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The Pauli Objection

Juan Leon[1] · **Lorenzo Maccone**[2]

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Abstract Schrödinger's equation says that the Hamiltonian is the generator of time translations. This seems to imply that any reasonable definition of time operator must be conjugate to the Hamiltonian. Then both time and energy must have the same spectrum since conjugate operators are unitarily equivalent. Clearly this is not always true: normal Hamiltonians have lower bounded spectrum and often only have discrete eigenvalues, whereas we typically desire that time can take any real value. Pauli concluded that constructing a general a time operator is impossible (although clearly it can be done in specific cases). Here we show how the Pauli argument fails when one uses an external system (a "clock") to track time, so that time arises as correlations between the system and the clock (conditional probability amplitudes framework). In this case, the time operator is conjugate to the clock Hamiltonian and not to the system Hamiltonian, but its eigenvalues still satisfy the Schrödinger equation for arbitrary system Hamiltonians.

Keywords Time · Quantum mechanics · Foundations of quantum mechanics · Time operator

1 Introduction

There are many different ways in which a time operator [1–6] can be introduced into quantum mechanics. These differences reflect the different physical meanings that "time" may have. We recall the main ones (the following list is, by necessity,

✉ Lorenzo Maccone
maccone@unipv.it

1 Instituto de Física Fundamental, CSIC, Serrano 113-B, 28006 Madrid, Spain

2 Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, 27100 Pavia, Italy

incomplete and clearly the following categorizations are not clear-cut): (1) typical time operators [7–11] represent a "time of arrival", whose measurement represents the time at which a system is in a certain location. This is dual [12] to the Newton–Wigner [13] position operator whose measurement represents the position at which a system is located at a certain time; (2) coordinate time [14–18]; (3) an arbitrary parameter (also reinterpreted as "Newtonian absolute time") [19–21]; (4) a dynamical variable that parametrizes different Hilbert spaces [22–24]; (5) a classical parameter that cannot be quantized [25–28]; (6) a parameter that can be quantized, but not using self-adjoint operators (observables) [14–16,29,30]; (7) proper time [31]; (8) clock time [32–54].

Here will be dealing with the clock time, mainly focusing on the Page–Wootters and Aharanov–Kaufherr (PWAK) approach [34–38]. In this framework, time is defined as "what is shown on a clock", where a clock is some (external) physical system that is taken as a time reference. Then, the measurement of time acts as a conditioning that outputs the position in time of some event that is being gauged by the clock: namely the emphasis is on the *correlations* between a system and the clock as in 'the state of a system *given that* the clock shows t'. As shown in [34–39] these correlations manifest themselves as a "static" entangled global state that satisfies a Wheeler-de Witt [55] equation. The PWAK approach is briefly reviewed in Sect. 2.

The Pauli objection [26] essentially states that since the energy is the generator of (continuous) time translations, any time operator must be conjugate to an energy operator (Hamiltonian) that has unbounded continuous spectrum, properties which are not satisfied by the Hamiltonian of typical systems. The standard textbook answer to this objection is that time in quantum mechanics cannot be represented by an operator and is a parameter, external to the theory, e.g. [25,40,41]. A different, but connected objection was put forth by Peres [56]: if the Hamiltonian is the generator of time translations and the momentum is the generator of space translations, then the Hamiltonian and the momentum must always commute, since space and time are independent degrees of freedom. In this paper we show how the PWAK formalism can easily bypass these objections and provide an acceptable time operator.

The main idea is simple: the global Hamiltonian must contain both the system Hamiltonian (which may have arbitrary spectrum) *and* the clock Hamiltonian, which for an ideal clock must have unbounded continuous spectrum (physical clocks can clearly only approximate this ideal situation). It is the clock Hamiltonian that is conjugate to the time operator, whereas it commutes with the system Hamiltonian which acts on a different Hilbert space. Then *the clock Hamiltonian H_c is the generator of clock shifts, hence of "time" translations*, whereas the system Hamiltonian H_s is the generator of translations only of the system state, and not of time. Then $[\hat{T}, H_s] = 0$, since the time operator $\hat{T}$ acts on the clock's Hilbert space whereas the system Hamiltonian H_s acts on the system Hilbert space. Hence, H_s and $\hat{T}$ do not need to have the same spectrum. To overcome Peres' objection, one notes that the system Hamiltonian indeed does commute with the momentum *of the clock*.

