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# Belief Fusion and Revision: An Overview Based on Epistemic Logic Semantics<sup>1</sup>

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*ABSTRACT.* In this paper, we formulate some approaches to belief fusion and revision using epistemic logic semantics. Fusion operators considered in this paper are majority merging, arbitration, and general merging. Some modalities corresponding to belief fusion and revision operators are incorporated into epistemic logics. The Kripke semantics of these extended logics are presented. While most existing approaches treat belief fusion and revision operators as meta-level constructs, we directly incorporate these operators into our object logic language. By doing so, we both extend the expressive power of epistemic logic and enhance the techniques of information fusion.

*KEYWORDS:* Epistemic logic, database merging, belief fusion, majority merging, arbitration, general merging, belief revision, multi-agent systems.

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## 1. Introduction

Philosophical analysis of knowledge and belief has stimulated the development of epistemic logic [HIN 62]. This kind of logic has attracted the attention of researchers from such diverse fields as artificial intelligence (AI), economics, linguistics, and theoretical computer science. AI researchers and computer scientists have developed some technically sophisticated formalisms and applied them to the analysis of distributed and multi-agent systems [FAG 96, MEY 95].

Epistemic logic, in relation to AI and computer science, emphasizes the interaction of agents. This results in the development of multi-agent epistemic logic. One representative example of such logic is proposed by Fagin et al. [FAG 96]. This ap-

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1. A preliminary version of this paper appeared in the proceedings of ECSQARU [LIA 03b].

proach uses the term “knowledge” in a broad sense to cover belief and information.<sup>1</sup> A novel feature of their logic is the consideration of common knowledge and distributed knowledge among a group of agents. Specifically, distributed knowledge can be deduced by pooling everyone’s knowledge. While proper knowledge must be true, the belief of an agent may be wrong. This may cause conflict when merging beliefs. In this case, everything can be deduced from the distributed beliefs due to the logical omniscience property of epistemic logic, so the merged result will be useless for further reasoning.

Instead of pooling all beliefs of agents, there are other techniques for knowledge base merging [CHO 94, CHO 97, KON 00, KON 98, KON 99, LIN 94, LIN 96, LIN 99]. Most of these techniques treat belief fusion operators as meta-level constructs. Given a set of knowledge bases, these fusion operators return the merged results. More precisely, a fusion operator combines a set of knowledge bases  $T_1, T_2, \dots, T_k$  into a merged knowledge base  $T$ , where each knowledge base is a theory in a logical language.

An operator closely related to belief fusion is belief revision. Given a knowledge base  $T$  and a sentence  $\varphi$ , a belief revision operator returns the result of revising  $T$  by  $\varphi$ . The best known framework for belief revision is the AGM theory proposed in [ALC 85, GÄR 88]. The belief revision operator is a special kind of belief fusion operator, in which the new information  $\varphi$  has higher priority than the original belief  $T$  [MAY 01].

Some of the above-mentioned works present concrete operators that can be used directly in the fusion process, while others stipulate the desirable properties of reasonable belief fusion operators by postulates. Few of the approaches provide the capability of multi-agent epistemic reasoning. Consequently, most information fusion logics can not represent nested beliefs. For example, we may have to express something like “John knows that Alice and Bob jointly know the key of the system.” However, this can not be achieved by meta-level information fusion operators. In this paper, we propose that belief fusion operators can be incorporated into the object language of the multi-agent epistemic logic. By doing so, we not only extend the expressive power of information fusion logic, but also circumvent the inconsistency belief problem of distributed belief operators in epistemic logic.

In next section, we review the multi-agent epistemic logic with distributed knowledge operators from [FAG 96] (with slightly different notations). The semantics of that logic is the basis of all logics developed in this paper. In Section 3 through 6, we present epistemic logics for majority merging, arbitration, general merging, and belief revision. We focus mainly on the semantics of these logics, although an axiomatic system for arbitration is also presented. The semantic models of these logics are all extensions of Kripke models of multi-agent epistemic logic with distributed

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1. More precisely, the logic for belief is called doxastic logic. However, here we use the three terms knowledge, belief, and information interchangeably, so we use epistemic logic to cover these three concepts.

knowledge. We also discuss related work for conditional logic in Section 6. Finally, in Section 7, we present our conclusions and suggest directions for further research.

## 2. Preliminaries

Let  $\mathcal{L}$  denote the multi-agent epistemic logic with distributed knowledge [FAG 96]. The alphabet of  $\mathcal{L}$  contains the following symbols:

- 1) a countable set  $\Phi_0 = \{p, q, r, \dots\}$  of atomic propositions
- 2) the propositional constants  $\perp$  (falsum or falsity constant) and  $\top$  (verum or truth constant)
- 3) the binary Boolean operator  $\vee$  (or), and unary Boolean operator  $\neg$  (not)
- 4) a set  $Ag = \{1, 2, \dots, n\}$  of agents
- 5) the modal operator-forming symbols  $[$  and  $]$

The set of well-formed formulas (wffs) is defined as the smallest set that contains  $\Phi_0 \cup \{\perp, \top\}$  and is closed under Boolean and modal operators:

$$WFF := p \mid \perp \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid [G]\varphi$$

where  $p \in \Phi_0$ ,  $G \subseteq Ag$ , and  $\varphi, \psi \in WFF$ . The intuitive meaning of  $[G]\varphi$  is “The group of agents  $G$  has distributed belief  $\varphi$ ”. According to [FAG 96], a group has distributed belief of  $\varphi$  if the belief of  $\varphi$  is distributed among its members, so that by pooling their beliefs, the members of the group can deduce  $\varphi$ , even though no member of the group individually believes  $\varphi$ .

Other classical Boolean connectives  $\wedge$  (and),  $\supset$  (implication), and  $\equiv$  (equivalence) are defined as abbreviations. Also, we write  $\langle G \rangle \varphi$  as an abbreviation of  $\neg[G]\neg\varphi$ . When  $G$  is a singleton  $\{i\}$ , we write  $[i]\varphi$  instead of  $[\{i\}]\varphi$ , so  $[i]\varphi$  means that agent  $i$  knows  $\varphi$ . Furthermore, the auxiliary symbols “(” and “)” (i.e. left and right parentheses) are used to avoid ambiguity of wffs.

For the semantics, a possible world model for  $\mathcal{L}$  is a structure

$$(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V),$$

where

- $W$  is a set of possible worlds,
- $\mathcal{R}_i \subseteq W \times W$  is a serial binary relation <sup>2</sup> over  $W$  for  $1 \leq i \leq n$ , and
- $V : \Phi_0 \rightarrow 2^W$  is a truth assignment that maps each atomic proposition to the set of worlds in which the proposition is true.

From the binary relations  $\mathcal{R}_i$ ’s, a derived relation  $\mathcal{R}_G$  for each nonempty  $G \subseteq Ag$  is:

$$\mathcal{R}_G = \bigcap_{i \in G} \mathcal{R}_i = \{(w, u) \mid \forall i \in G, (w, u) \in \mathcal{R}_i\}.$$

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2. A binary relation  $\mathcal{R}$  is serial if  $\forall w \exists u \mathcal{R}(w, u)$ .

Let  $\mathcal{R}(w)$  denote the set of possible worlds  $\{u \mid (w, u) \in \mathcal{R}\}$  for any binary relation  $\mathcal{R}$ . Intuitively,  $\mathcal{R}_i(w)$  is the set of worlds that agent  $i$  considers possible under  $w$  according to his belief, so  $\mathcal{R}_G(w)$  is the set of worlds that are considered possible by all agents in  $G$ . Such intuition is reflected in the definition of the satisfaction relation. Let  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  be a model and  $\Phi$  be the set of wffs for  $\mathcal{L}$ , then the satisfaction relation  $\models_M \subseteq W \times \Phi$  is defined by the following inductive rules (we use the infix notation for the relation and omit the subscript  $M$  for brevity):

- 1) for each  $p \in \Phi_0$ ,  $w \models p$  iff  $w \in V(p)$ ,
- 2)  $w \not\models \perp$  and  $w \models \top$ ,
- 3)  $w \models \neg\varphi$  iff  $w \not\models \varphi$ ,
- 4)  $w \models \varphi \vee \psi$  iff  $w \models \varphi$  or  $w \models \psi$ ,
- 5)  $w \models [G]\varphi$  iff for all  $u \in \mathcal{R}_G(w)$ ,  $u \models \varphi$ .

A set of wffs  $\Sigma$  is satisfied in a world  $w$ , written as  $w \models \Sigma$ , if  $w \models \varphi$  for all  $\varphi \in \Sigma$ . We write  $\Sigma \models_M \varphi$  if, for each possible world  $w$  in  $M$ ,  $w \models \Sigma$  implies  $w \models \varphi$ , and  $\Sigma \models_{\mathcal{L}} \varphi$  if  $\Sigma \models_M \varphi$  for each  $\mathcal{L}$  model  $M$ . When  $\Sigma = \emptyset$ , it can be omitted. We say that a wff  $\varphi$  is valid in  $M$  if  $\models_M \varphi$  and  $\varphi$  is valid if  $\models_{\mathcal{L}} \varphi$ . For brevity, the subscript is usually omitted.

In this paper, we use the notations of pre-order extensively. Let  $S$  be a set, then a pre-order over  $S$  is a reflexive and transitive binary relation  $\leq$  on  $S$ . A pre-order over  $S$  is called total (or connected) if for all  $x, y \in S$ , either  $x \leq y$  or  $y \leq x$  holds. We will write  $x < y$  as the abbreviation of  $x \leq y$  and  $y \not\leq x$ . For a subset  $S'$  of  $S$ ,  $\min(S', \leq)$  is defined as the set  $\{x \in S' \mid \forall y \in S', y \not\leq x\}$ .

