Relevant implication

by

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In [4] I offered an analysis of what it means to be (entirely) about a subject matter. I first repeat that analysis. Then I define several relations of relevance, for instance, between the premise and conclusion of an implication. I show that whenever a premise implies a conclusion, in the ordinary sense of truth-preservation, then also the premise is relevant to the conclusion in the sense of the present analysis. *Pace* Anderson and Belnap [1], there can be no such thing as a truth-preserving "fallacy of relevance". Finally I remark that this does not by any means do away with all motivations for relevant logic.

Subject Matters

We can think of a subject matter, sometimes, as a part of the world: the 17th Century is a subject matter, and also a part of this world. Or better, we can think of a subject matter as a part of the world *in intension*: a function which picks out, for any given world, the appropriate part—as it might be, that world's 17th Century. (If for some reason the world had no 17th Century, the function would be undefined.) We can say that two worlds are exactly alike with respect to a given subject matter. For instance two worlds are alike with respect to the 17th Century iff their 17th Centuries are exact intrinsic duplicates (or if neither one has a 17th Century).

This being exactly alike is an equivalence relation. So instead of thinking of a subject matter as a part of the world in intension, we

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can think of it instead as the equivalence relation. This seems a little artificial. But in return it is more general, because some subject matters—for instance, demography—do not seem to correspond to parts of the world. Or if they do, it is because some contentious theory of "abstract" parts of the world is true.

The equivalence relation on worlds partitions the worlds into equivalence classes. The equivalence classes are propositions, ways things might possibly be. An equivalence class is a maximally specific way things might be with respect to the subject matter. So a third way to think of a subject matter, again general, is as the partition of equivalence classes.

We can associate the partition with a question, more or less as in [2]. The partition gives all the alternative complete answers to the question; the question asks which cell of the partition is the true one. (Which cell does our world fall into?) So a fourth way to think of a subject matter, again general, is as a question. The other way around, we can think of some questions as taking the form: what is the whole truth about such-and-such subject matter? (We often ask easier questions, of course. We demand not the whole truth, but only, say, a paragraph-length answer that hits the highlights.) Sometimes the best way to denote a subject matter is by a clause derived from a question. One subject matter is the question: how many stars there are.

Inclusion of Subject Matters

A big subject matter, the 17th Century, includes the smaller, more specialized subject matter, the 1680's. A big subject matter, how many stars there are, includes the smaller subject matter, whether there are finitely or infinitely many stars.

When we think of subject matters as parts of the world, or rather parts of the world in intension, we can say that M *includes* N iff, for each world w, N(w) is part of M(w) (or both are undefined, or N(w) alone is undefined). This definition will apply to the 17th Century and the 1680's; not, or not in any obvious and uncontentious way, to how many stars there are and whether there are finitely or infinitely many. The definitions to follow will apply generally.

When we think of subject matters as equivalence relations, we can say that M includes N iff whenever M(v,w) then also N(v,w). If two worlds are alike with respect to the bigger subject matter, *a fortiori* they are alike with respect to the smaller. It is easier to be alike with respect to the smaller subject matter, so more worlds manage to do the easier thing. (Caution!—if M is more inclusive than N *qua* subject matters, then N is more inclusive than M *qua* equivalence relation, i.e. *qua* set of ordered pairs. I take it that when we talk of "inclusion" of subject matters, then we are speaking not literally but analogically; not of the genuine relation of part to whole, but of a relation that formally imitates it.)

When we think of subject matters as partitions, we can say that M includes N iff every cell of N is a union of cells of M. The bigger subject matter is a refinement of the smaller. A class of worlds all alike with respect to the smaller subject matter may yet subdivide with respect to the bigger.

Aboutness

A proposition is *about* a subject matter, and it is a subject matter *of* the proposition, iff the truth value of that proposition supervenes on that subject matter. When we think of subject matters as equivalence relations, we can say that P is about M iff, whenever M(w,v), both w and v give P the same truth value. Contrapositively, if two worlds give the proposition different truth values although they are entirely alike with respect to a subject matter, then that proposition cannot have been entirely about that subject matter. When we think of subject matters as partitions, we can say that P is about M iff each cell of M either implies or contradicts P. The cells are maximally specific propositions about the subject matter, and accordingly must imply or contradict any other proposition about the same subject matter.

(If a proposition is, in some sense, partly about one subject matter and partly about another, then we would not expect its truth value to supervene on either one. Our supervenience definition, therefore, defines what it means to be *entirely* about a subject matter. For some senses of partial aboutness, see [4].)