There are some arguments whose most extreme formulation says that in quantum theory "time is not a quantity at all" [57], i.e., there is no way to attribute values to time, be it an operator or a parameter indistinctly. All these arguments assume that the spectrum of the Hamiltonian generating the time evolution in the state space is bounded from below. Indeed Halvorson ruled out the existence of subspaces of states $s(t_1, t_2)$ that can be associated to time intervals (t_1, t_2) and hence, dispensing with the traditional notion of the passage of time in quantum theory. He concretely derived [57] the contradiction that, as a consequence of the Hegerfeldt theorem [58], $\forall|v\rangle \in s(t_1, t_2) \Rightarrow |v\rangle = 0$. Again, the PWAK approach is immune to these kind of arguments that, as Pauli's, require boundedness of the Hamiltonian spectrum, which is not the case of the clock Hamiltonian considered here.

While the basic mechanism to overcome the Pauli and Peres objections presented here is clear, one has to be careful, since in the PWAK formalism (reviewed in Sect. 2) the dynamics is imposed as a constraint and one must check that even in the space of physical states the above properties still hold true: indeed, in the space of physical states, the Wheeler-de Witt equation "forces" the clock Hamiltonian to coincide with the system Hamiltonian. This analysis is given in Sects. 3 and 4 (that contains the more technical parts). Even though the system Hamiltonian and the time operator commute in this framework, it is still possible to give a time-energy uncertainty relation, as shown in Sect. 5.

2 The PWAK Mechanism

The PWAK mechanism was initially proposed by Page and Wootters [34–36] and soon after by Aharanov and Kaufherr [37] (but similar previous approaches can be found, e.g. in [39,43–46]). A recent review, together with the solution to the objections that were moved against it, can be found in [38].

To provide a quantization of time, one can simply define time as "what is shown on a clock" and then use a quantum system as a clock. If one wants a continuous time that goes from $-\infty$ to $+\infty$, a good candidate clock is to use the position of a 1-d particle [32,37,39]. Nonetheless, introducing explicitly a physical system is not necessary (although it may help in visualizing the mechanism), and one can only consider the time Hilbert space as an abstract space with no physical meaning, namely one that does not describe any physical system. This abstract space is by no means arbitrary as the global system must satisfy the constraint equation, but one can consider it as a space that is disconnected from an actual physical system. In a sense this is analogous to what happens when one introduces a purification Hilbert space to view mixed states as pure states in a larger Hilbert space: the purification space can be either seen as a true physical space and the we consider the system as mixed because we are ignoring the correlations between the system and this additional physical system, or alternatively it can be seen as a completely abstract space with no physical meaning whatsoever.

The global Hilbert space is then $\mathcal{H}_{TS} = \mathcal{H}_T \otimes \mathcal{H}_S$, where T represents the "time" Hilbert space, typically the one for a particle on a line, $\mathcal{L}^2(\mathbb{R})$. In $\mathcal{H}_T$ we introduce the position operator $\hat{T}$ and its conjugate variable $\hat{\Omega}$, with $[\hat{T}, \hat{\Omega}] = i$. We associate $\hat{\Omega}$ to the energy of the clock (for a particle, this can be a good approximation for sufficiently massive non relativistic particles [37], Appendix E). We can enforce that $\hat{T}$ represents the time operator which describes the evolution of a system by imposing the following constraint equation, namely by requiring that the only states $|\Psi\rangle\rangle$ of the joint Hilbert space $\mathcal{H}_{TS}$ that represent physically relevant situations are the ones that satisfy

$$\left(\hbar\hat{\Omega}\otimes I_S + I_T\otimes H_s\right)|\Psi\rangle\rangle = 0\ , \tag{1}$$