### 3. Merging by Majority

#### 3.1. Review of basic concepts

Majority voting is used to resolve conflicts between agents. As an example, take three knowledge bases  $T_1 = \{p\}$ ,  $T_2 = \{p\}$ , and  $T_3 = \{\neg p\}$ . When combined, the result obtained is  $\{p\}$  because there are two votes for  $p$ , and only one vote against it. One of the most general majority merging functions is defined in [LIN 96]. A function *Merge* is applied to weighted knowledge bases. Let  $wt : \{T_i \mid 1 \leq i \leq k\} \rightarrow \mathfrak{R}^+$  be a weight function that assigns a positive real number<sup>3</sup> to each component knowledge base. A total pre-order over the set of propositional interpretations is defined as:

$$w \preceq_{(\{T_1, T_2, \dots, T_k\}, wt)} w' \text{ iff } \sum_{i=1}^k dist(w, T_i) \cdot wt(T_i) \leq \sum_{i=1}^k dist(w', T_i) \cdot wt(T_i),$$

3. In fact, they allow zero as the weight of a knowledge base. However, we require that a weight is a strictly positive number.

where  $dist$  is a function denoting the distance between a propositional interpretation and a knowledge base. When the propositional language is finite, the Dalal distance (or Hamming distance) between two interpretations of the language is used [DAL 88]. This distance is defined as the number of atoms whose valuations differ between these two interpretations. Let  $dist(w, w')$  denote the Dalal distance between two interpretations  $w$  and  $w'$ . Then, the distance from  $w$  to a theory  $T$ , i.e.  $dist(w, T)$ , is defined as:

$$dist(w, T) = \min\{dist(w, w') \mid w' \models T\}.$$

The merged result  $Merge(T_1, T_2, \dots, T_k; wt)$  is defined as:

$$\{\varphi \mid \forall w \in \min(\Omega, \preceq), w \models \varphi\},$$

where  $\Omega$  is the set of all propositional interpretations and  $\preceq$  is  $\preceq_{(\{T_1, T_2, \dots, T_k\}, wt)}$ . In other words, the models of  $Merge(T_1, T_2, \dots, T_k; wt)$  are the propositional interpretations that have the shortest weighted total distance to the knowledge bases.

### 3.2. Epistemic logic for majority merging

The weighted merging operator proposed in [LIN 96] can be incorporated into epistemic logic in the following way. Syntactically, a new class of modal operators,  $[\oplus(G, wt)]$ , for any nonempty  $G \subseteq \{1, 2, \dots, n\}$  and any list  $wt$  consisting of  $n$  positive real numbers, is added to our logic language. The wff of the majority merging logic ( $\mathcal{L}_{mm}$ ) is defined as:

$$WFF ::= p \mid \perp \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid [G]\varphi \mid [\oplus(G, wt)]\varphi$$

where  $p \in \Phi_0$ ,  $G \subseteq Ag$ , and  $\varphi, \psi \in WFF$ , and  $wt$  is a list of  $n$  positive real numbers. We also assume  $\Phi_0$  is finite for the language of  $\mathcal{L}_{mm}$ .

The list  $wt$  is used to encode the importance or reliability of agents. If agent  $i$  is considered more important or more reliable than agent  $j$ , then the  $i$ th element of  $wt$  is greater than its  $j$ th element. It is tempting to propagate the weights into a group of agents, so that we have a weight  $wt(G)$  for each group  $G$ . This weight may be useful in the belief fusion of two groups of agents. However, we do not really need this because if we want to merge the beliefs of two groups  $G_1$  and  $G_2$ , we can simply merge the beliefs of agents in  $G_1 \cup G_2$ .

Let  $(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  be a model for  $\mathcal{L}$ , then for each  $w \in W$ , define  $V_w : \Phi_0 \rightarrow \{0, 1\}$  and  $A_w : Ag \rightarrow 2^W$  as follows:

$$V_w(p) = 1 \Leftrightarrow w \in V(p) \text{ and } A_w(i) = \mathcal{R}_i(w),$$

for each  $p \in \Phi_0$  and  $i \in Ag$ . A *generalized Dalal distance (or Hamming distance)*  $\delta : W \times W \rightarrow \{0, 1, \dots, |\Phi_0| + n\}$  is defined as

$$\delta(w, u) = |\{p \in \Phi_0 \mid V_w(p) \neq V_u(p)\}| + |\{i \in Ag \mid A_w(i) \neq A_u(i)\}|,$$

for each  $w, u \in W$ . Note that the range of  $\delta$  is finite.

Semantically, an  $\mathcal{L}_{mm}$  model is an  $\mathcal{L}$  model  $(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  satisfying the following two conditions:

1) (Saturation) For each  $w \in W$  and  $p \in \Phi_0$ , there exists an  $u \in W$  such that  $\delta(w, u) = 1$  and  $V_u(p) = 1 - V_w(p)$ .

2) (Indiscernibility) For each  $i \in Ag$  and  $w, u, u' \in W$ , if  $u \in \mathcal{R}_i(w)$  and  $\delta(u, u') = 0$ , then  $u' \in \mathcal{R}_i(w)$ .

Note that we can not use the original Dalal distance to measure the distance between possible worlds, because possible worlds are not merely truth assignments. If there exist two possible worlds  $w$  and  $w'$  such that  $w \in V(p)$  iff  $w' \in V(p)$  for any  $p \in \Phi_0$  and  $\mathcal{R}(w) \neq \mathcal{R}(w')$ , then the original Dalal distance between  $w$  and  $w'$  is zero. However, it is misleading to consider  $w$  and  $w'$  as equivalent worlds since epistemic sentences may have different truth values in these two worlds. The generalized Dalal distance takes the epistemic aspect of possible worlds into account.

The first condition for  $\mathcal{L}_{mm}$  models says that for each atomic proposition  $p$  and each possible world  $w$ , there exists another world  $u$ , such that the only difference between  $w$  and  $u$  is their truth assignment to  $p$ . Thus, the model is saturated with possible worlds with different classical truth assignments. The second condition guarantees that no agent can discern between worlds with zero distance.

The distance from a possible world  $u$  to the belief state of an agent  $i$  in the possible world  $w$  is then defined as:

$$dist_w(u, i) = \min_{w' \in W} \{\delta(u, w') \mid (w, w') \in \mathcal{R}_i\}.$$

A total pre-order  $\preceq_{(G, wt)}^w$  over the possible worlds for each possible world  $w$  and modal operator  $[\oplus(G, wt)]$  is defined as:

$$u \preceq_{(G, wt)}^w u' \text{ iff } \sum_{i \in G} dist_w(u, i) \cdot wt(i) \leq \sum_{i \in G} dist_w(u', i) \cdot wt(i),$$

where  $wt(i)$  denotes the  $i$ th number of the list  $wt$ . We can define the satisfaction of the wff  $[\oplus(G, wt)]\varphi$  by

$$w \models [\oplus(G, wt)]\varphi \text{ iff for all } u \in \min(W, \preceq_{(G, wt)}^w), u \models \varphi.$$

The validity in  $\mathcal{L}_{mm}$  is defined as that for  $\mathcal{L}$  and denoted by  $\models_{\mathcal{L}_{mm}}$ .

The following theorem shows several important validity results in  $\mathcal{L}_{mm}$ . For brevity, we omit the subscript in  $\models_{\mathcal{L}_{mm}}$ .

**Theorem 1** For any  $G \subseteq Ag$  and any list  $wt$  consisting of  $n$  positive real numbers, we have:

$$1) \models [\oplus(G, wt)]\varphi \supset [G]\varphi$$

- 2)  $\models \neg[G]\perp \supset ([G]\varphi \supset [\oplus(G, wt)]\varphi)$   
 3)  $\models \neg[\oplus(G, wt)]\perp$   
 4) if  $G' \subseteq G$  is a group such that  $\sum_{i \in G'} wt(i) > \sum_{i \in G} wt(i)/2$ , then

$$\models (\wedge_{i \in G'} [i]l) \supset [\oplus(G, wt)]l,$$

where  $l$  is an atomic proposition or its negation (i.e. a literal)

- 5) if  $i \in G$  is an agent such that  $wt(i) > \sum_{j \in G - \{i\}} wt(j)$ , then

$$\models [i]\varphi \supset [\oplus(G, wt)]\varphi$$

**Proof:** Let  $M$  be any  $\mathcal{L}_{mm}$  model and  $w$  be any possible world in  $M$ . We prove that each result is satisfied in  $w$ .

1) We show  $\cap_{i \in G} \mathcal{R}_i(w) \subseteq \min(W, \preceq_{(G, wt)}^w)$ . Let  $u \in \cap_{i \in G} \mathcal{R}_i(w)$ , then  $dist_w(u, i) = 0$  for each  $i \in G$  by definition. Thus,  $u \in \min(W, \preceq_{(G, wt)}^w)$  due to  $\sum_{i \in G} dist_w(u, i) = 0$ . The result is then satisfied in  $w$  by the definition of satisfaction.

2) We show that  $\cap_{i \in G} \mathcal{R}_i(w) \neq \emptyset$  implies  $\min(W, \preceq_{(G, wt)}^w) \subseteq \cap_{i \in G} \mathcal{R}_i(w)$ . Let  $u \in \min(W, \preceq_{(G, wt)}^w)$ . Observing the proof of 1), we can find that  $\sum_{i \in G} dist_w(u, i) = 0$  holds. Since each  $dist_w(u, i)$  is nonnegative, we have  $dist_w(u, i) = 0$  for each  $i \in G$ . This means that for each  $i \in G$ , there exists  $w' \in \mathcal{R}_i(w)$  such that  $\delta(u, w') = 0$ . By the indiscernibility condition of the model, we have  $u \in \cap_{i \in G} \mathcal{R}_i(w)$ . Again, the result is satisfied in  $w$  by the definition of satisfaction.

3) Because the range of  $\delta$  is finite, there does not exist any infinitely descending chain for the ordering  $\preceq_{(G, wt)}^w$ . Consequently,  $\min(W, \preceq_{(G, wt)}^w)$  must be nonempty and the result holds by the definition of satisfaction.

