Hindawi
Complexity
Volume 2018, Article ID 6504590, 13 pages
https://doi.org/10.1155/2018/6504590

Research Article

Almost Periodic Synchronization for Quaternion-Valued Neural Networks with Time-Varying Delays

Yongkun Li,[1] **Xiaofang Meng**,[1] **and Yuan Ye**[2]

[1]*Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China*
[2]*Graduate School, Yunnan University, Kunming, Yunnan 650091, China*

Correspondence should be addressed to Yuan Ye; yye_yd@aliyun.com

Received 19 November 2017; Accepted 13 March 2018; Published 29 April 2018

Academic Editor: Arturo Buscarino

This paper focuses on the global exponential almost periodic synchronization of quaternion-valued neural networks with time-varying delays. By virtue of the exponential dichotomy of linear differential equations, Banach's fixed point theorem, Lyapunov functional method, and differential inequality technique, some sufficient conditions are established for assuring the existence and global exponential synchronization of almost periodic solutions of the delayed quaternion-valued neural networks, which are completely new. Finally, we give one example with simulation to show the applicability and effectiveness of our main results.

1. Introduction

It is well known that the quaternion is a mathematical concept discovered by Hamilton in 1843, which did not arouse much attention for quite a long time, let alone real world applications. Due to the noncommutativity of quaternion multiplication, the study of quaternion is much more difficult than that of plurality, which is one of the reasons for the slow development of quaternion. Fortunately, over the past two decades, the quaternion theory has achieved a rapid development, especially in algebra, and found many applications in the real world, like attitude control, quantum mechanics, robotics, computer graphics, and so on [1–5]. For example, in the application of color image compression technology, one can apply the quaternion theory to encode and improve the color image; see [5]. In recent years, the quaternion-valued neural networks, as an extension of the real-valued neural networks and the complex-valued neural networks [6, 7], research has become a hot topic. It should be pointed out that, at present, almost all the investigations on quaternion-valued neural networks are mainly dealing with the stability, robustness, or dissipation of the equilibrium of the neural networks; see [8–17].

On the one hand, since the concept of drive-response synchronization for coupled chaotic systems was proposed by Pecora and Carroll in [18], chaos synchronization has become a hot research topic due to its potential applications in secure communication, automatic control, biological systems, and information science; see, for instance, [19–26].

On the other hand, the almost periodicity is a generalization of periodicity and is much more common than the periodicity, and so, almost periodic oscillation is a very important dynamic phenomenon for nonautonomous neural networks [27–30]. However, so far, few results have been available for the almost periodicity of complex-valued neural networks [31] and quaternion-valued neural networks. Moreover, in the past few years, although the problem of periodic synchronization for real-valued systems has been extensively studied by many scholars (see, for instance, [32–43]), there are few articles dealing with the almost periodic synchronization for real-valued systems, let alone dealing with the almost periodic synchronization for complex-valued or quaternion-valued neural networks. Very recently, some authors study the synchronization for complex-valued neural networks with time delays [44, 45]. Based on the LMI method and mathematical analysis technique, they obtain some sufficient conditions for synchronization of complex-valued neural networks. However, to the best of our knowledge, there has been no paper published on the almost periodic synchronization of the quaternion-valued neural networks, which remains as an open challenge. So, it is necessary to investigate the synchronization of quaternion-valued neural networks.

Motivated by the above statement, the main purpose of this paper is to study the synchronization of quaternion-valued neural networks. The main contributions of this paper can be summarized as follows: (1) this is the first time to investigate the almost periodic synchronization of neural networks. (2) Compared with other results, the results of this paper are the ones about quaternion value and with time delays. Therefore, the results are less conservative and more general. (3) Our methods used in this paper can easily apply to study the almost periodic synchronization for other types of neural networks with delays.