(H_s being the arbitrary Hamiltonian of the system s and I the identity) which can be interpreted as a Wheeler-de Witt equation [55]. The double ket notation serves only as a reminder that $|\Psi\rangle\rangle$ is a state on the joint Hilbert space $\mathcal{H}_{TS}$. Note that, the system Hamiltonian H_s may have arbitrary but *bounded* spectrum. As eigenstates of the Wheeler-de Witt equation, the physical states $|\Psi\rangle\rangle$ are "static" in the sense that they do not evolve with respect to an "external" time. However, the system evolves with respect to the clock and viceversa, in the sense that the correlations (entanglement) between system and clock track the system evolution. Indeed the solutions of (1) are

$$|\Psi\rangle\rangle = \int_{-\infty}^{+\infty} d\omega\ |\omega\rangle|\tilde{\psi}(\omega)\rangle\ , \tag{2}$$

where $|\omega\rangle$ is the eigenstate of $\hat{\Omega}$ with eigenvalue ω, $|\tilde{\psi}(\omega)\rangle$ is the (un-normalized) Fourier transform of the system state $|\psi(t)\rangle$. [Note that the state (2) is not uniform in $|\omega\rangle$, as the eventual weight is implicit in the norm of $|\tilde{\psi}(\omega)\rangle$, e.g. such weight selects the solutions of (1).] Indeed,

$$|\tilde{\psi}(\omega)\rangle = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\ e^{-i\omega t}|\psi(t)\rangle \Rightarrow \tag{3}$$

$$H_s|\tilde{\psi}(\omega)\rangle = \sqrt{2\pi}\sum_k \delta(\omega_k+\omega)\psi_k\hbar\omega_k|e_k\rangle = -\hbar\omega|\tilde{\psi}(\omega)\rangle, \tag{4}$$

where we have used the expansion $|\psi(t)\rangle = \sum_k \psi_k e^{-i\omega_k t}|e_k\rangle$ in terms of the system Hamiltonian eigenstates $|e_k\rangle$ of eigenvalue $\hbar\omega_k$. If the system Hamiltonian has a continuous spectrum, an analogous expression holds:

$$\begin{aligned} H_s|\tilde{\psi}(\omega)\rangle &= \sqrt{2\pi}\int d\omega'\delta(\omega'+\omega)\psi(\omega')\hbar\omega'|\omega'\rangle \\ &= -\hbar\omega\sqrt{2\pi}\,\psi(-\omega)|-\omega\rangle = -\hbar\omega|\tilde{\psi}(\omega)\rangle, \end{aligned} \tag{5}$$

where $|\omega\rangle$ is the δ-normalized energy eigenstate of eigenvalue $\hbar\omega$. It is clear from these expressions that $|\tilde{\psi}(\omega)\rangle$ is the null vector if ω is not an eigenvalue of the system Hamiltonian. The solutions (2) can be written as

$$|\Psi\rangle\rangle = \int_{-\infty}^{+\infty} d\omega\ |\omega\rangle|\tilde{\psi}(\omega)\rangle = \int_{-\infty}^{+\infty} dt\ |t\rangle|\psi(t)\rangle\ , \tag{6}$$

where $|t\rangle = \int d\omega\ e^{-i\omega t}|\omega\rangle/\sqrt{2\pi}$ is the position eigenstate in $\mathcal{H}_T$ and $|\psi(t)\rangle$ is the system state at time t in $\mathcal{H}_S$, with normalization $\langle\psi(t)|\psi(t)\rangle = 1$ for all t. [Note that any nontrivial probability amplitude $\phi(\omega)$ in the integral (2) can be absorbed in the definition of the system state $|\tilde{\psi}(\omega)\rangle$ as $\psi_k \rightarrow \phi(\omega_k)\psi_k$.] The states (6) are improper (non-normalizable) states that reduce to the "momentum" eigenstate $\sqrt{2\pi}|\omega=0\rangle = \int dt|t\rangle$ in $\mathcal{H}_T$ whenever the system is in an eigenstate of its Hamiltonian H_s. Starting from the state (6) and conditioning the clock system to the state $|t\rangle$, we recover the Schrödinger equation: indeed, Eq. (1) in the position representation becomes

$$\langle t|\hbar\hat{\Omega} + H_s|\Psi\rangle\rangle = 0 \Leftrightarrow \left(-i\hbar\tfrac{\partial}{\partial t} + H_s\right)|\psi(t)\rangle = 0\ , \tag{7}$$

where we wrote the "momentum" $\hat{\Omega}$ in the position representation $\langle t|\hat{\Omega} = (-i\partial/\partial t)\langle t|$, and we used $\langle t|\Psi\rangle\rangle = |\psi(t)\rangle$ which follows from (6). One can similarly also derive the unitary evolution for the system [38].