Least Subject Matters

If P is about M, then also P is about any subject matter that includes M. So we can speak of *a* subject matter of P, but not unequivocally of *the* subject matter of P. The closest we can come is to define the *least* subject matter of P as that subject matter M, if such there be, such that M is a subject matter of P, M is included in any other subject matter of P, and there is no other subject matter of which the same is true.

For any proposition P, we have a question: whether or not P. If this question is a genuine subject matter, then we can say that P is entirely about M iff M includes the subject matter: whether or not P. And if so, then also the least subject matter of P will be the question: whether or not P.

But sometimes we might well decline to count the question whether or not P as a genuine subject matter. For instance, when P is noncontingent, that question is a thoroughly *degenerate* subject matter. It is the question whether the necessary proposition or the impossible proposition is true, or in other words the universal equivalence relation on worlds, the one-celled partition. A degenerate subject matter, or not a genuine subject matter at all? This is a verbal question only. I choose the second alternative for convenience later. We may want to be restrictive in other ways too, for instance in excluding unduly gerrymandered, unnatural subject matters. Whenever we decide, for whatever reason, not to count the question whether P as a genuine subject matter, then P may or may not turn out to have something else as its least subject matter. It may turn out, therefore, that a proposition has subject matters, but has no least subject matter.

Noncontingent Propositions

When P is a noncontingent proposition, necessary or impossible, probably it has no least subject matter (given that we don't count the degenerate subject matter just mentioned). But it does have subject matters. It turns out that a noncontingent proposition is about *any* subject matter M, since whenever M(w,v) then w and v give it the same truth value. This is an immediate consequence of our definition of aboutness as supervenience.

It is a surprising consequence, no doubt. But there is more to say in its favor. First, for the case that P is necessary. Then P is contentless; it gives us no information about anything, it rules out nothing at all. Now observe that our conception of aboutness as supervenience is a double-negative notion. A proposition is entirely about a subject matter iff *none* of its content is *not* about that subject matter. We do not require complete coverage, rather we forbid straying outside. A proposition about turkeys is entirely about poultry even if it gives us no information about chickens, for it gives us no information about anything else but poultry. But a contentless proposition does better still. It gives us no information about anything, *a fortiori* no information about non-poultry. It never strays anywhere, thereby it keeps inside whatever boundary you please.

The same cannot be said when P is impossible. Then P seems to give us altogether too much information, ruling out all possibilities for all subject matters, straying out of all bounds. However, in all other cases, it is intuitively compelling that a proposition and its negation should be exactly alike with respect to what they are about; and in all cases including this, a proposition and its negation supervene on exactly the same subject matters. Therefore it is best, on the whole, to say that an impossible proposition also is about the same subject matters as its denial; and its denial is necessary, and so is vacuously about every subject matter.

We have four conflicting intuitive *desiderata*. (1) Impossible propositions seem never to be entirely about subject matter M (unless it be the greatest subject matter, the equivalence relation on which no two worlds are equivalent). (2) Necessary propositions seem always to be entirely about any subject matter M, since they never give any information not about M. (3) Necessary propositions and their impossible negations should come out alike in subject matter. Finally, (4) noncontingent propositions should not get special treatment, but should fall in line with the contingent propositions, for which aboutness as supervenience works smoothly. We can't have all four. Respecting (1) and (2) together loses both (3) and (4).

Prefering (1) to (2) saves (3) but still loses (4). We do best to drop (1) and save the rest.

You may well protest that we have a deeper problem. Even granted that necessary propositions should be about the same subject matters as their impossible negations, it still seems that we ought to be able to distinguish the different subject matters of different necessary propositions, and likewise of different impossible propositions. (In which case we must first admit more than just two noncontingent propositions, one necessary and one impossible. No worries; many conceptions of propositions are available, some intensionally and some hyperintensionally individuated, and—as always—we should choose a conception that suits the job at hand.) Pity the poor mathematics librarian! How can he classify his books? Some are about number theory, some are about topology, . . ., and yet their propositional content is noncontingent through and through. Now our definition of aboutness as supervenience may seem bankrupt. Not so. Before we are done, we shall find out how to solve the librarian's problem. But for now, I accept your protest as fair, leave it unanswered, and move on.