4) The proof is analogous to that of Theorem 4.1 in [LIN 96]. Assume that the result is not satisfied, then  $w \models \wedge_{i \in G'} [i]l$  and  $w \not\models [\oplus(G, wt)]l$  hold. The former implies that for all  $i \in G'$  and  $u \in \mathcal{R}_i(w)$ ,  $u \models l$ , whereas the latter implies that there exists  $u_0 \in \min(W, \preceq_{(G, wt)}^w)$  such that  $u_0 \models \neg l$ . By the saturation condition, there exists another world  $u' \models l$  and  $\delta(u_0, u') = 1$ . This implies  $dist_w(u', i) = dist_w(u_0, i) - 1$  for all  $i \in G'$ , and  $dist_w(u', i) \leq dist_w(u_0, i) + 1$  for all  $i \in G - G'$ . Thus,

$$\begin{aligned} \sum_{i \in G} dist_w(u', i) \cdot wt(i) &\leq \sum_{i \in G} dist_w(u_0, i) \cdot wt(i) + \sum_{i \in G - G'} wt(i) - \sum_{i \in G'} wt(i) \\ &< \sum_{i \in G} dist_w(u_0, i) \cdot wt(i). \end{aligned}$$

This contradicts  $u_0 \in \min(W, \preceq_{(G, wt)}^w)$ , so the result must be satisfied. Note that the last inequality holds due to the assumption about the  $wt$  list.

5) We show that the assumption implies  $\min(W, \preceq_{(G, wt)}^w) \subseteq \mathcal{R}_i(w)$ . Let  $u \in \min(W, \preceq_{(G, wt)}^w)$  be a world not in  $\mathcal{R}_i(w)$  and  $w'$  be a world in  $\mathcal{R}_i(w)$  such that  $dist_w(u, i) = \delta(u, w')$ . By the indiscernibility condition of the model,  $\delta(u, w')$  is strictly positive. According to the definition of  $\delta$  and  $dist_w$ , we can prove  $dist_w(w', j) \leq dist_w(u, j) + \delta(u, w')$  for each  $j \neq i$ . Thus,

$$\begin{aligned} \sum_{j \in G} dist_w(w', j) \cdot wt(j) &\leq \sum_{j \in G - \{i\}} (dist_w(u, j) + \delta(u, w')) \cdot wt(j) \\ &< \sum_{i \in G} dist_w(u, i) \cdot wt(i). \end{aligned}$$

This means  $w' \prec_{(G, wt)}^w u$  and contradicts  $u \in \min(W, \preceq_{(G, wt)}^w)$ . Therefore,  $\min(W, \preceq_{(G, wt)}^w) \subseteq \mathcal{R}_i(w)$  holds and the result follows by the definition of satisfaction.

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The first result shows that majority merging is logically stronger than distributed belief, and the second shows that they are actually equivalent when the distributed belief of the merged agents is consistent. The third shows that majority merging can indeed resolve conflicts between the beliefs of different agents, while the fourth shows that the merged belief reflects the majority view on literals. Note that the fourth result holds only for literals, instead of general formulas. The last result shows that if one agent has dominating power, his belief will be fully kept in the merged belief. We expect that these results can serve as the basis of an axiomatic system for  $\mathcal{L}_{mm}$ , although it is still far from being complete.

The next theorem shows that  $\mathcal{L}_{mm}$  is a conservative extension of  $\mathcal{L}$ .

**Theorem 2** For each wff  $\varphi$  in  $\mathcal{L}$ , we have  $\models_{\mathcal{L}} \varphi$  iff  $\models_{\mathcal{L}_{mm}} \varphi$ .

**Proof:** The theorem is based on the following two facts. First, because each  $\mathcal{L}_{mm}$  model is also an  $\mathcal{L}$  model,  $\models_{\mathcal{L}} \varphi$  implies  $\models_{\mathcal{L}_{mm}} \varphi$ . Second, we show that if  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model and  $w \in W$  such that  $w \models_M \varphi$ , then there exists an  $\mathcal{L}_{mm}$  model  $M'$  such that  $\varphi$  is satisfiable in  $M'$ . The construction of  $M'$  from  $M$  and  $w$  is a little complicated, so we divide it into four smaller steps.

1) Unraveling (or unwinding, or unfolding): This is a well-known technique in modal logic [BLA 01]. By using such a technique,  $M$  can be transformed into a *tree-like* model<sup>4</sup> with the root being  $w$  ([BLA 01], proposition 2.15). Therefore, without loss of generality, we can assume that  $M$  is a tree-like model and  $w$  is the root of the tree.

4. In our context, a model  $(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is tree-like if  $(W, \cup_{i \in Ag} \mathcal{R}_i)$  is a tree in the graph-theoretic sense, where  $W$  is viewed as the set of nodes and  $\cup_{i \in Ag} \mathcal{R}_i$  is viewed as the set of arcs.



2) Cutting: Let us first define the modal depth of wffs in  $\mathcal{L}$  as:

- a)  $depth(p) = depth(\perp) = depth(\top) = 0$  if  $p$  is a propositional atom,
- b)  $depth([G]\psi) = depth(\psi) + 1$  for any nonempty  $G \subseteq Ag$ ,
- c)  $depth(\neg\psi) = depth(\psi)$ ,
- d)  $depth(\psi_1 \vee \psi_2) = \max(depth(\psi_1), depth(\psi_2))$ .

Let  $depth(\varphi) = m$ , then we cut off the worlds not reachable from the root  $w$  within  $m$  steps. More precisely, let

$$U = \{w' \mid \exists(w_0, w_1, \dots, w_k) \text{ such that } k \leq m, w_0 = w, w_k = w', \\ \text{and } \forall 0 \leq j \leq k-1, (w_j, w_{j+1}) \in \cup_{i \in Ag} \mathcal{R}_i\}$$

Let us define the model  $M_0 = (W_0, (\mathcal{R}_i^0)_{1 \leq i \leq n}, V^0)$ , where

- $W_0 = W - U$ ,
- $\mathcal{R}_i^0 = (\mathcal{R}_i \cap (W_0 \times W_0)) \cup \{(u, u) \mid \exists u' \in W_0 \text{ such that } (u, u') \in \mathcal{R}_i\}$ ,
- and  $V^0(p) = V(p) \cap W_0$  for each atom  $p$ .

Note that  $M_0$  is a tree-like model of depth  $m$  except each of its leaf nodes has a self-loop for each agent  $i$ . It can be seen that we still have  $w \models_{M_0} \varphi$ .

3) Adding arcs: To make the model satisfy the indiscernibility condition, we have to add some necessary arcs into the finite depth tree  $M_0$ . This can be achieved by scanning the nodes backward from the leaf nodes to the root. It can be seen that the indiscernibility condition is satisfied for the leaf nodes, which have distance  $m$  from the root. For  $1 \leq k \leq m$ , let us define  $M_k = (W_0, (\mathcal{R}_i^k)_{1 \leq i \leq n}, V^0)$ , where

$$\mathcal{R}_i^k = \mathcal{R}_i^{k-1} \cup \{(u, u_2) \mid u \text{ is a node with distance } (m-k) \text{ from the root and} \\ \exists u_1, u_2 \in W_0 \text{ such that } [\delta_{k-1}(u_1, u_2) = 0, (u, u_1) \in \mathcal{R}_i^{k-1}, \\ \text{and } (u, u_2) \notin \mathcal{R}_i^{k-1}]\}$$

In the definition above, we use  $\delta_{k-1}$  to denote the generalized Dalal distance in the model  $M_{k-1}$ . When all necessary arcs are added in this way,  $M_m$  is a model satisfying the indiscernibility condition and we still have  $w \models_{M_m} \varphi$ .

4) Saturating: For each  $u \in W_0$  and  $p \in \Phi_0$ , if there does not exist  $u' \in W_0$  such that  $\delta_m(u, u') = 1$  and  $V_u(p) = 1 - V_{u'}(p)$ , then take a new world  $u_0$  and call it the  $p$ -flipping world of  $u$ . Now, our model  $M'$  is defined as  $(W', (\mathcal{R}'_i)_{1 \leq i \leq n}, V')$ , where

$$W' = W_0 \cup \{u_0 \mid u_0 \text{ is the } p\text{-flipping world of some } u \in W_0 \text{ for some } p \in \Phi_0\}$$

$$\mathcal{R}'_i = \mathcal{R}_i^m \cup \{(u_0, u') \mid u_0 \text{ is the } p\text{-flipping world of some} \\ u \in W_0 \text{ for some } p \in \Phi_0 \text{ and } (u, u') \in \mathcal{R}_i^m\}$$

$$V'(p) = V_0(p) \cup \{u_0 \mid u_0 \text{ is the } p\text{-flipping world of some } u \notin V_0(p)\}$$

Obviously,  $M'$  satisfies both the saturation and the indiscernibility conditions, so it is an  $\mathcal{L}_{mm}$  model. Furthermore, we have  $w \models_{M'} \varphi$ . Therefore, our result is proved.

■

Let us close this section with an example to show the expressive power of  $\mathcal{L}_{mm}$ .