This paper is organized as follows: In Section 2, we give the model description and introduce some definitions and preliminary lemmas and transform the quaternion-valued system (9) into an equivalent real-valued system. In Section 3, we establish some sufficient conditions for the existence and global exponential synchronization of almost periodic solutions of (9). In Section 4, we give an example to demonstrate the feasibility of our results. This paper ends with a brief conclusion in Section 5.

2. Model Description and Preliminaries

In this section, we shall first recall some fundamental definitions and lemmas which are used in what follows.

First, we give some notations of this paper. Let $\mathbb{R}$ and $\mathbb{Q}$ stand for the real field and the skew field of quaternions, respectively. $\mathbb{R}^{n\times n}$, $\mathbb{Q}^{n\times n}$ denote the set of all $n\times n$ real-valued and quaternion-valued matrices, respectively. The skew field of quaternion is denoted by

$$\mathbb{Q} := \left\{x = x^R + ix^I + jx^J + kx^K\right\}, \tag{1}$$

where x^R, x^I, x^J, x^K are real numbers and the elements i, j and k obey Hamilton's multiplication rules:

$$\begin{aligned} ij &= -ji = k, \\ jk &= -kj = i, \\ ki &= -ik = j, \\ i^2 &= j^2 = k^2 = ijk = -1. \end{aligned} \tag{2}$$

Definition 1 (see [46], let $f \in \mathrm{BC}(\mathbb{R}, \mathbb{R}^n)$). Function f is said to be almost periodic if, for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$; there exists a number $\tau = \tau(\epsilon)$ in this interval such that $|f(t+\tau) - f(t)| < \epsilon$, for all $t \in \mathbb{R}$.

We denote by $\mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$ the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$.

Definition 2. Quaternion-valued function $f = f^R + if^I + jf^J + kf^K$ is an almost periodic function if and only if f^R, f^I, f^J and f^K are almost periodic functions.

Consider the following linear homogenous system

$$x'(t) = A(t)x(t) \tag{3}$$

and linear nonhomogenous system

$$x'(t) = A(t)x(t) + f(t), \tag{4}$$

where $A(t)$ is an almost periodic matrix function and $f(t)$ is an almost periodic vector function.

Definition 3 (see [46]). System (3) is said to admit an exponential dichotomy if there exist a projection P and positive constants α, β such that the fundamental solution matrix $X(t)$ satisfies

$$\begin{aligned} \left|X(t)PX^{-1}(s)\right| &\le \beta e^{-\alpha(t-s)}, \quad t \ge s, \\ \left|X(t)(I-P)X^{-1}(s)\right| &\le \beta e^{-\alpha(s-t)}, \quad t \le s. \end{aligned} \tag{5}$$

Lemma 4 (see [46]). *If the linear system (3) admits an exponential dichotomy, then system (4) has a unique almost periodic solution that can be expressed as*

$$\begin{aligned} x(t) = &\int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)\,\mathrm{d}s \\ &- \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s)\,\mathrm{d}s, \end{aligned} \tag{6}$$

where $X(t)$ is the fundamental solution matrix of (3).

Lemma 5 (see [46]). *Let c_p be an almost periodic function on $\mathbb{R}$ and*

$$M\left[c_p\right] := \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} c_p(s)\,\mathrm{d}s > 0, \quad p = 1, 2, \ldots, n. \tag{7}$$

Then the linear system

$$x'(t) = \mathrm{diag}\left(-c_1(t), -c_2(t), \ldots, -c_n(t)\right)x(t) \tag{8}$$

admits an exponential dichotomy on $\mathbb{R}$.