Mereology of Subject Matters

Once we know what it means to say that one subject matter includes another, we can also say that two subject matters *overlap* (again, in a non-literal and analogical sense) iff they have some subject matter as a common part, included in both. Otherwise they are *disjoint*. We can define the *sum* of subject matters M_1, M_2, . . . as the subject matter, if such there be, that includes all of the M's, and is included in any other subject matter of which the same is true. We can define the *intersection* of subject matters M_1, M_2, . . . as the subject matter, if such there be, that is included in all the M's, and includes any other subject matter of which the same is true.

(If we had counted the degenerate subject matter—the universal relation on worlds, the one-membered partition—it would have been included in every subject matter. So in a trivial way, all subject matters would have overlapped. It was to avoid this, and to avoid having to talk always of "non-trivial overlap" instead of overlap *simpliciter*, that I chose not to count the degenerate subject matter.)

Orthogonality and Connection

Two subject matters M_1 and M_2 are *orthogonal* iff, roughly, and way for M_1 to be is compatible with any way for M_2 to be. If we think of subject matters as equivalence relations, orthogonality means that for any worlds w and v there is a world u such that $M_1(u,w)$ and $M_2(u,v)$. If we think of subject matters as partitions, orthogonality means that M_1 and M_2 cut across each other: each cell of M_1 intersects each cell of M_2. Subject matters are *connected* iff they are not orthogonal.

If M_1 is included in N_1, and M_2 is included in N_2, then if M_1 and M_2 are connected, so are N_1 and N_2. *Proof.* We show the contrapositive. If N_1 and N_2 are orthogonal, then for any w and v, we have u such that $N_1(u,w)$ and $N_2(u,v)$. By the two inclusions, also $M_1(u,w)$ and $M_2(u,v)$, so M_1 and M_2 also are orthogonal. *QED*

Whenever two subject matters overlap, they are connected. *Proof.* Suppose not. Then we have N included in M_1 and M_2, which are orthogonal. Let W and V be two equivalence classes of N (since N is non-degenerate). Let w be in W, and v in V; then we have u such tht $M_1(u,w)$ and $M_2(u,v)$. Because N is included in M_1 and M_2, $N(u,w)$ and $N(u,v)$; so $N(w,v)$, contradicting the supposition that w and v fall in different equivalence class of N. *QED*

(What about the converse? If arbitrary non-degenerate equivalence relations may count as subject matters, then there are subject matters that are connected without overlapping. *Example:* eight worlds like this, where dotted lines indicate M_1-equivalence and dashed lines indicate M_2-equivalence.

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0----0
.    .
.    .
.    .
0----0----0
.    .    .
.    .    .
.    .    .
0----0----0
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But I find it hard to think of a natural example of connection without overlap. Maybe such cases ought to be excluded by a constraint on which equivalence relations count as genuine subject matters.)

Relevance of Propositions

We can define four relations of relevance between propositions, as follows.

Identity: P and Q have the same least subject matter; more generally, all and only the subject matters of P are subject matters of Q.

Inclusion: the least subject matter of P is included in the least subject matter of Q; more generally, some subject matter of P is included in all subject matters of Q; equivalently, every subject matter of Q is a subject matter of P.

Overlap: the least subject matter of P overlaps the least subject matter of Q; more generally, every subject matter of P overlaps every subject matter of Q.

Connection: the least subject matter of P is connected with the least subject matter of Q; more generally, every subject matter of P is connected with every subject matter of Q.

Identity implies inclusion; overlap implies connection. Inclusion, or even identity, does not imply overlap; for a counterexample, suppose M and N are subject matters of both P and Q, but no common part of M and N counts as a genuine subject matter. Inclusion, or even identity, does not imply connection; for a counterexample, let one or both of P and Q be noncontingent. But for contingent P and Q, inclusion (or identity) does imply connection. *Proof.* P and Q are relevant by inclusion in one or the other direction; let it be that every subject matter of Q is a subject matter of P. Since P is contingent, P is true at a world w and false at a world v. Suppose for *reductio* that some subject matter M_1 of P is orthogonal to some subject matter M_2 of Q. M_2 is a subject matter also of P. There is a world u such that $M_1(u,w)$ and $M_2(u,v)$; but then P is both true and false at w, which is impossible. So P and Q are connected. *QED*

Let us say that two propositions are *relevant* to one another iff at least one of the four relations of relevance holds between them; equivalently, iff either inclusion or connection holds between them; equivalently, iff either connection holds between them or else one of them is noncontingent.

Implication

Proposition P *implies* proposition Q iff every world that makes P true makes Q true as well.