**Example 1** Let  $Ag = \{1, 2, \dots, 5\}$ ,  $\Sigma = \{[1][2]p, [1][3]p, [1][4]p, [1][5]\neg p\}$ ,  $wt_1 = (0.2, 0.1, 0.2, 0.3, 0.7)$ , and  $wt_2 = (0.2, 0.1, 0.2, 0.4, 0.4)$ , then we have

$$\Sigma \models_{\mathcal{L}_{mm}} [1]([1]p \supset [\oplus(G, wt_1)]p \wedge [1]\neg p \supset [\oplus(G, wt_1)]\neg p),$$

and

$$\Sigma \models_{\mathcal{L}_{mm}} [1][\oplus(G, wt_2)]p.$$

The first sentence shows that agent 1 believes that his belief is crucially important to the merged belief, according to the weight distribution  $wt_1$ . The second sentence shows that agent 1 believes that his belief does not have any influence on the merged belief because, whether he believes  $p$  or not, the merged belief always contains  $p$ , according to the weight distribution  $wt_2$ . The reasoning of the importance of an agent by itself is not possible in the meta-level approach to information fusion. ■

## 4. Arbitration

### 4.1. Review of basic concepts

Distance measure between possible worlds is also used in arbitration, which is another type of merging operator [LIB 95, REV 93, REV 97]. Arbitration is the process of settling a conflict between two or more parties. The arbitration operator between knowledge bases was first proposed in [REV 93] and then further articulated in [REV 93, LIB 95, LIB 98]. Here, we are interested in the semantic characterization of arbitration given in [LIB 95, LIB 98]. A knowledge base in [LIB 95, LIB 98] is identified with a set of propositional interpretations. The semantic characterization for this kind of arbitration is given by assigning to each subset of models  $A$  a binary relation  $\leq_A$  over the set of model sets satisfying the five conditions in Figure 1 (the subscript is omitted when it means all binary relations of the form  $\leq_A$ )

- 1) transitivity: if  $A \leq B$  and  $B \leq C$  then  $A \leq C$
- 2) if  $A \subseteq B$  then  $B \leq A$
- 3)  $A \leq A \cup B$  or  $B \leq A \cup B$
- 4)  $B \leq_A C$  for every  $C$  iff  $A \cap B \neq \emptyset$
- 5)  $A \leq_{C \cup D} B \Leftrightarrow \begin{cases} C \leq_{A \cup B} D \text{ and } A \leq_C B \text{ or} \\ D \leq_{A \cup B} C \text{ and } A \leq_D B \end{cases}$

**Figure 1.** Conditions for  $\leq_A$

To understand these conditions, let us imagine a pseudo-distance measure  $\mu : W \times W \rightarrow \mathbb{R}^+ \cup \{0\}$ , which satisfies at least the following two properties for any  $w, u \in W$ : (i)  $\mu(w, u) = 0$  iff  $w = u$  and (ii)  $\mu(w, u) = \mu(u, w)$ . The measure  $\mu$  is extended to the domain  $2^W \times 2^W$  as  $\mu(S, T) = \inf_{w \in S, u \in T} \mu(w, u)$  for any  $S, T \subseteq W$ . Note that we have  $\mu(S, T_1 \cup T_2) = \min(\mu(S, T_1), \mu(S, T_2))$  by this definition. The ordering  $A \leq_C B$  can be defined as

$$A \leq_C B \Leftrightarrow \mu(A, C) \leq \mu(B, C). \quad (1)$$

Then, the five conditions are fairly obvious properties of this definition.

The arbitration between two sets of models  $A$  and  $B$  is defined as

$$A \Delta B = \min(A, \leq_B) \cup \min(B, \leq_A) \quad (2)$$

Note that although the relation  $\leq_A$  is defined between sets of models, only singleton comparison is used in the definition of arbitration. By slightly abusing the notation,  $\leq_A$  may also denote an ordering between models.

#### 4.2. Epistemic logic for arbitration

To incorporate the arbitration operator of [LIB 95, LIB 98] into epistemic logic, first note that according to (2), arbitration is commutative, but not necessarily associative. Thus, the arbitration operator should be a binary operator between two agents. We can add a class of modal operators for arbitration into our logic just as in the case of majority merging. However, to be more expressive, we will also consider the interaction between arbitration and other epistemic operators. We therefore define the set of *arbitration expressions* (AE) over  $Ag$  recursively as the smallest set containing  $Ag$  and closed under the binary operators  $+$ ,  $\cdot$ , and  $\Delta$ :

$$AE ::= i \mid a + b \mid a \cdot b \mid a \Delta b$$

where  $i \in Ag$  and  $a, b \in AE$ .

Here,  $+$  and  $\cdot$  correspond respectively to the distributed belief and the “everybody knows” operators in multi-agent epistemic logic [FAG 96]. The wff for arbitration logic ( $\mathcal{L}_{ar}$ ) is defined as:

$$WFF ::= p \mid \perp \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid [a]\varphi$$

where  $p \in \Phi_0$ ,  $a \in AE$ , and  $\varphi, \psi \in WFF$ . We no longer need the operator  $[G]$  because it is a special case of operators  $[a]$ . In other words, we can take  $[i_1 + i_2 + \dots + i_k]$  as  $[G]$  where  $G = \{i_1, i_2, \dots, i_k\}$ . For example,  $[1 + 2 + 3]\varphi$  means that  $\varphi$  can be deduced if the beliefs of agents 1, 2, and 3 are pooled, even though none of the three agents individually believes  $\varphi$ , and  $[2 \cdot 5]\varphi$  means that both agent 2 and agent 5 believe  $\varphi$ , whereas  $[1 \Delta 3]\varphi$  means that the arbitration of the beliefs of agent 1 and agents 3 can deduce  $\varphi$ .

For semantics, an  $\mathcal{L}_{ar}$  model is

$$(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$$

where

–  $(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model,

–  $\leq$  is a function that assigns a binary relation  $\leq_A \subseteq 2^W \times 2^W$  satisfying the five conditions in Figure 1 to each subset of possible worlds  $A \subseteq W$ .

Note that the first two conditions in Figure 1 imply that  $\leq_A$  is a pre-order over  $2^W$ . For each arbitration expression, we can define the binary relations  $\mathcal{R}_{a+b}$ ,  $\mathcal{R}_{a \cdot b}$  and  $\mathcal{R}_{a \Delta b}$  over  $W$  recursively as:

$$\mathcal{R}_{a+b} = \mathcal{R}_a \cap \mathcal{R}_b \quad (3)$$

$$\mathcal{R}_{a \cdot b} = \mathcal{R}_a \cup \mathcal{R}_b \quad (4)$$

$$\mathcal{R}_{a \Delta b}(w) = \min(\mathcal{R}_a(w), \leq_{\mathcal{R}_b(w)}) \cup \min(\mathcal{R}_b(w), \leq_{\mathcal{R}_a(w)}) \quad (5)$$

The satisfaction for wff  $[a]\varphi$  is:

$$u \models [a]\varphi \text{ iff for all } w \in \mathcal{R}_a(u), w \models \varphi.$$

### 4.3. An axiomatic system for $\mathcal{L}_{ar}$

Since a set of possible worlds  $W$  may be infinite in our logic, the minimal models in (5) may not exist. We define *coherent models* as those satisfying the limit assumption [ARL 92] for each binary relation  $\leq_A$  such that  $A \subseteq W$ :

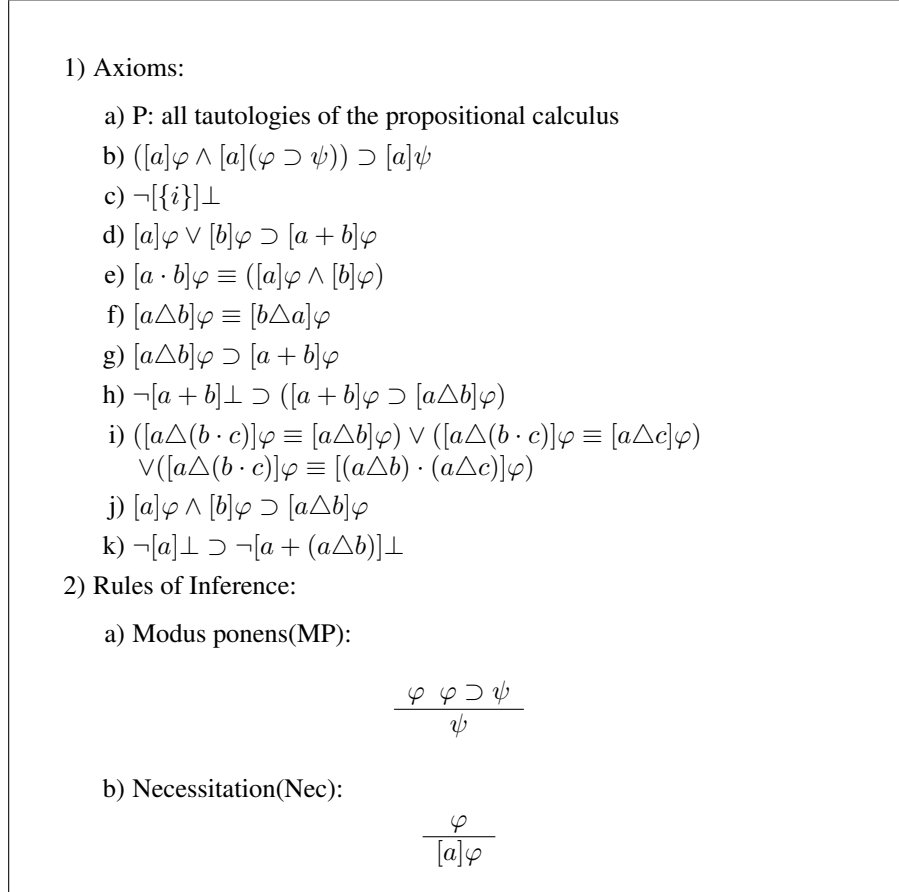
$$\text{for any nonempty } U \subseteq W, \min(U, \leq_A) \text{ is nonempty.}$$

An axiomatic system for valid inference in coherent models is presented in Figure 2, where  $a, b$ , and  $c$  are meta-variables for arbitration expressions,  $i$  is meta-variable for agents, and  $\varphi$  and  $\psi$  are meta-variables for wffs. Axioms (a) through (e) are basic axioms of epistemic logic. Axioms (f) through (k) correspond to the postulates 2 through 4, and 6 through 8 of [LIB 95]. Postulate 5 of [LIB 95] corresponds to the following derived theorem:

$$[a \Delta b] \perp \supset [a] \perp \wedge [b] \perp,$$

which can be derived by using axioms (a), (d) and (k).