Consider the following quaternion-valued neural networks with time-varying delays:

$$\begin{aligned} x_p'(t) = &-d_p(t)x_p(t) + \sum_{q=1}^{n} a_{pq}(t)f_q\left(x_q(t)\right) \\ &+ \sum_{q=1}^{n} b_{pq}(t)g_q\left(x_q\left(t-\tau_{pq}(t)\right)\right) + u_p(t), \end{aligned} \tag{9}$$

where $p \in \{1, 2, \ldots, n\} := T$ and $x_p(t) \in \mathbb{Q}$ is the state of the pth neuron at time t; $d_p(t) > 0$ is the self-feedback connection weight; $a_{pq}(t) \in \mathbb{Q}$ and $b_{pq}(t) \in \mathbb{Q}$ are the connection weight and the delay connection weight from neuron q to neuron p at time t, respectively; $u_p(t) \in \mathbb{Q}$ is an external input on the pth unit at time t; $f_q(x_q(t)) \in \mathbb{Q}$ and $g_q(x_q(t-\tau_{pq}(t))) \in \mathbb{Q}$ denote the activation functions without and with time-varying delays, respectively; $\tau_{pq}(t)$ represents the time-varying delay and satisfies $0 \le \tau_{pq}(t) \le \tau$.

The initial value of (9) is given by

$$x_p(s) = \phi_p(s), \quad s \in [-\tau, 0], \ p \in T, \tag{10}$$

where $\phi_p \in C([-\tau, 0], \mathbb{Q})$.

The response system of (9) is designed as

$$\begin{aligned} y_p'(t) = {} & -d_p(t) y_p(t) + \sum_{q=1}^{n} a_{pq}(t) f_q(y_q(t)) \\ & + \sum_{q=1}^{n} b_{pq}(t) g_q(y_q(t - \tau_{pq}(t))) + u_p(t) \\ & + \theta_p(t), \end{aligned} \tag{11}$$

where $p \in T$, $y_p(t) \in \mathbb{Q}$ denotes the state of the pth neuron at time t of the response system and $\theta_p(t) \in \mathbb{Q}$ is a controller.

The initial condition associated with (11) is of the form

$$y_p(s) = \psi_p(s), \quad s \in [-\tau, 0], \ p \in T, \tag{12}$$

where $\psi_p \in C([-\tau, 0], \mathbb{Q})$.

In order to overcome the inconvenience of the noncommutativity of quaternion multiplication, in the following, we always assume the following:

(H_1) Let $x_q = x_q^R + ix_q^I + jx_q^J + kx_q^K$, $x_q^R, x_q^I, x_q^J, x_q^K \in \mathbb{R}$. Then $f_q(x_q)$ and $g_q(x_q)$ can be expressed as

$$\begin{aligned} f_q(x_q) = {} & f_q^R(x_q^R, x_q^I, x_q^J, x_q^K) + i f_q^I(x_q^R, x_q^I, x_q^J, x_q^K) \\ & + j f_q^J(x_q^R, x_q^I, x_q^J, x_q^K) \\ & + k f_q^K(x_q^R, x_q^I, x_q^J, x_q^K), \\ g_q(x_q) = {} & g_q^R(x_q^R, x_q^I, x_q^J, x_q^K) + i g_q^I(x_q^R, x_q^I, x_q^J, x_q^K) \\ & + j g_q^J(x_q^R, x_q^I, x_q^J, x_q^K) \\ & + k g_q^K(x_q^R, x_q^I, x_q^J, x_q^K), \end{aligned} \tag{13}$$

where $f_q^\nu, g_q^\nu : \mathbb{R}^4 \to \mathbb{R}$, $q \in T$, $\nu \in \{R, I, J, K\} := E$.

According to (H_1), system (9) can be transformed into the following real-valued system:

