $f(k) - f(1)$ just after the definition of d_k.

3-4. In the usual ordering of $\mathbb{Q}$ and whatsoever be the current value of r once completed (XXIII), the rational $f(1)+r$ is not less than any rational greater than $f(1)$. In fact, the rational $f(1)+0.1\times r$, for instance, is greater than $f(1)$ and less then $f(1)+r$

3-5. Evidently **3-3** and **3-4** are contradictory results, and the cause of the contradiction can only be the assumed completeness of the *uncompletable* totalities involved in the argument. The contradiction, in fact, only arises under the assumption of the actual infinity. Under the assumption of the potential infinity only finite totalities can be considered and in this case the above computation, now of a finite number n of steps, will always ends with a finite sequence $\langle d_i\rangle_{i=1,2,\ldots n}$ and a value of r that will be either 1 or the less element in $\{f(2), f(3)\ldots f(n)\}$ greater than $f(1)$.

4. **Reinterpreting the paradoxes of reflexivity**

Perhaps with the exception of self-reference, no other concept in the history of science is comparable to infinity in its ability to produce paradoxes. And taking into account the subtle and frequently confusing frontier between paradoxes and contradictions, the suspicious of inconsistency inevitably falls on the actual infinity. We have just found some contradictions showing that, in fact, the actual infinity could be an inconsistent notion. We will end by analyzing that possibility from a basic set theoretical perspective.

4-1. Most of infinity paradoxes arise from the violation of the Euclidian axiom of the whole and the part, among them, the so called paradoxes of reflexivity in which a whole is put into a one to one correspondence with one of its proper parts [44], [14]. Galileo's paradox [20] is perhaps the best known example of a reflexive paradox, although authors as Proclus, J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. de Rimini, W. of Ockham etc. found many others examples [44]. Set theory was finally founded on that violation (Dedekind definition of infinite set [13]).

4-2. As is well known, an exhaustive injection (bijection or one to one correspondence) between two sets A and B is a correspondence between the elements of both sets such that every element of A is paired with a different element of B and all elements of A and B result paired. When at least one element of the set B result unpaired we say the injection is not exhaustive. Exhaustive and not exhaustive injections can be used as instruments to compare the cardinality of finite sets. If the sets are infinite, however, their cardinality can only be compared by exhaustive injections.

4-3. It seems, in fact, reasonable to assume that if after pairing every element of a set A with a different element of a set B all elements of the set B result paired (exhaustive injection) then A and B have the same number of elements. But it seems also reasonable, and for the same reasons, that if after pairing every element of a set A with a different element of a set B one or more elements of the set B remain unpaired then A and B have not the same number of elements. For finite sets we can supervise the pairings but for infinite sets we cannot. For infinite sets we have to *assume* the completion of the infinitely many pairings, which is usually indicated by the inevitable etcetera or ellipsis (...). Notice that, in any case, exhaustive and not exhaustive injections make use of the same basic method of pairing elements, so that no arithmetic operation is carried out.

4-4. In accordance with **4-3** both exhaustive and not exhaustive injections should have the same legitimacy when used as instruments to compare the cardinality of infinite sets. However, only exhaustive injections do have it. And no reason has ever been given to explain that arbitrary distinction. It is worth noting that the existence of both exhaustive and not exhaustive injections between two sets could be indicating the existence of a contradiction (to have and not to have the same cardinality), in whose case the distinction in favor of exhaustive injections would be the distinction of a term of a contradiction to the detriment of the other.

4-5. Assume for a moment that exhaustive and non-exhaustive injections are equally legitimate as instruments to compare the cardinality of infinite sets. In these conditions, let B be a denumerable infinite set; by definition, there exists a proper subset A of B and an exhaustive injection f from A to B so that both sets have the same number $\aleph_0$ of elements. Consider now the injection g from A to B defined by:

$$g(x) = x,\ \forall x \in A \qquad \text{(XXVI)}$$

which evidently is non-exhaustive (the elements of the non empty set B-A are not paired). Injections f and g would be proving respectively that A and B have and do not have the same number of elements.

4-6. We must therefore decide if exhaustive and non-exhaustive injections have the same legitimacy when used as instrument to compare the cardinality of infinite sets. If they have, then the actual infinite sets are inconsistent. If they don't, at least one reason should be given to explain why they don't. And, if no reason can be given, then the arbitrary distinction in favor of exhaustive injections should be arbitrarily declared in an appropriate ad hoc axiom. Until then, the foundation of set theory will not be completed.

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S01012010

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