One of the many advantages of this approach is that it renders explicit the problem that, when an event is gauged by a quantum clock or a system is controlled by a quantum clock, a feedback (disturbance) to the clock must occur [33].

3 Bypassing the Pauli Objection

The Pauli objection is just an argument and is not really rigorous. There are many counterexamples in the literature (e.g. [59]), but it can also be made into a rigorous statement if one is careful enough (e.g. [60]). It basically says that if one introduces a time operator, then time and energy are conjugate operators through the Schrödinger equation. Then their spectrum must be the same. This is a consequence of the Stone-von Neumann theorem: if $\left[\hat{T}, H_s\right] = i$ then $\hat{T}$ and H_s have the same spectrum.

The PWAK mechanism is immune to this, since we are requesting that $[\hat{T}, \hat{\Omega}] = i$ and then enforcing that $\hat{\Omega}$ is equal to H_s only on the physical states through the constraint Eq. (1). Such equation is saying that in this subspace $\hat{\Omega} = H_s$! So it seems that in the space of physical states, the Pauli argument should apply: $\hat{T}$ has the same spectrum as $\hat{\Omega}$ which (in the subspace) has the same spectrum as H_s. So we must conclude that in the subspace of physical states $\hat{T}$ has the same spectrum as H_s!

Luckily this statement is false, although it is not immediately trivial to see. To see why that statement is false, we must formalize it very carefully. We start by defining $\hat{T}$ and $\hat{\Omega}$ as

$$\begin{aligned} \hat{T} &\equiv \int_{-\infty}^{+\infty} dt\ t\ |t\rangle\langle t|,\ \ \hat{\Omega} \equiv \int_{-\infty}^{+\infty} d\omega\ \omega\ |\omega\rangle\langle\omega| \\ |\omega\rangle &\equiv \int_{-\infty}^{+\infty} \tfrac{dt}{\sqrt{2\pi}}\ e^{-i\omega t}|t\rangle,\ \Rightarrow |t\rangle = \int_{-\infty}^{+\infty} \tfrac{d\omega}{\sqrt{2\pi}}\ e^{i\omega t}|\omega\rangle. \end{aligned} \tag{8}$$

The Pauli objection can be formalized as follows:

1. The definitions of $\hat{T}$ and $\hat{\Omega}$ imply that $[\hat{T}, \hat{\Omega}] = i$.
2. Introduce the Hilbert space of physical states $\mathcal{H}_c$ as the ones that satisfy Eq. (1), $\left(\hbar\hat{\Omega} + H_s\right)|\Psi\rangle\rangle = 0$.

3. Since $(\hbar\hat{\Omega} + H_s)|\Psi\rangle\rangle = 0$, then* also $\hat{T}(\hbar\hat{\Omega} + H_s)|\Psi\rangle\rangle = 0$, and $\langle\langle\Phi|\hat{T}(\hbar\hat{\Omega} + H_s)|\Psi\rangle\rangle = 0$ for all $|\Phi\rangle\rangle, |\Psi\rangle\rangle \in \mathcal{H}_{TS}$.
4. The point above implies that

$$0 = \langle\langle\Phi|\hat{T}(\hbar\hat{\Omega} + H_s)|\Psi\rangle\rangle - \langle\langle\Phi|(\hbar\hat{\Omega} + H_s)\hat{T}|\Psi\rangle\rangle \tag{9}$$

$$= \langle\langle\Phi|\hbar[\hat{T}, \hat{\Omega}] + [\hat{T}, H_s]|\Psi\rangle\rangle, \tag{10}$$

$$\text{so} \quad \langle\langle\Phi|\hbar[\hat{T}, \hat{\Omega}]|\Psi\rangle\rangle = -\langle\langle\Phi|[\hat{T}, H_s]|\Psi\rangle\rangle \tag{11}$$

5. Since $\left[\hat{T}, \hat{\Omega}\right] = i$, this means that, when restricting to the physical states space $\mathcal{H}_c$, we have $[\hat{T}, H_s] = -i\hbar$, which through the Stone-von Neumann theorem implies that $\hat{T}$ has the same spectrum as H_s in this Hilbert space $\mathcal{H}_c$, the Pauli objection!