The derivability in the axiomatic system is defined as follows. A wff  $\varphi$  is derivable from  $\mathcal{L}_{ar}$  (or simply  $\varphi$  is a *theorem* of  $\mathcal{L}_{ar}$ ), if there is a finite sequence  $\varphi_1, \dots, \varphi_m$  such that  $\varphi = \varphi_m$  and every  $\varphi_i$  is an instance of an axiom schema, or is obtained from earlier  $\varphi_j$ 's by the application of an inference rule. We write  $\vdash_{\mathcal{L}_{ar}} \varphi$  if  $\varphi$  is a theorem of  $\mathcal{L}_{ar}$ . Let  $\Sigma \cup \{\varphi\}$  be a subset of wffs, then  $\varphi$  is derivable from  $\Sigma$  in the system



**Figure 2.** An axiomatic system for the logic of arbitration

$\mathcal{L}_{ar}$ , written as  $\Sigma \vdash_{\mathcal{L}_{ar}} \varphi$ , if there is a finite subset  $\Sigma'$  of  $\Sigma$  such that  $\vdash_{\mathcal{L}_{ar}} \bigwedge \Sigma' \supset \varphi$ . Theorem 3 shows the soundness of the axiomatic system, though, as yet, it is unclear whether the system is complete.

**Theorem 3** For any wff  $\varphi$ ,  $\vdash_{\mathcal{L}_{ar}} \varphi$  implies  $\models_{\mathcal{L}_{ar}} \varphi$ , where  $\models_{\mathcal{L}_{ar}} \varphi$  denotes that  $\varphi$  is valid in all coherent  $\mathcal{L}_{ar}$  models.

**Proof:** The validity of axioms (a) through (e) is easily proven from the semantics of  $\mathcal{L}$ . Also, it is easy to show that the inference rules are validity-preserving. The proof of the validity of axioms (f) through (k) is essentially the same as that for theorem 5 of [LIB 98]. Note that the limit assumption for coherent models is needed for the validity of axiom (k).■

Recalling that we can abbreviate  $[i_1+i_2+\dots+i_k]$  as  $[G]$  where  $G = \{i_1, i_2, \dots, i_k\}$ , each  $\mathcal{L}$  wff is also an  $\mathcal{L}_{ar}$  wff. The next theorem shows that  $\mathcal{L}_{ar}$  is a conservative extension of  $\mathcal{L}$ .

**Theorem 4** For each  $\mathcal{L}$  wff  $\varphi$ ,  $\models_{\mathcal{L}} \varphi$  iff  $\models_{\mathcal{L}_{ar}} \varphi$ .

**Proof:** To prove this theorem, we only need to show the following two facts:

1) If  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  is a coherent  $\mathcal{L}_{ar}$  model, then  $M' = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model such that for all  $w \in W$  and  $\mathcal{L}$  wff  $\varphi$ , we have  $w \models_M \varphi$  iff  $w \models_{M'} \varphi$ .

2) If  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model, then there exist  $\leq$  such that  $M' = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  is a coherent  $\mathcal{L}_{ar}$  model, and for all  $w \in W$  and  $\mathcal{L}$  wff  $\varphi$ , we have  $w \models_M \varphi$  iff  $w \models_{M'} \varphi$ . To prove this, let  $\leq$  be defined as<sup>5</sup>:

$$A \leq_C B \Leftrightarrow A \cap C \neq \emptyset \vee B \cap C = \emptyset.$$

It is then easily verified that  $\leq$  satisfies the conditions given in Figure 1. Furthermore, for any nonempty  $U \subseteq W$ ,

$$\min(U, \leq_A) = \begin{cases} U \cap A & \text{if } U \cap A \neq \emptyset \\ U & \text{otherwise} \end{cases}$$

according to the definition. Therefore,  $M'$  satisfies the limit assumption. ■

The next example shows some valid inference in  $ar$ .

**Example 2** This example is related to the notion of trust[LIA 03a]. One consequence of agent 1 trusting agent 2 is that the former believes in the judgement of the latter. In epistemic logic, this is written as  $[1]([2]\varphi \supset \varphi)$ . Now, let

$$\Sigma = \{[1]([3](\varphi \wedge \psi) \supset (\varphi \wedge \psi)), [2]([3](\varphi \wedge \psi) \supset (\varphi \wedge \neg\psi))\}.$$

This means that agent 1 fully trusts agent 3 for his judgement on  $\varphi \wedge \psi$ , whereas agent 2 only partially trusts agent 3 for his judgement on the same thing. Therefore, if both agent 1 and 2 receive information  $\varphi$  and  $\psi$  from agent 3, then due to different judgements on the reliability of information source 3, their beliefs are in conflict. By using arbitration, we can extract their common beliefs, i.e.  $\varphi$ . Formally, we have

$$\Sigma \models_{\mathcal{L}_{ar}} ([1][3]\varphi \wedge [1][3]\psi \wedge [2][3]\varphi \wedge [2][3]\psi) \supset [1\Delta 2]\varphi.$$

■

5. Alternatively, we can define  $\mu : W \times W \rightarrow \{0, 1\}$  as  $\mu(x, y) = 0$  iff  $x = y$  and then define  $\leq$  by equation 1.

## 5. General Merging

### 5.1. Review of basic concepts

In [KON 98], an axiomatic framework unifying the majority merging and arbitration operators is presented. A set of postulates common to majority and arbitration operators is proposed to characterize the general merging operators, and additional postulates for differentiating them are considered. In [KON 98], a knowledge base is also a finite set of propositional sentences. The general merging operator is defined as a mapping from a multi-set<sup>6</sup> of knowledge bases (called a *knowledge set*) to a knowledge base. Therefore, the arbitration operator defined using this approach can merge more than two knowledge bases, whereas the definition of the arbitration operator in [LIB 95, LIB 98] can only merge two knowledge bases. The merging operator is denoted by  $\Delta$ . For each knowledge set  $E$ ,  $\Delta(E)$  is a knowledge base. A semantic characterization based on *syncretic assignment* is given for the merging operators. A syncretic assignment maps each knowledge set  $E$  to a pre-order  $\leq_E$  over interpretations such that conditions reflecting the postulated properties of the merging operators must be satisfied. Then  $\Delta(E)$  is the knowledge base whose models are the minimal interpretations according to  $\leq_E$ .

This framework is further extended to deal with integrity constraints in [KON 99]. Let  $E$  be a knowledge set and  $\varphi$  be a propositional sentence denoting the integrity constraints. The merging of knowledge bases in  $E$  with integrity constraint  $\varphi$ ,  $\Delta_\varphi(E)$ , is a knowledge base that implies  $\varphi$ . The models of  $\Delta_\varphi(E)$  are characterized by  $\min(\text{Mod}(\varphi), \leq_E)$ .  $\Delta_\varphi(E)$  is called an IC merging operator. According to the semantics, it is obvious that  $\Delta(E)$  is a special case of IC merging operator  $\Delta_\top(E)$ . When  $E$  contains exactly one knowledge base, the operator is reduced to the AGM revision operator proposed in [ALC 85]. Therefore, IC merging is able to cover majority merging, arbitration, and the AGM revision operator.

### 5.2. Epistemic logic for general merging

The wff of IC merging logic ( $\mathcal{L}_{ic}$ ) is defined as:

$$WFF ::= p \mid \perp \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid \Box\varphi \mid [G]\varphi \mid [\Delta_\varphi(G)]\psi$$

where  $p \in \Phi_0$ ,  $G \subseteq Ag$ , and  $\varphi, \psi \in WFF$ . Note that in addition to the distributed belief operator and the fusion operator, we also add the alethic modal operator  $\Box$ , which will be useful in the statement of some valid properties of the logic. As usual, we write  $\neg\Box\neg\varphi$  as  $\diamond\varphi$ .

6. A multi-set, also called a bag, is a collection of elements over a domain which allows multiple occurrences of elements.

We call a subset of possible worlds a *belief state*. Let  $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$  denote a multi-set of belief states, then  $\bigcap \mathcal{U} = U_1 \cap \dots \cap U_k$ . An  $\mathcal{L}_{ic}$  model is a quadruple

$$(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$$

where

–  $(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model,  
–  $\leq$  is an assignment mapping each multi-set of belief states  $\mathcal{U}$  to a total pre-order  $\leq_{\mathcal{U}}$  over  $W$  satisfying the following conditions:

- 1) If  $w, w' \in \bigcap \mathcal{U}$ , then  $w \leq_{\mathcal{U}} w'$ ,
- 2) If  $w \in \bigcap \mathcal{U}$  and  $w' \notin \bigcap \mathcal{U}$  then  $w <_{\mathcal{U}} w'$ ,
- 3) For any  $w \in U_1$ , there exists  $w' \in U_2$ , such that  $w' \leq_{\{U_1, U_2\}} w$ , where  $U_1$  and  $U_2$  are two belief states,
- 4) If  $w \leq_{\mathcal{U}_1} w'$  and  $w \leq_{\mathcal{U}_2} w'$ , then  $w \leq_{\mathcal{U}_1 \sqcup \mathcal{U}_2} w'$ , where  $\sqcup$  denotes the union of two multi-sets,
- 5) If  $w <_{\mathcal{U}_1} w'$  and  $w \leq_{\mathcal{U}_2} w'$ , then  $w <_{\mathcal{U}_1 \sqcup \mathcal{U}_2} w'$ .

These conditions are model-theoretic counterparts of those for syncretic assignments in [KON 98, KON 99]. Condition 1 states that possible worlds appearing in the belief states of all agents are equally plausible. Condition 2 states that a possible worlds that appears in the belief states of all agents is more plausible than those that don't. Condition 3 requires that all agents are treated fairly. Therefore, even if agent 1 considers  $w$  possible, it is not more plausible than all other worlds in the belief state of agent 2. Conditions 4 and 5 require that if two groups of agents agree on the ordering of  $w$  and  $w'$ , then the united group of these two groups does not reverse the ordering.