Whenever P implies Q, P and Q are relevant. *Proof.* If either P or Q is noncontingent, it is about all subject matters, in which case P and Q are relevant by inclusion in one or the other direction. Otherwise, P is true at some world w and Q is false at some world v. If P and Q were not relevant, then some subject matter M_1 of P would have to be orthogonal to some subject matter M_2 of Q. Then there would be a world u such that $M_1(u,w)$ and $M_2(u,v)$; so P would be true and Q false at u, contradicting the implication of P by Q. *QED*

We can extend the result: whenever either P or its negation implies either Q or its negation, P and Q are relevant. *Proof.* A proposition and its negation must supervene on exactly the same subject matters, hence must stand in exactly the same relations of relevance. *QED*

What of the converse? If neither P nor its negation implies either Q or its negation—if P and Q are *logically independent*—and if the question whether or not P and the question whether or not Q are genuine subject matters, then these two subject matters are orthogonal; and further, P and Q are contingent; whence it follows that P and Q are not relevant. But if we reject these questions as genuine subject matters, then it may happen that P and Q are relevant despite their independence.

Example. Let P be the proposition that Fred is either in Carneys Point or in Ellerslie or in Germiston or in Mundrabilla or in

Northcote; and let Q be the proposition that Fred is either in Ellerslie or in Heby or in Noke or in Northcote or in Zumbrota. P and Q are logically independent. If the two orthogonal questions whether P and whether Q are genuine subject matters, the P and Q turn out not to be relevant—which may seem wrong. It may well seem better, then, not to count the two questions as genuine subject matters, due to the miscellaneously disjunctive character of P and Q. A much more natural subject matter is close at hand: Fred's whereabouts. This is a subject matter of P and of Q; perhaps it is the least among the genuine subject matters of the two propositions, once we throw away the gerrymanders. If so, then P and Q, despite their independence, are relevant by identity.

Argument-forms

An *argument-form* is a schema for a pair of sentences. (We disregard multi-premise arguments; let the premises be conjoined.) It is *truth-preserving* iff, whenever a pair of sentences is an instance of it, the proposition expressed by the first implies the proposition expressed by the second. It is *relevant* iff, whenever a pair of sentences is an instance of it, the proposition expressed by the first is relevant to the proposition expressed by the second. Some argument-forms are neither truth-preserving nor relevant; these are fallacies of relevance. *Example:* "If you do not agree that A, I shall beat you with this stick. Therefore A." Some argument-forms are relevant but not truth-preserving. These are not fallacies of relevance, but they are still fallacies. *Example:* "A. Therefore not-A." Since whenever one proposition implies another the two are relevant, every truth-preserving argument-form is also relevant. A Pittsburgh "fallacy of relevance", as denounced in [1], would be an argument form that was truth-preserving but not relevant; according to the present treatment, there are none of those.

Ex falso quodlibet (for short, *quodlibet*) is the argument-form "A and not-A. Therefore B." It is truth-preserving, since in every instance of it, the contradictory premise expresses the impossible proposition. Since the impossible proposition is about every subject matter, *a fortiori* it is about every subject matter of the proposition expressed by the conclusion, so we have relevance by inclusion.

Quodlibet is supposed to be the Pittsburgh "fallacy of relevance" *par excellence,* but according to the present treatment, it can be nothing of the kind. The very last place to look for irrelevance will be an argument-form where either the premise or the conclusion is either a contradiction or a tautology.

Relevant Logic

It may seem by now that I am bashing relevant logic. Not really. Despite appearances, relevant logic is not an ideological movement, or even a philosophical position. It is a certain body of technology that can be applied in the service of quite a wide range of different philosphical positions—some more interesting than others, some more plausible than others. For one safe and dull application, see [5]; for a daring and interesting one, see [6]. (The technology may have applications outside philosophy, for instance to efficient automated reasoning.) It is not even very helpful to say that the philosophical positions served by relevant logic are united by a common animus against *quodlibet*. The complaints lodged against *quodlibet* are just too varied: it is truth-preserving but irrelevant, it is not truth-preserving in paradoxical cases, it does not preserve truth-according-to-a-story, it does not preserve truth-on-some-disambiguation, it is not a way we actually reason, it would be dangerous if we did reason that way, it just is not what we call "implication".

I mean to bash only one of all the motivations for relevant logic, only one of all the complaints against *quodlibet*: namely, the idea that a good argument-form must have two separate virtues, truth-preservation and relevance, and *quodlibet* has the first but lacks the second.