$$\begin{aligned} (x_p^R)'(t) = {} & -d_p(t) x_p^R(t) + \sum_{q=1}^{n} \big(a_{pq}^R(t) f_q^R[t, x] \\ & - a_{pq}^I(t) f_q^I[t, x] - a_{pq}^J(t) f_q^J[t, x] \\ & - a_{pq}^K(t) f_q^K[t, x]\big) + \sum_{q=1}^{n} \big(b_{pq}^R(t) g_q^R[t - \tau_{pq}(t), x] \\ & - b_{pq}^I(t) g_q^I[t - \tau_{pq}(t), x] \\ & - b_{pq}^J(t) g_q^J[t - \tau_{pq}(t), x] \\ & - b_{pq}^K(t) g_q^K[t - \tau_{pq}(t), x]\big) + u_p^R(t), \quad p \in T, \\ (x_p^I)'(t) = {} & -d_p(t) x_p^I(t) + \sum_{q=1}^{n} \big(a_{pq}^R(t) f_q^I[t, x] \\ & + a_{pq}^I(t) f_q^R[t, x] + a_{pq}^J(t) f_q^K[t, x] \\ & - a_{pq}^K(t) f_q^J[t, x]\big) + \sum_{q=1}^{n} \big(b_{pq}^R(t) g_q^I[t - \tau_{pq}(t), x] \\ & + b_{pq}^I(t) g_q^R[t - \tau_{pq}(t), x] \\ & + b_{pq}^J(t) g_q^K[t - \tau_{pq}(t), x] \\ & - b_{pq}^K(t) g_q^J[t - \tau_{pq}(t), x]\big) + u_p^I(t), \quad p \in T, \\ (x_p^J)'(t) = {} & -d_p(t) x_p^J(t) + \sum_{q=1}^{n} \big(a_{pq}^R(t) f_q^J[t, x] \\ & + a_{pq}^J(t) f_q^R[t, x] - a_{pq}^I(t) f_q^K[t, x] \\ & + a_{pq}^K(t) f_q^I[t, x]\big) + \sum_{q=1}^{n} \big(b_{pq}^R(t) g_q^J[t - \tau_{pq}(t), x] \\ & + b_{pq}^J(t) g_q^R[t - \tau_{pq}(t), x] \\ & - b_{pq}^I(t) g_q^K[t - \tau_{pq}(t), x] \\ & + b_{pq}^K(t) g_q^I[t - \tau_{pq}(t), x]\big) + u_p^J(t), \quad p \in T, \\ (x_p^K)'(t) = {} & -d_p(t) x_p^K(t) + \sum_{q=1}^{n} \big(a_{pq}^R(t) f_q^K[t, x] \\ & + a_{pq}^K(t) f_q^R[t, x] + a_{pq}^I(t) f_q^J[t, x] \\ & - a_{pq}^J(t) f_q^I[t, x]\big) + \sum_{q=1}^{n} \big(b_{pq}^R(t) g_q^K[t - \tau_{pq}(t), x] \\ & + b_{pq}^K(t) g_q^R[t - \tau_{pq}(t), x] \\ & + b_{pq}^I(t) g_q^J[t - \tau_{pq}(t), x] \\ & - b_{pq}^J(t) g_q^I[t - \tau_{pq}(t), x]\big) + u_p^K(t), \quad p \in T, \end{aligned} \tag{14}$$

where $f_q^\nu[t, x] \triangleq f_q^\nu(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t))$, $g_q^\nu[t - \tau_{pq}(t), x] \triangleq g_q^\nu(x_q^R(t - \tau_{pq}(t)), x_q^I(t - \tau_{pq}(t)), x_q^J(t - \tau_{pq}(t)), x_q^K(t - \tau_{pq}(t)))$, $p, q \in T$, $\nu \in E$, and

$$\begin{aligned} a_{pq}(t) &= a_{pq}^R(t) + i a_{pq}^I(t) + j a_{pq}^J(t) + k a_{pq}^K(t), \\ & \qquad p, q \in T, \\ b_{pq}(t) &= b_{pq}^R(t) + i b_{pq}^I(t) + j b_{pq}^J(t) + k b_{pq}^K(t), \\ & \qquad p, q \in T, \\ u_p(t) &= u_p^R(t) + i u_p^I(t) + j u_p^J(t) + k u_p^K(t), \quad p \in T. \end{aligned} \tag{15}$$