For a group of agents  $G$  and a possible world  $u$ , let us define a total pre-order  $\leq_G^u$  over  $W$  as:

$$w \leq_G^u w' \text{ iff } w \leq_{\{\mathcal{R}_i(u) \mid i \in G\}} w'.$$

The truth condition of  $[\Delta_\varphi(G)]\psi$  is defined as that for conditional logic [BOU 94b, BOU 94a]. Formally,  $u \models [\Delta_\varphi(G)]\psi$  iff

- (i) there are no possible worlds in  $W$  satisfying  $\varphi$ , or
- (ii) there exists  $w_0 \in W$  such that  $w_0 \models \varphi$  and for any  $w \leq_G^u w_0$ ,  $w \models \varphi \supset \psi$ .

Furthermore, the satisfaction of the alethic modal formula  $\Box\varphi$  is defined as:  $u \models \Box\varphi$  iff for all  $w \in W$ ,  $w \models \varphi$ .

Note that in IC merging, a knowledge set consists of a multi-set of objective sentences, whereas for the modal operator  $[\Delta_\varphi(G)]$ ,  $G$  is a set of agents whose beliefs may contain any epistemic sentences. Also, an integrity constraint in [KON 99] must be an objective sentence, whereas  $\varphi$  may be arbitrary complex wffs of our extended language. Furthermore, instead of selecting minimal models of  $\varphi$ , since the set of



possible worlds may be infinite in our case, we adopt the system-of-spheres semantics as described in [BOU 94b, BOU 94a] for the fusion operator  $[\Delta_\varphi(G)]$ .

The next theorem shows several important valid wffs in  $\mathcal{L}_{ic}$ .

**Theorem 5**

- 1)  $\models [\Delta_\varphi(G)]\varphi$
- 2)  $\models \diamond\varphi \supset \neg[\Delta_\varphi(G)]\perp$
- 3)  $\models \neg[G]\neg\varphi \supset ([G](\varphi \supset \psi) \equiv [\Delta_\varphi(G)]\psi)$
- 4)  $\models \Box(\varphi_1 \equiv \varphi_2) \supset ([\Delta_{\varphi_1}(G)]\psi \equiv [\Delta_{\varphi_2}(G)]\psi)$
- 5)  $\models [\Delta_{\varphi_1 \wedge \varphi_2}(G)]\psi \supset [\Delta_{\varphi_1}(G)](\varphi_2 \supset \psi)$
- 6)  $\models \neg[\Delta_{\varphi_1}(G)]\neg\varphi_2 \supset ([\Delta_{\varphi_1}(G)]\psi \supset [\Delta_{\varphi_1 \wedge \varphi_2}(G)]\psi)$

**Proof:** We only prove 3) and the remaining ones can be proven analogously. Let  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  be an  $\mathcal{L}_{ic}$  model and  $w \in W$ . Assume  $w \models \neg[G]\neg\varphi$  holds, then there exists  $u_0 \in \bigcap_{i \in G} \mathcal{R}_i(w)$  such that  $u_0 \models \varphi$  holds. First, by conditions 1) and 2) for  $\mathcal{L}_{ic}$  models, we have  $u \leq_G^w u_0$  iff  $u \in \bigcap_{i \in G} \mathcal{R}_i(w)$  for any  $u \in W$ . Thus, if  $w \models [G](\varphi \supset \psi)$  holds, then for any  $u \leq_G^w u_0$ ,  $u \models \varphi \supset \psi$ . Therefore,  $w \models [\Delta_\varphi(G)]\psi$ . Second, if  $w \models [\Delta_\varphi(G)]\psi$  holds, then there exists  $w_0 \models \varphi$  such that for any  $u \leq_G^w w_0$ ,  $u \models (\varphi \supset \psi)$ . Again, by conditions 1) and 2) for  $\mathcal{L}_{ic}$  models, we have  $u \leq_G^w w_0$  for any  $u \in \bigcap_{i \in G} \mathcal{R}_i(w)$ . Therefore,  $w \models [G](\varphi \supset \psi)$  holds. ■

Note that the set of  $\mathcal{L}$  wffs is a subset of  $\mathcal{L}_{ic}$  wffs. The next theorem shows that  $\mathcal{L}_{ic}$  is a conservative extension of  $\mathcal{L}$ .

**Theorem 6** For each  $\mathcal{L}$  wff  $\varphi$ ,  $\models_{\mathcal{L}} \varphi$  iff  $\models_{\mathcal{L}_{ic}} \varphi$ .

**Proof:** To prove this theorem, we only need to show the following two facts:

1) If  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  is an  $\mathcal{L}_{ic}$  model, then  $M' = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model such that for all  $w \in W$  and  $\mathcal{L}$  wff  $\varphi$ , we have  $w \models_M \varphi$  iff  $w \models_{M'} \varphi$ .

2) If  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model, then there exist  $\leq$  such that  $M' = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  is an  $\mathcal{L}_{ic}$  model, and for all  $w \in W$  and  $\mathcal{L}$  wff  $\varphi$ , we have  $w \models_M \varphi$  iff  $w \models_{M'} \varphi$ . To prove this, let  $\leq$  be defined as:

$$w \leq_{\mathcal{U}} w' \Leftrightarrow |\{U \in \mathcal{U} \mid w \in U\}| \geq |\{U \in \mathcal{U} \mid w' \in U\}|.$$

It is then easily verified that  $\leq$  indeed satisfies the conditions for  $\mathcal{L}_{ic}$  models. ■

The next example is a realistic scenario to show the expressive power of  $\mathcal{L}_{ic}$ .

**Example 3** Let us consider the following scenario. There are two managers (agents 1 and 2) in a company. Agent 1 receives reports from three assistants (agents 3,4 and 5), whereas agent 2 is responsible for collecting external information from sources 6 and 7. Agent 1 forms his belief by merging the beliefs of his assistants, but agent 2

only believes the information that both agents 6 and 7 believe. These constraints are written as two schemata:

$$S_1(\varphi) = [\Delta_{\top}(\{3, 4, 5\})]\varphi \supset [1]\varphi$$

and

$$S_2(\varphi) = [2]\varphi \supset ([6]\varphi \wedge [7]\varphi).$$

Note that  $S_1(\varphi)$  and  $S_2(\varphi)$  are schemata, so any instance of them should be respected. Now, assume these two managers have to meet to make a decision about if  $\psi$  holds, and the decision problem depends on a finite number of issues  $p_1, p_2, \dots, p_k$ . Let  $\varphi$  denote  $\bigwedge_{1 \leq i \leq k} (S_1(p_i) \wedge S_2(p_i))$ , then the decision problem of the managers is to test if  $\Sigma \models [\Delta_{\varphi}(\{1, 2\})]\psi$  holds, where  $\Sigma$  contains the beliefs from the assistants and the external sources. Note that the problem is not expressible by the original IC merging operator because the integrity constraints contain sentences that express the inter-relationship among the beliefs of the different agents. ■

## 6. Belief Change and Conditional Logic

### 6.1. Review of basic concepts

Unlike belief fusion, where the component knowledge bases are equally important, belief change is an asymmetric operator, where new information always outweighs old. The two main belief change operators are belief revision and update. They are characterized by different postulates [ALC 85, KAT 91a, KAT 91b]. In [KAT 91a], a uniform model-theoretic framework is provided for the semantic characterization of the revision and update operators. In that context, a knowledge base is a finite set of propositional sentences, so it can also be represented by a single sentence (i.e. the conjunction of all sentences in the knowledge base).

For a revision operator, it is assumed that there is a total pre-order  $\leq_{\psi}$  over the propositional interpretations for each knowledge base  $\psi$ . The revision operators satisfying the AGM postulates in [ALC 85] select the minimal models of  $\varphi$  with respect to the ordering  $\leq_{\psi}$ . More precisely, let  $\psi$  be a knowledge base and  $\varphi$  denote the new information. The result of revising  $\psi$  by  $\varphi$ , denoted by  $\psi \circ \varphi$ , will have the set of models

$$Mod(\psi \circ \varphi) = \min(Mod(\varphi), \leq_{\psi}).$$

For an update operator, assume that, for each propositional interpretation  $w$ , there exists a partial pre-order  $\leq_w$  over the interpretations for closeness to  $w$ . Update operators select for each model  $w$  in  $Mod(\psi)$  the set of models from  $Mod(\varphi)$  that are closest to  $w$ . The updated theory is characterized by the union of all such models. That is,

$$Mod(\psi \diamond \varphi) = \bigcup_{w \in Mod(\psi)} \min(Mod(\varphi), \leq_w),$$

where  $\psi \diamond \varphi$  is the result of updating the knowledge base  $\psi$  by  $\varphi$ .

Both belief revision and update may occur in the observation of new information  $\varphi$ . For belief revision, it is assumed that the world is static, so if new information is incompatible with the agent's original beliefs, the agent may have an incorrect belief about the world. The agent will try to accommodate the new information by minimally changing his original beliefs. However, for belief update, it is assumed that the world is dynamic, so the agent's belief may become outdated, though it may have been correct for the original world. The agent will assume that possible worlds are those resulting from the minimal change of the original world. In [BOU 95], a generalized update model is proposed which combines aspects of both revision and update. It shows that a belief update model is inadequate without modelling the dynamic aspect (i.e. the events causing the update) simultaneously. Since modelling the dynamic change of external worlds is beyond the scope of this paper, we will not model belief update in our logic. Therefore, we will only focus on the belief revision operator.