We note that Dirac had introduced an equation of the type (1) in [61], but he did not consider it as a constraint on the physical states. This meant that he ran into an inconsistency similar to the one emphasized above. Dirac never gave a solution [54]. We show here that a solution is provided by the PWAK mechanism.

This above argument is clearly wrong since $\left[\hat{T}, H_s\right] = 0$ because they are operators acting on different Hilbert spaces. In fact, the implication indicated with an asterisk at point 3 fails: even though it is true that $\left(\hbar\hat{\Omega} + H_s\right)|\Psi\rangle\rangle = 0$, this does *not* imply that $\hat{T}(\hbar\hat{\Omega} + H_s)|\Psi\rangle\rangle = 0$. This comes from the fact that the spectrum of $\hat{T}$ is unbounded, see the definition (8). We prove this in the following section (using two different regularizations for the physical states $|\Psi\rangle\rangle$ which are un-normalizable).

One can also give a physical interpretation to this: one should expect that the expectation value of $\hat{T}$ must be undefined in the space of physical states. In fact $\langle\hat{T}\rangle$ has as value the result to the question "what is the time?" which is a meaningless question per se in physics. Meaningful questions are "what is the time when the spin is up?" or "what is the time now that you're reading this?", etc. So, one must expect that $\langle\langle\Phi|\hat{T}|\Psi\rangle\rangle$ will be undefined in the space of physical states, which is indeed what happens.

4 Regularization

Here we provide the regularizations necessary to prove the relations introduced in Sect. 3.

The state (6) is the solution of the eigenvalue equation (1). The eigenvalue $\lambda = 0$ is an essential eigenvalue of the self-adjoint constraint operator $\hat{\mathbb{J}} = \hbar\hat{\Omega} + H_s$. This can be shown through Weyl's criterion [62, Sect. 7], since $\|(\hat{\mathbb{J}} - \lambda)|\Psi_n\rangle\rangle\| \to 0$ for $n \to \infty$ where $|\Psi_n\rangle\rangle$ is a Weyl sequence, i.e. a normalized sequence of vectors that weakly converges to zero (namely, $\forall|\theta\rangle\rangle \in \mathcal{H}$ we have $\langle\langle\theta|\Psi_n\rangle\rangle \to 0$).

We will show this using two different Weyl sequences which can be considered as approximate eigenvectors (as expected, both give the same results):

$$|\Psi_n\rangle\rangle \equiv \left(\tfrac{2}{\pi n}\right)^{1/4}\int dt\ e^{-t^2/n}|t\rangle|\psi(t)\rangle \tag{12}$$

$$|\Psi'_m\rangle\rangle \equiv \tfrac{1}{\sqrt{m}}\int dt\ \beta(t/m)|t\rangle|\psi(t)\rangle\ , \tag{13}$$

where the first uses a Gaussian whose width diverges for $n\to\infty$, the second uses the box function β whose width diverges for $m\to\infty$, with $\beta(x)=1$ if $-\frac{1}{2}<x<\frac{1}{2}$, $\beta(-\frac{1}{2})=\beta(\frac{1}{2})=\frac{1}{2}$, and $\beta(x)=0$ otherwise. It has derivative $\partial\beta(x)/\partial x=\delta(x+1/2)-\delta(x-1/2)$. These are both Weyl sequences (see [62], pp. 71 and 74 respectively).