It follows from (14) that

$$X_p'(t) = -d_p X_p(t) + \sum_{q=1}^{n} A_{pq}(t) F_q[t,x] + \sum_{q=1}^{n} B_{pq}(t) G_q\left[t-\tau_{pq}(t), x\right] + U_p(t), \quad p \in T, \tag{16}$$

where

$$A_{pq}(t) = \begin{bmatrix} a_{pq}^R(t) & -a_{pq}^I(t) & -a_{pq}^J(t) & -a_{pq}^K(t) \\ a_{pq}^I(t) & a_{pq}^R(t) & -a_{pq}^K(t) & a_{pq}^J(t) \\ a_{pq}^J(t) & a_{pq}^K(t) & a_{pq}^R(t) & -a_{pq}^I(t) \\ a_{pq}^K(t) & -a_{pq}^J(t) & a_{pq}^I(t) & a_{pq}^R(t) \end{bmatrix},$$

$$B_{pq}(t) = \begin{bmatrix} b_{pq}^R(t) & -b_{pq}^I(t) & -b_{pq}^J(t) & -b_{pq}^K(t) \\ b_{pq}^I(t) & b_{pq}^R(t) & -b_{pq}^K(t) & b_{pq}^J(t) \\ b_{pq}^J(t) & b_{pq}^K(t) & b_{pq}^R(t) & -b_{pq}^I(t) \\ b_{pq}^K(t) & -b_{pq}^J(t) & b_{pq}^I(t) & b_{pq}^R(t) \end{bmatrix},$$

$$F_q[t,x] = \begin{bmatrix} f_q^R[t,x] \\ f_q^I[t,x] \\ f_q^J[t,x] \\ f_q^K[t,x] \end{bmatrix}, \tag{17}$$

$$G_q\left[t-\tau_{pq}(t),x\right] = \begin{bmatrix} g_q^R\left[t-\tau_{pq}(t),x\right] \\ g_q^I\left[t-\tau_{pq}(t),x\right] \\ g_q^J\left[t-\tau_{pq}(t),x\right] \\ g_q^K\left[t-\tau_{pq}(t),x\right] \end{bmatrix},$$

$$X_p(t) = \left(x_p^R(t), x_p^I(t), x_p^J(t), x_p^K(t)\right)^T,$$

$$U_p(t) = \left(u_p^R(t), u_p^I(t), u_p^J(t), u_p^K(t)\right)^T.$$

The initial condition associated with (16) is of the form

$$X_p(s) = \Phi_p(s), \quad s \in [-\tau, 0], \ p \in T, \tag{18}$$

where $\Phi_p(s) = (\varphi_p^R(s), \varphi_p^I(s), \varphi_p^J(s), \varphi_p^K(s))^T, \varphi_p^\nu(s) \in C([-\tau, 0], \mathbb{R}), \nu \in E$.

Similarly, the response system (11) can be transformed into the following real-valued system:

$$Y_p'(t) = -d_p(t) Y_p(t) + \sum_{q=1}^{n} A_{pq}(t) F_q[t,y] + \sum_{q=1}^{n} B_{pq}(t) G_q\left[t-\tau_{pq}(t), y\right] + U_p(t) + \Theta_p(t), \quad p \in T, \tag{19}$$

where $Y_p(t) = (y_p^R(t), y_p^I(t), y_p^J(t), y_p^K(t))^T, \Theta_p(t) = (\theta_p^R(t), \theta_p^I(t), \theta_p^J(t), \theta_p^K(t))^T$.

The initial condition associated with (19) is of the form

$$Y_p(s) = \Psi_p(s), \quad s \in [-\tau, 0], \ p \in T, \tag{20}$$

where $\Psi_p = (\psi_p^R, \psi_p^I, \psi_p^J, \psi_p^K)^T, \psi_p^\nu(s) \in C([-\tau, 0], \mathbb{R}), \nu \in E$.