We saw how to make the case that *quodlibet* is as relevant an argument-form as any, and indeed that every truth-preserving argument-form is relevant. The trivial way that *quodlibet* is relevant is very like the trivial way that it preserves truth. The proposition expressed by a contradiction is about any subject matter because, since there is no way at all for two worlds to give it different truth values, *a fortiori* there is no way for two worlds to give it different truth values without differing with respect to the subject matter. An argument from a contradiction preserves truth because, since there is no way at all for a world to make the premise true, *a fortiori* there is no way for a world to make the premise true without making the conclusion true.

I do not say that you have to like these two arguments, or that you cannot resist them. Far from it! But, given their similarity, it seems highly arbitrary to resist the first and accept the second. The reason why *quodlibet* is relevant and the reason why it is truth-preserving go together like hand and glove.

If you want to resist both arguments, you know what to do. (See [3] pp. 40–44, [6], [7], [8].) Extend the class of worlds to a broader class of *worlds**, as we may call them: ways, not necessarily possible, for things to be. A *proposition** is a class of worlds*. It is *noncontingent** iff it has the same truth value at all worlds*; *necessary** if everywhere true, *impossible** if everywhere false. One proposition* *implies** another iff no world* makes the first true without making the second true. Extend semantics so that a sentence *expresses** a proposition*. We can think of a *subject matter** now as an equivalence relation extended from worlds to worlds*. A proposition* P is *about** a subject matter* M iff, for any two worlds* w and v, if M(w,v) then w and v give P the same truth value. Two subject matters* M_1 and M_2 are *connected** iff it is not the case that for any worlds* w and v there is a world* u such that $M_1(u,w)$ and $M_2(u,v)$. One proposition* is *relevant** to another iff either every subject matter* of the first is a subject matter* of the second, or *vice versa*, or else any subject matters* of the two are connected*; equivalently, iff either any subject matters* of the two propositions* are connected* or else at least one of the two propositions* is noncontingent*. An argument-form is *truth-preserving**, or is *relevant**, iff, in any instance of it, the proposition* expressed* by the premise implies*, or is relevant* to, the proposition* expressed* by the conclusion.

Splattering stars all over the page cannot affect our argument. Just as before, a noncontingent* proposition* is about* every subject matter*. An impossible* proposition* implies*, and also is relevant* to, every proposition*. Whenever one proposition* implies* another, then the first is relevant* to the second. Any truth-preserving* argument-form is relevant*.

None of that changes. But it is up for grabs how it applies to the arguments we take for instances of *quodlibet*. If a proposition* expressed by some sentence we take for a contradiction is not impossible* but merely impossible, false at all worlds but true at some paradoxical worlds*, then there is no reason why this proposition* should either imply* or be relevant* to every proposition*. If we count such cases as genuine instances of *quodlibet* (or better yet, if *all* instances of *quodlibet* are of this kind), then it turns out that *quodlibet* is neither relevant nor truth-preserving. Now the reason why *quodlibet* is irrelevant and the reason why it is not truth-preserving go together like hand and glove. Again, it would be arbitrary in the extreme to say that it is one but not the other.

It is also up for grabs whether the content of the books in the mathematics library is contingent*, despite not being contingent. If it is, the librarian may carry on classifying them without in any way departing from a supervenience conception of aboutness*.

(But his troubles may not yet be at an end. For someday he may have to classify some books on mathematics*: a subject which parallels ordinary mathematics, and which is noncontingent* in the same way that mathematics itself is noncontingent. When that day comes he may hope to discover a broader class of worlds**. . . .)

Needless to say, the undoing of *quodlibet* and the distinguishing of noncontingent subject matters are both of them easier said than done. What I just said is somewhere between "pure" and "applied"—a wave of the hand in the direction of many very different philosphical applications of relevant logic. To finish the job, we would be obliged to take up several questions.

(1) What is the nature of the worlds*? Are they the same sort of things as actual world, the big thing consisting of us and all our surroundings? Or are they mathematical constructions out of parts of the actual world? Or out of parts of all the possible worlds? Do we want to allow only subtly, as opposed to blatantly, impossible worlds*? If so, how do we draw the line?

(2) How do we extend semantics so that sentences not only express propositions, but also express* propositions*? Is this extension arbitrary, or is it governed by pre-existing meanings? (If one wishes to criticize earlier views or comment on ordinary-language arguments, it had better be the latter!)

(3) How do we extend the equivalence relations of subject matters so that they partition worlds*? Again, is this extension arbitrary?

(4) Finally, why should argument-forms be evaluated in terms of truth-preservation* and relevance*?

Different appliers of relevant logic, with different philosophical views, can be expected to undertake this agenda in very different ways.

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