All states in $\mathcal{H}_c$ can be obtained from these as

$$|\Psi\rangle\rangle=\lim_{n\to\infty}\left(\tfrac{\pi n}{2}\right)^{1/4}|\Psi_n\rangle\rangle=\lim_{m\to\infty}\sqrt{m}|\Psi'_m\rangle\rangle. \tag{14}$$

Note that the state $|\Psi\rangle\rangle$ is un-normalizable: it does not live in a Hilbert space, but one has to resort to rigged Hilbert spaces, where the Hilbert space containing normalized vectors (and the limit of sequences of normalized vectors) is incremented with vectors of infinite norm [63].

First we show that $|\Psi_n\rangle\rangle$ and $|\Psi'_m\rangle\rangle$ are indeed Weyl sequences for $\hat{\mathbb{J}}$ and $\lambda=0$, namely they are "proper" approximations of the improper eigenvectors of $\hat{\mathbb{J}}$ with eigenvalue $\lambda=0$. (We already know that, this is just a consistency check.) Let us start with $|\Psi_n\rangle\rangle$. We have to show that $\|(\hat{\mathbb{J}}-\lambda)|\Psi_n\rangle\rangle\|\to 0$ for $n\to\infty$. Indeed,

$$\lim_n(\hbar\hat{\Omega}+H_s)|\Psi_n\rangle\rangle=\lim_n\tfrac{2i}{n}\left(\tfrac{2}{\pi n}\right)^{1/4}\int dt\ t\ e^{-t^2/n}|t\rangle|\psi(t)\rangle.$$

That this is a null vector can be seen by taking its modulus:

$$\|(\hbar\hat{\Omega}+H_s)|\Psi_n\rangle\rangle\|^2=\tfrac{4}{n^2}\sqrt{\tfrac{2}{\pi n}}\int dt\ t^2\ e^{-2t^2/n}\to 0,$$

where we used the fact that $\int dt\ t^2\ e^{-at^2}=\sqrt{\pi/a^3}/2$ and $\langle\psi(t)|\psi(t)\rangle=1$. The same result applies using the other regularization: $(\hbar\hat{\Omega}+H_s)|\Psi'_m\rangle\rangle\to 0$ for $m\to\infty$. Indeed, for all vectors $|\theta\rangle\rangle=\int dt\theta(t)|t\rangle|\phi(t)\rangle$ in the Hilbert space, we find

$$\begin{aligned}&|\langle\langle\theta|(\hbar\hat{\Omega}+H_s)|\Psi\rangle\rangle|\\&\quad=\lim_m|\langle\langle\theta|\int dt\ |t\rangle|\psi(t)\rangle\left[\delta\left(t-\tfrac{m}{2}\right)-\delta\left(t+\tfrac{m}{2}\right)\right] \qquad (15)\\&=\lim_m|\langle\phi(0)|\psi(0)\rangle\left[\theta^*\left(\tfrac{m}{2}\right)-\theta^*\left(-\tfrac{m}{2}\right)\right]|=0\ , \qquad (16)\end{aligned}$$

where we used (14), and the fact that $\langle\phi(t)|\psi(t)\rangle$ is constant and that all square integrable functions $\theta(t)\to 0$ for $t\to\pm\infty$.

Now the crucial point: what happens when we multiply these null vectors by the unbounded operator $\hat{T}$? We obtain a non-null vector! Indeed,

$$\|\hat{T}(\hbar\hat{\Omega} + H_s)|\Psi_n\rangle\rangle\|^2 = \tfrac{4}{n^2}\sqrt{\tfrac{2}{\pi n}} \int dt\ t^4\ e^{-2t^2/n} = \tfrac{3}{4}\ ,$$

for all n, since $\int dt\ t^4\ e^{-at^2} = 3\sqrt{\pi/a^5}/4$. This implies that $|\Psi_n\rangle\rangle$ is not an approximate eigenvector for the $\lambda = 0$ eigenvalue of the operator $\hat{T}(\hbar\hat{\Omega} + H_s)$, even though it was an approximate eigenvector for the operator $(\hbar\hat{\Omega} + H_s)$. This also means that in the rigged Hilbert space we cannot consider $|\Psi\rangle\rangle$ as eigenvector of this operator.