Let $Z_p(t) = Y_p(t) - X_p(t)$, $Z_p(t) = (z_p^R(t), z_p^I(t), z_p^J(t), z_p^K(t))^T, p \in T$. Then, from (16) and (19), we obtain the following error system:

$$Z_p'(t) = -d_p(t) Z_p(t) + \sum_{q=1}^{n} A_{pq}(t) \cdot \left(F_q[t,y] - F_q[t,x]\right) + \sum_{q=1}^{n} B_{pq}(t) \cdot \left(G_q\left[t-\tau_{pq}(t), y\right] - G_q\left[t-\tau_{pq}(t), x\right]\right) + \Theta_p(t), \quad p \in T. \tag{21}$$

In order to realize the almost periodic synchronization of the drive-response system, we choose the controller

$$\begin{aligned} \theta_p^R(t) &= \epsilon_p z_p^R(t), \\ \theta_p^I(t) &= \varepsilon_p z_p^I(t), \\ \theta_p^J(t) &= \rho_p z_p^J(t), \\ \theta_p^K(t) &= \varrho_p z_p^K(t), \end{aligned} \tag{22}$$

where $\epsilon_p, \varepsilon_p, \rho_p, \varrho_p \in \mathbb{R}^+$.

Definition 6. The response system (19) and the drive system (16) can be globally exponentially synchronized, if there exist positive constants $M \geq 1$, $\lambda > 0$ such that

$$\|Z(t)\| \leq \|\psi - \varphi\| M e^{-\lambda t}, \quad t \geq 0, \tag{23}$$

where $\|Z(t)\| = \max_{1\leq p\leq n}\{\max_{\nu\in E}\{|z_p^\nu(t)|\}\}$, $\|\psi - \varphi\| = \max_{1\leq p\leq n}\{\max_{\nu\in E}\{\sup_{s\in[-\tau,0]}|\psi_p^\nu(s) - \varphi_p^\nu(s)|\}\}$.

Let $\mathbb{X} = \{x \mid x = (x_1^R, x_1^I, x_1^J, x_1^K, x_2^R, x_2^I, x_2^J, x_2^K, \ldots, x_n^R, x_n^I, x_n^J, x_n^K)^T := (x_1, x_2, \ldots, x_n)^T \in AP(\mathbb{R}, \mathbb{R}^{4n})\}$ with the norm $\|x\|_{\mathbb{X}} = \max_{p\in T}\{\max_{\nu\in E}\{\sup_{t\in\mathbb{R}}|x_p^\nu(t)|\}\}$; then $\mathbb{X}$ is a Banach space.

Throughout this paper, we assume that the following conditions hold:

(H_2) Function $d_p \in C(\mathbb{R}, \mathbb{R}^+)$ with $M[d_p] > 0, U_p \in C(\mathbb{R}, \mathbb{R}^{4\times1})$, $A_{pq}, B_{pq} \in C(\mathbb{R}, \mathbb{R}^{4\times4})$ and $\tau_{pq} \in C^1(\mathbb{R}, \mathbb{R}^+)$ with $\sup_{t\in\mathbb{R}} \tau_{pq}'(t) = \beta < 1$ are almost periodic, where $p, q \in T$.

(H_3) There exist positive constants l_q^ν, m_q^ν such that for all $x^\nu, y^\nu \in \mathbb{R}$,

$$\begin{aligned}
&\left|f_q^\nu\left(x_q^R, x_q^I, x_q^J, x_q^K\right) - f_q^\nu\left(y_q^R, y_q^I, y_q^J, y_q^K\right)\right| \\
&\quad \le l_q^R\left|x_q^R - y_q^R\right| + l_q^I\left|x_q^I - y_q^I\right| + l_q^J\left|x_q^J - y_q^J\right| \\
&\qquad + l_q^K\left|x_q^K - y_q^K\right|, \\
&\left|g_q^\nu\left(x_q^R, x_q^I, x_q^J, x_q^K\right) - g_q^\nu\left(y_q^R, y_q^I, y_q^J, y_q^K\right)\right| \\
&\quad \le m_q^R\left|x_q^R - y_q^R\right| + m_q^I\left|x_q^I - y_q^I\right| + m_q^J\left|x_q^J - y_q^J\right| \\
&\qquad + m_q^K\left|x_q^K - y_q^K\right|,
\end{aligned} \tag{24}$$

and $f_q^\nu(0,0,0,0) = g_q^\nu(0,0,0,0) = 0$, where $q \in T$, $\nu \in E$.