## 6.2. Epistemic logic for belief revision

Let us now consider the possibility of incorporating the belief revision operator into epistemic logic. In addition to the original definition of revising a knowledge base  $\psi$  with new information  $\varphi$ , there is an alternative reading for the revision operator. That is, we can consider  $\circ$  as a prioritized belief fusion operator that gives priority to its second argument [MAY 01]. In knowledge base revision, these two interpretations are essentially equivalent. However, from the perspective of our logic in multi-agents systems, they may be quite different. Roughly speaking,  $i \circ \varphi$  denotes the result of revising the beliefs of agent  $i$  with new information  $\varphi$ , whereas  $i \circ j$  is the result of merging the beliefs of agents  $i$  and  $j$  by giving priority to  $j$ . Therefore, in the logic of revision ( $\mathcal{L}_{rv}$ ), we define the set of *revision expressions* (RE) and the set of wffs in the following way:

$$RE ::= i \mid r \circ i \mid r \circ \varphi$$

where  $i \in Ag$  is an agent symbol and  $\varphi$  is any wff in  $\mathcal{L}$ , and

$$WFF ::= p \mid \perp \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid \Box\varphi \mid [G]\varphi \mid [r]\varphi$$

where  $i \in Ag$ ,  $p \in \Phi_0$ ,  $G \subseteq Ag$ ,  $r \in RE$ , and  $\varphi, \psi \in WFF$ .

Note that a revision expression allows us to represent a revision sequence, which is directly related to iterated revision in [BOU 93, DAR 97]. Furthermore, since an agent index  $i$  is also a revision expression, the wff  $[i]\varphi$  has a different meaning than  $[\{i\}]\varphi$ . It appears that  $[\{i\}]\varphi$  can no longer be abbreviated as  $[i]\varphi$ . However, it turns out that our semantics makes  $[\{i\}]\varphi$  and  $[i]\varphi$  equivalent.

To interpret the modal operator in our semantic framework, we define an  $\mathcal{L}_{rv}$  model as

$$(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$$

where

–  $(W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model,  
–  $\leq$  is an assignment mapping each belief state (i.e. subset of possible worlds)  $U$  to a total pre-order  $\leq_U$  over  $W$  such that

- if  $w, w' \in U$ , then  $w \leq_U w'$ , and
- if  $w \in U$  and  $w' \notin U$ , then  $w <_U w'$ .

Let  $S = (U_1, U_2, \dots, U_k)$  denote a sequence of belief states, then  $S \cdot U$  denotes the sequence  $(U_1, U_2, \dots, U_k, U)$ . The assignment  $\leq$  is extended to sequences of belief states in the following way:

- 1)  $\leq_{(U)} = \leq_U$
- 2)  $w <_{S \cdot U} w'$  if  $w \in U$  and  $w' \notin U$ ,
- 3)  $w \leq_{S \cdot U} w'$  iff  $w \leq_S w'$ , when both  $w, w' \in U$  or both  $w, w' \notin U$ .

For each wff  $\varphi$ , let the truth set of  $\varphi$ , denoted by  $|\varphi|$ , be defined as  $\{w \in W \mid w \models \varphi\}$ . For each possible world  $u$ , define a function  $\tau_u$  mapping any revision expression  $r$  into a sequence of belief states  $\tau_u(r)$  as follows:

- 1)  $\tau_u(i) = (\mathcal{R}_i(u))$ ,
- 2)  $\tau_u(r \circ i) = \tau_u(r) \cdot \mathcal{R}_i(u)$ ,
- 3)  $\tau_u(r \circ \varphi) = \tau_u(r) \cdot |\varphi|$ .

The satisfaction condition for the wff  $[r]\varphi$  is

$$u \models [r]\varphi \text{ iff there exists } w_0 \in W \text{ such that for any } w \leq_{\tau_u(r)} w_0, w \models \varphi.$$

According to the semantics, the satisfaction of  $[i]\varphi$  is  $u \models [i]\varphi$  iff there exists  $w_0 \in W$  such that for any  $w \leq_{\mathcal{R}_i(u)} w_0$ ,  $w \models \varphi$ . This is equivalent to the original  $\forall w \in \mathcal{R}_i(u), w \models \varphi$  by the definition of  $\leq_{\mathcal{R}_i(u)}$ . Furthermore,  $[i \circ \varphi]\psi$  is semantically equivalent to  $[\Delta_\varphi(\{i\})]\psi$  described in Section 5.

The next theorem shows important valid wffs in  $\mathcal{L}_{rv}$ .

### Theorem 7

- 1)  $\models \neg[r]\perp$
- 2)  $\models \diamond\varphi \supset [r \circ \varphi]\varphi$
- 3)  $\models [r](\varphi \supset \psi) \supset ([r]\varphi \supset [r]\psi)$
- 4)  $\models [r \circ \top]\varphi \equiv [r]\varphi$
- 5)  $\models [r \circ \perp]\varphi \equiv [r]\varphi$
- 6)  $\models \Box(\varphi_1 \equiv \varphi_2) \supset ([r \circ \varphi_1]\psi \equiv [r \circ \varphi_2]\psi)$
- 7)  $\models [r \circ (\varphi \wedge \psi_1)]\psi_2 \supset [r \circ \varphi](\psi_1 \supset \psi_2)$
- 8)  $\models (\diamond\varphi \wedge \neg[r \circ \varphi]\neg\psi_1) \supset ([r \circ \varphi](\psi_1 \supset \psi_2) \supset [r \circ (\varphi \wedge \psi_1)]\psi_2)$
- 9)  $\models \neg[\{i, j\}]\perp \supset ([i \circ j]\varphi \equiv [\{i, j\}]\varphi)$
- 10)  $\models [i]\varphi \supset [r \circ i]\varphi$

**Proof:** Let  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  be any  $\mathcal{L}_{rv}$  model and  $u$  be any world in  $W$ . We show that all these wffs are satisfied in  $u$ .

- 1) and 2): Follow from the definition of the satisfaction condition immediately.
- 3): Holds because  $\leq_S$  is a *total* pre-order for any sequence of belief states  $S$ .
- 4) and 5): Hold because  $\leq_{\tau_u(r \circ \top)}$  and  $\leq_{\tau_u(r \circ \perp)}$  are both equal to  $\leq_{\tau_u(r)}$ .
- 6): If  $u \models \Box(\varphi_1 \equiv \varphi_2)$ , then we have  $|\varphi_1| = |\varphi_2|$  and  $\tau_u(r \circ \varphi_1) = \tau_u(r \circ \varphi_2)$ . Therefore, the result follows immediately.
- 7): If  $u \models [r \circ (\varphi \wedge \psi_1)]\psi_2$ , then there exists  $w_0 \in W$  such that

$$\forall w \leq_{\tau_u(r \circ (\varphi \wedge \psi_1))} w_0, w \models \psi_2. \quad (6)$$

There are two cases:

Case 1: If  $|\varphi \wedge \psi_1| \neq \emptyset$ , then without loss of generality, we can assume  $w_0 \models \varphi \wedge \psi_1$  (otherwise, we can find another  $w'_0 \leq_{\tau_u(r \circ (\varphi \wedge \psi_1))} w_0$  satisfying the condition). From (6), this implies

$$\forall w \leq_{\tau_u(r)} w_0, w \models \varphi \wedge \psi_1 \Rightarrow w \models \psi_2. \quad (7)$$

Let us now consider any  $w \leq_{\tau_u(r \circ \varphi)} w_0$ . By the assumption for this case, we have  $w \models \varphi$  and  $w \leq_{\tau_u(r)} w_0$ . If  $w \models \neg\psi_1$  holds, then we have  $w \models \psi_1 \supset \psi_2$ . If  $w \models \psi_1$  holds, then we have  $w \models \psi_2$  by (7). Therefore, for any  $w \leq_{\tau_u(r \circ \varphi)} w_0$ , we have  $w \models \psi_1 \supset \psi_2$ . This means  $u \models [r \circ \varphi](\psi_1 \supset \psi_2)$ .

Case 2: If  $|\varphi \wedge \psi_1| = \emptyset$ , then we have two subcases. If  $|\varphi| = \emptyset$ , then the result follows from 3), 5) and 6). If  $|\varphi| \neq \emptyset$ , then let us take a  $u_0 \models \varphi$ . For any  $w \leq_{\tau_u(r \circ \varphi)} u_0$ , we have  $w \models \varphi$ , so  $w \models \neg\psi_1$  and  $w \models \psi_1 \supset \psi_2$  hold. This means  $u \models [r \circ \varphi](\psi_1 \supset \psi_2)$ .

- 8): The proof is analogous to that of 7) and is omitted.
- 9): Follows from the fact that  $\min(W, \leq_{\tau_u(i \circ j)}) = \mathcal{R}_i(u) \cap \mathcal{R}_j(u)$  if  $\mathcal{R}_i(u) \cap \mathcal{R}_j(u) \neq \emptyset$ .
- 10): Due to the seriality assumption,  $\mathcal{R}_i(u)$  is nonempty. Let us take a  $w_0 \in \mathcal{R}_i(u)$ , then for any  $w \leq_{\tau_u(r \circ i)} w_0$ , we have  $w \in \mathcal{R}_i(u)$ . Therefore,  $u \models [i]\varphi$  implies  $u \models [r \circ i]\varphi$ .

■

As in the cases of  $\mathcal{L}_{ar}$  and  $\mathcal{L}_{ic}$ , we also have the following theorem.

**Theorem 8** For each  $\mathcal{L}$  wff  $\varphi$ ,  $\models_{\mathcal{L}} \varphi$  iff  $\models_{\mathcal{L}_{rv}} \varphi$ .

**Proof:** To prove this theorem, it is sufficient to show the following two facts:

1) If  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  is an  $\mathcal{L}_{rv}$  model, then  $M' = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model such that for all  $w \in W$  and  $\mathcal{L}$  wff  $\varphi$ , we have  $w \models_M \varphi$  iff  $w \models_{M'} \varphi$ .