One can also show that

$$\langle\langle\Psi_n|(\hbar\hat{\Omega} + H_s)\hat{T}|\Psi_n\rangle\rangle \tag{17}$$

$$= i\sqrt{\tfrac{2}{\pi n}} \int dt \left(e^{-2t^2/n} - \tfrac{2t^2}{n}e^{-2t^2/n}\right) = \tfrac{i}{2}\ , \tag{18}$$

which suggests that $|\Psi\rangle\rangle$ is an (improper) eigenstate of $(\hbar\hat{\Omega} + H_s)\hat{T}$ with eigenvalue $i/2$. Indeed, the above results imply

$$\|[(\hbar\hat{\Omega} + H_s)\hat{T} - \lambda]|\Psi_n\rangle\rangle\| \to 0 \tag{19}$$

for $\lambda = i/2$ (actually the modulus is equal to 0 for all n). Note that this is the value that is necessary in Eq. (9) to avoid the contradiction!

Analogous considerations hold for the other regularization since

$$\langle\langle\theta|\hat{T}\left(\hbar\hat{\Omega} + H_s\right)|\Psi\rangle\rangle = \lim_m i\tfrac{m}{2}\left[\theta^*\left(\tfrac{m}{2}\right) + \theta^*\left(-\tfrac{m}{2}\right)\right]\ , \tag{20}$$

where we used $|\Psi\rangle\rangle = \lim_m \sqrt{m}|\Psi'_m\rangle\rangle$. This does not tend to zero as $m \to \infty$ for all square integrable functions $\theta(t)$, since square integrable functions must go to zero faster than $1/\sqrt{t}$ for $t \to \infty$.

In conclusion, not only we have shown that point 3 of the Pauli argument fails, but we have also recovered the expected values of the scalar products $\langle\langle\Psi|\hat{T}(\hbar\hat{\Omega}+H_s)|\Psi\rangle\rangle = i/2$ that are necessary in Eq. (9) if it has to be consistent with the fact that $[\hat{T}, \hat{\Omega}] = i$.

5 Unbounded-Energy Clocks?

The way we the PWAK mechanism bypasses the Pauli objection is by using a clock with an unbounded Hamiltonian equal to its "momentum" [37]. Clearly this is unphysical and one could object that our resolution is not a resolution after all. However, it is important to notice that all quantum experiments to date have been performed with macroscopic "classical" clocks (except for especially crafted situations [53]). These have energy so large compared to the time uncertainties that can be tracked in practice that their spectrum can be considered unbounded for all practical purposes. Moreover, macroscopic systems get very quickly correlated to astronomical distances (e.g., the motion of one gram of matter on the star Sirius by one meter sensibly influences the particle trajectories in a box of gas on earth on a time-scale of μs after the transit time [64]) so that a pure-state analysis as performed above will break down unless one is able to track all the correlated degrees of freedom, a practical impossibility.

In this section we study how good is the approximation of considering a clock with unbounded spectrum. We show that if the energy spread is ΔE the time can be measured up to a precision $\Delta t = \hbar/2\Delta E$. This is a direct consequence of the time-energy uncertainty relation [28,65,66] which says that, if the energy spread is ΔE, then the minimum time interval it takes to evolve to an orthogonal state is $\tau \geqslant \hbar/2\Delta E$. Hence no smaller time interval can be measured with accuracy.

Clearly, a spread in energy is by itself insufficient to obtain a clock: one also needs good time correlation. However, in the absence of energy spread, a clock in the state $|\Psi\rangle\rangle$ of (2) cannot keep time, and with limited energy spread, it can only keep time up to some accuracy since the correlation in time cannot be sufficiently high. Consider first what happens if we keep the unbounded Hamiltonian of the clock $\hbar\hat{\Omega}$, but reduce the energy spread by making explicit the spectral function $\phi(\omega)$ which was absorbed into $|\tilde{\psi}(\omega)\rangle$ in (2) as $|\tilde{\psi}(\omega)\rangle = \phi(\omega)|\tilde{\chi}(\omega)\rangle$. Consider $\Delta\omega = \Delta E/\hbar$ the standard deviation of the probability $|\phi(\omega)|^2$. Since the clock and the system are entangled, this spectral function *does not refer exclusively to the clock*, but to both the clock and the system. As expected from the time-energy uncertainty relation, a limited-bandwidth spectral function $\phi(\omega)$ will reduce the speed of evolution (time resolution of the global system). Indeed (neglecting multiplicative constants) we have