(H_4) There exists a positive constant κ such that

$$\begin{aligned}
&\max\left\{\frac{\Omega_p\kappa + u_p^{\nu^+}}{d_p^-}\right\} \le \kappa, \\
&\qquad\qquad \max\left\{\frac{\Omega_p}{d_p^-}\right\} := r < 1, \quad p \in T, \ \nu \in E,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\Omega_p &= \Lambda_p + \Delta_p, \quad p \in T, \\
\Lambda_p &= \sum_{q=1}^n \left(a_{pq}^{R^+} + a_{pq}^{I^+} + a_{pq}^{J^+} + a_{pq}^{K^+}\right)\left(l_q^R + l_q^I + l_q^J + l_q^K\right), \\
&\qquad p \in T, \\
\Delta_p &= \sum_{q=1}^n \left(b_{pq}^{R^+} + b_{pq}^{I^+} + b_{pq}^{J^+} + b_{pq}^{K^+}\right) \\
&\quad \cdot \left(m_q^R + m_q^I + m_q^J + m_q^K\right), \quad p \in T.
\end{aligned} \tag{26}$$

(H_5) There exists a positive constant λ such that for $p \in T$,

$$\begin{aligned}
\lambda - d_p^- + \epsilon_p + \Lambda_p + \frac{\Delta_p}{1-\beta}e^{\lambda\tau} &< 0, \\
\lambda - d_p^- + \varepsilon_p + \Lambda_p + \frac{\Delta_p}{1-\beta}e^{\lambda\tau} &< 0, \\
\lambda - d_p^- + \rho_p + \Lambda_p + \frac{\Delta_p}{1-\beta}e^{\lambda\tau} &< 0, \\
\lambda - d_p^- + \varrho_p + \Lambda_p + \frac{\Delta_p}{1-\beta}e^{\lambda\tau} &< 0.
\end{aligned} \tag{27}$$

3. Main Results

In this section, we will study the existence of almost periodic solutions of system (16).

Theorem 7. *Let (H_1)–(H_4) hold. Then system (16) has a unique almost periodic solution in the region* $\mathbb{X}^* = \{\varphi \mid \varphi \in \mathbb{X}, \|\varphi\|_{\mathbb{X}} \le \kappa\}$.

Proof. For any $\varphi \in \mathbb{X}$, we consider the linear almost periodic differential equation system

$$\begin{aligned}
X_p'(t) &= -d_p X_p(t) + \sum_{q=1}^n A_{pq}(t) F_q[t,\varphi] \\
&\quad + \sum_{q=1}^n B_{pq}(t) G_q\left[t - \tau_{pq}(t), \varphi\right] + U_p(t), \\
&\qquad p \in T.
\end{aligned} \tag{28}$$

Combining (H_2) and Lemma 5, we can conclude that the linear system

$$X_p'(t) = -d_p(t) X_p(t), \quad p \in T \tag{29}$$

admits an exponential dichotomy on $\mathbb{R}$; furthermore, by Lemma 4, we conclude that system (28) has a unique almost periodic solution that can be expressed as

$$X^\varphi(t) = \left(X_1^\varphi(t), X_2^\varphi(t), \ldots, X_n^\varphi(t)\right)^T, \tag{30}$$

where

$$\begin{aligned}
X_p^\varphi(t) &= \int_{-\infty}^t e^{-\int_s^t d_p(u)\mathrm{d}u}\left(\sum_{q=1}^n A_{pq}(s) F_q[s,\varphi]\right. \\
&\quad \left. + \sum_{q=1}^n B_{pq}(s) G_q\left[s - \tau_{pq}(s), \varphi\right] + U_p(s)\right)\mathrm{d}s, \\
&\qquad p \in T.
\end{aligned} \tag{31}$$