2) If  $M = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)$  is an  $\mathcal{L}$  model, then there exists  $\leq$  such that  $M' = (W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)$  is an  $\mathcal{L}_{rv}$  model and for all  $w \in W$  and  $\mathcal{L}$  wff  $\varphi$ , we have  $w \models_M \varphi$  iff  $w \models_{M'} \varphi$ . To prove this, let  $\leq$  be defined as

$$w \leq_U w' \Leftrightarrow w \in U \vee w' \notin U,$$

for any  $U \subseteq W$ . It is then easy to verify that  $\leq_U$  is indeed a total pre-order satisfying the required conditions. ■

**Example 4** Let us use a simple scenario to illustrate the expressive power of  $\mathcal{L}_{rv}$ . Assume a decision maker bases his decision on information from three agents (1, 2 and 3). The most reliable information provider is agent 1, the least reliable one is agent 3, and the reliability of agent 2 lies somewhere between that of agents 1 and 2. As in the case of Example 3, these information providers may obtain information from other sources, so their beliefs have to satisfy some constraint  $\varphi$ . Note that  $\varphi$  may contain epistemic sentences, as well as objective sentences, as in the case of Example 3. So, to make a decision, the decision maker has to test the validity of

$$\Sigma \models_{\mathcal{L}_{rv}} [3 \circ 2 \circ 1 \circ \varphi] \psi$$

where  $\Sigma$  contains source beliefs and  $\psi$  is the decision problem of the decision maker. ■

### 6.3. Related work in conditional logic

There have been various attempts to formalize the belief change process through modal logic, or conditional logic systems [BOU 91, BOU 92a, BOU 92b, BOU 93, FRI 94, RYA 96, SEG 95]. In [BOU 92a], a modal logic  $\text{CO}^*$  is proposed for modelling the belief revision.  $\text{CO}^*$  is an extension of the logic  $\text{CO}$  proposed in [BOU 91]. In  $\text{CO}^*$ , revision of a theory by a sentence is represented using a conditional connective. The connective is not primitive, but rather defined using two unary modal operators  $\square$  and  $\bar{\square}$ <sup>7</sup>. These modal operators are interpreted with respect to a total pre-order  $R$  over the possible worlds, which is assumed to rely on a background theory  $K$ . Thus,  $w \models \square \varphi$  iff  $\varphi$  is true in all possible worlds that are as plausible as  $w$  given the theory  $K$ , and  $w \models \bar{\square} \varphi$  iff  $\varphi$  is true in all possible worlds that are less plausible than  $w$  given  $K$ . By defining  $\bar{\square} \varphi$  as  $\square \varphi \wedge \bar{\square} \varphi$  and  $\bar{\diamond} \varphi$  as  $\neg \bar{\square} \neg \varphi$ , the conditional  $\varphi \xrightarrow{KB} \psi$  is defined as

$$\bar{\square} \neg \varphi \vee \bar{\diamond} (\varphi \wedge \square (\varphi \supset \psi)),$$

7. The same symbol  $\square$  is also used in  $\mathcal{L}_{ic}$  and  $\mathcal{L}_{rv}$  with different semantics.

where  $KB$  is a finite representation of the theory  $K$ . Since there is only one global ordering  $R$  in a  $CO^*$  model, the logic is appropriate only for the revision of a single theory  $K$ . On the other hand, our logic allows the revisions of many agents' belief states. Furthermore, since the ordering  $R$  in a  $CO^*$  model is global,  $\varphi \xrightarrow{KB} \psi$  is true in a world iff it is true in all worlds, thus no iterated revisions are allowed in the model. In [BOU 93], this restriction is lifted by allowing the revision of the ordering  $R$  to  $R'$  simultaneously. The idea is to move the most plausible  $\varphi$ -models with respect to  $R$  to the most plausible level of  $R'$  and keep the rest of  $R$  unchanged. Our assignment of a total pre-order to a sequence of belief states is based on the same idea. However, while the definition of [BOU 93] presumes the existence of minimal models for any propositional formulas, we do not need this assumption.

In [FRI 94], a logic with conditional and epistemic operators is used in belief revision. Conditional and epistemic sentences are interpreted in an abstract belief change system (BCS). The two basic components of a BCS are a set of belief states and a belief change function. The latter maps each belief state and sentence of some base language into a new belief state. In a more concrete preferential interpretation, each belief state  $s$  is interpreted as a subset of possible world  $K(s)$ , and a pre-order  $\preceq_s$  over the possible worlds is associated with  $s$ . In this regard,  $\preceq_s$  corresponds to  $\leq_{K(s)}$  in our semantic models and the conditional wff  $\varphi > \psi$  in the logic  $\mathcal{L}^>$  of [FRI 94] is roughly equivalent to our wff  $[i \circ \varphi]\psi$  for a fixed agent  $i$ . However, since in  $\mathcal{L}^>$ , the antecedent  $\varphi$  of a conditional is restricted to a wff in the base language  $\mathcal{L}$ , it does not allow epistemic wffs of the form  $B\varphi$ . It is argued that the antecedent must be observable whereas conditional wffs are unobservable, so we should not allow conditional wffs in the antecedent. However, in multi-agent systems, one agent may learn the beliefs of other agents through communication, so we should not exclude this flexibility. Another significant difference between BCS and our logic is that BCS only allows revision of a belief state by a sentence, while in our system the prioritized fusion of two belief states held by two agents is incorporated.

A dynamic doxastic logic for belief revision is proposed in [SEG 95] and further developed in [SEG 01]. By using the notations of [SEG 95], the doxastic operator  $B$  and two kinds of dynamic modal operators  $[+\varphi]$  and  $[-\varphi]$  for propositional wff  $\varphi$  are taken as the basic constructs of the language. The operators  $[+\varphi]$  and  $[-\varphi]$  correspond to the expansion and contraction operators of AGM theory respectively. Thus, the revision operator  $[\circ\varphi]$  is defined as  $[-(\neg\varphi)][+\varphi]$  according to Levi's identity [ALC 85]. The wffs of the language are interpreted with respect to a hypertheory. A hypertheory  $H$  is a set of subsets of possible worlds linearly ordered by inclusion, and is similar to the widening ranked model defined in [LEH 95]. However, the latter assumes that the subsets of models are indexed by natural numbers. A hypertheory is said to be replete if  $W$  (i.e. the set of all possible worlds) is in  $H$ . From the hypertheory  $H$ , a pre-order  $\leq_H$  over  $W$  can be defined as:

$$w' \leq_H w \leftrightarrow \forall U \in H \wedge w \in U \exists U' \in H, U' \subseteq U \wedge w' \in U'.$$

When  $H$  is replete, the ordering  $\leq_H$  is total. When the wffs  $[+\varphi]\psi$  and  $[-\varphi]\psi$  are evaluated with respect to a hypertheory  $H$ , it causes evaluation of  $\psi$  in some revised hypertheory  $H'$ . The semantics are essentially equivalent to that proposed in [BOU 93], although the revisions of the corresponding pre-order are somewhat different in the two approaches. Therefore, the logic only allows the belief revision of a single agent by some new information, and the prioritized fusion of multi-agent beliefs can not be represented in such logic.

## 7. Concluding Remarks

In this paper, we assume that agent belief states are represented as a subset of possible worlds, i.e.  $\mathcal{R}_i(w)$  is the belief state of agent  $i$  in world  $w$ . However, more fine-grained representations have also been proposed, such as total pre-orders over the set of possible worlds [BOU 93, DAR 97, LEH 95, SEG 95], ordinal conditional functions [BOU 95, SPO 88, WIL 94], possibility distributions [BEN 97, BEN 03, DUB 92, DUB 00], belief functions [SME 00] and pedigreed belief states [MAY 01]. Perhaps the most popular representation among them is an ordering of the possible worlds. While a set of possible worlds can be seen as the minimal worlds with respect to a given ordering, the fusion of two orderings is more general than the revision of an ordering by a set of possible worlds [MAY 01]. Thus, AGM revision is in fact a special case of the fusion operator in [MAY 01]. Indeed, in our extended models, the assignment  $\leq$  has mapped each subset of possible worlds to a total pre-order between worlds. However, to fully utilize the semantic power of an ordering, the logic language should be extended to cover the conditional connectives. Since the focus of our paper is on integrating belief fusion operators into an epistemic reasoning framework, this extension is beyond its scope. Nevertheless, the development of logical systems that incorporate fusion operators based on fine-grained representations of belief states is an interesting research direction.

We present the semantics of epistemic logics for information fusion. However, to do practical reasoning, we must develop proof methods for these logics. There have been previous works on developing axiomatic or Gentzen-style calculi for information fusion logics [BOL 95, BOL 96, BOL 97, BOL 99, CHO 01, CHO 02, CHO 03b, LIA 04].

In [BOL 95, BOL 96, BOL 97, BOL 99], logics for information fusion based on possibility theory are proposed. The Hilbert-style or Gentzen-style proof systems of those logics are also presented. The logic  $\mathbf{PL}_n^\otimes$  in [BOL 99] is an extension of QML in [LIA 92, LIA 93, LIA 96] with a distributed belief operator, so the fusion operator in  $\mathbf{PL}_n^\otimes$  is different to the merging operators used in this paper. An axiomatic system and theorem prover for the majority fusion logic  $MF$  are developed in [CHO 01, CHO 02, CHO 03b]. The logic  $MF$  and its proof system have been applied to the analysis of the STANAG 2002 recommendations of NATO about information evaluation [CHO 03a]. The belief bases in  $MF$  are sets of literals, so it does not allow nested modalities. In



spite of these differences, the development of proof theory for epistemic fusion logics could take these previous works as good starting points.

Furthermore, we have presented several validity formulas for each logic introduced in this paper. Most of them are translated from postulates of corresponding information fusion operators. In the case of arbitration, all postulates in [LIB 95] can be translated into corresponding wffs in  $\mathcal{L}_{ar}$ . Therefore, we make them into an axiomatic system. Though we can not yet prove the completeness of the system, we tend to believe it is indeed complete. In other cases, not all postulates in corresponding information fusion operators can be expressed in our framework. This suggests that further extensions of syntax and semantics of  $\mathcal{L}_{mm}$ ,  $\mathcal{L}_{ic}$  and  $\mathcal{L}_{rv}$  may be needed.

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