$$|\Psi\rangle\rangle = \int d\omega\, \phi(\omega)|\omega\rangle|\tilde{\chi}(\omega)\rangle \propto \int dt dt'\, \tilde{\phi}(t-t')|t\rangle|\chi(t')\rangle,$$

where $\tilde{\phi}$ and $|\chi(t)\rangle$ are the Fourier transforms of ϕ and $|\tilde{\chi}(\omega)\rangle$. Even though this seems to be incompatible with Eq. (6), it is not as can be seen by writing

$$|\psi(t)\rangle \propto \int dt' \tilde{\phi}(t-t')|\chi(t')\rangle\ . \tag{21}$$

This can be interpreted as if $|\psi(t)\rangle$ is obtained by "averaging" $|\chi(t)\rangle$ over time with a probability amplitude $\tilde{\phi}$. Then the smallest time interval during which $|\psi(t)\rangle$ can vary appreciatively is of the order of $\hbar/\Delta E$, i.e. the inverse of the spread of the probability $|\phi(\omega)|^2$. Indeed

$$\langle\psi(t)|\psi(t')\rangle \propto \int d\tau d\tau'\, \tilde{\phi}(t-\tau)\tilde{\phi}^*(t'-\tau')\langle\chi(\tau')|\chi(\tau)\rangle,$$

whence, even supposing instantaneous change of $|\chi\rangle$, namely $\langle\chi(\tau')|\chi(\tau')\rangle \propto \delta(\tau-\tau')$, we have

$$\langle\psi(t)|\psi(t')\rangle \propto \int d\omega |\phi(\omega)|^2 e^{i\omega(t-t')}. \tag{22}$$

If $|\phi(\omega)|^2$ has a spread $\simeq \Delta E/\hbar$, then its Fourier transform will have a spread of the order of $\hbar/\Delta E$. This means that the scalar product $\langle\psi(t)|\psi(t')\rangle$ cannot change appreciatively in a smaller interval, namely the time scale of change of the system state must be larger than $\hbar/\Delta E$, in accordance with the time-energy uncertainty relation.

Thanks to its linearity, the Schrödinger equation holds for the "averaged" $|\psi(t)\rangle$ of Eq. (21), which implies that eventual imperfect correlations between system and clock will not induce a fundamental decoherence effect in this case.

However, if we drop the (unphysical) assumption that the clock Hamiltonian is unbounded $\hat{H}_c = \hbar\hat{\Omega}$, then the above analysis fails. Indeed, the clock's energy boundedness will induce a, model-dependent decoherence effect in the system evolution: we will not be able to reproduce the Schrödinger equation. Since it is a model-dependent effect (e.g., see [37], Appendix E), we will not further elaborate here, except to note that in practical situations this effect will be completely negligible for all quantum experiments performed to date where classical clocks (i.e., macroscopic quantum systems) have been employed to time the experiment. In these cases, the spectrum of the clock is so large as to make it effectively unbounded over the limited time duration of any reasonable quantum experiment. So any decoherence effect due to the limited energy spectrum of the clock will be completely negligible. Moreover, we point out that macroscopic clocks will quickly correlate with other systems, so it quickly becomes meaningless to consider as "clock" just the actual clock, which very rapidly becomes entangled to external degrees of freedom of the rest of the laboratory and beyond: when considering the clock energy, one would then have to consider the energy of all these additional degrees of freedom. Then, the approximation of considering the clock energy as unbounded is even better than what would be by considering the clock by itself.

We note that this fundamental decoherence is of a different nature with respect to the one present in the Gambini et al. framework [50–52] which also presents an interesting fundamental decoherence due to the unobservability of the time parameter.

6 Conclusions

In this paper we have shown how one can easily bypass the Pauli and the Peres objections to a quantum operator for time using the conditional probability amplitude framework of Page, Wootters, Aharanov and Kaufherr. Moreover we have detailed how the time-energy uncertainty relation arises in this framework.

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