Now, we define an operator $\Phi : \mathbb{X}^* \to \mathbb{X}$ as follows:

$$\Phi\varphi = X^\varphi. \tag{32}$$

First, we show that, for any $\varphi \in \mathbb{X}^*$, we have $\Phi\varphi \in \mathbb{X}^*$. From (32), we have

$$
\begin{aligned}
&\left|(\Phi\varphi)_p^R(t)\right| = \left|\int_{-\infty}^{t} e^{-\int_s^t d_p(u)\mathrm{d}u}\left(\sum_{q=1}^{n}\left(a_{pq}^R(s)\, f_q^R[s,\varphi] - a_{pq}^I(s)\, f_q^I[s,\varphi] - a_{pq}^J(s)\, f_q^J[s,\varphi] - a_{pq}^K(s)\, f_q^K[s,\varphi]\right)\right.\right.\\
&\quad + \sum_{q=1}^{n}\left(b_{pq}^R(s)\, g_q^R\left[s-\tau_{pq}(s),\varphi\right] - b_{pq}^I(s)\, g_q^I\left[s-\tau_{pq}(s),\varphi\right] - b_{pq}^J(s)\, g_q^J\left[s-\tau_{pq}(s),\varphi\right] - b_{pq}^K(s)\, g_q^K\left[s-\tau_{pq}(s),\varphi\right]\right)\\
&\quad \left.\left. + u_p^R(s)\right)\mathrm{d}s\right| \le \int_{-\infty}^{t} e^{-\int_s^t d_p(u)\mathrm{d}u}\left(\sum_{q=1}^{n}\left(a_{pq}^{R^+}\left|f_q^R[s,\varphi]\right| + a_{pq}^{I^+}\left|f_q^I[s,\varphi]\right| + a_{pq}^{J^+}\left|f_q^J[s,\varphi]\right| + a_{pq}^{K^+}\left|f_q^K[s,\varphi]\right|\right)\right.\\
&\quad + \sum_{q=1}^{n}\left(b_{pq}^{R^+}\left|g_q^R\left[s-\tau_{pq}(s),\varphi\right]\right| + b_{pq}^{I^+}\left|g_q^I\left[s-\tau_{pq}(s),\varphi\right]\right| + b_{pq}^{J^+}\left|g_q^J\left[s-\tau_{pq}(s),\varphi\right]\right| + b_{pq}^{K^+}\left|g_q^K\left[s-\tau_{pq}(s),\varphi\right]\right|\right)\\
&\quad \left. + u_p^{R^+}\right)\mathrm{d}s \le \int_{-\infty}^{t} e^{-\int_s^t d_p(u)\mathrm{d}u}\left(\sum_{q=1}^{n}\left(a_{pq}^{R^+} + a_{pq}^{I^+} + a_{pq}^{J^+} + a_{pq}^{K^+}\right)\times\left(l_q^R + l_q^I + l_q^J + l_q^K\right)\|\varphi\|_{\mathbb{X}}\right.\\
&\quad \left. + \sum_{q=1}^{n}\left(b_{pq}^{R^+} + b_{pq}^{I^+} + b_{pq}^{J^+} + b_{pq}^{K^+}\right)\left(m_q^R + m_q^I + m_q^J + m_q^K\right)\|\varphi\|_{\mathbb{X}} + u_p^{R^+}\right)\mathrm{d}s \le \frac{1}{d_p^-}\left(\Lambda_p\kappa + \Delta_p\kappa + u_p^{R^+}\right) = \frac{\Omega_p\kappa + u_p^{R^+}}{d_p^-},\\
&\qquad p \in T.
\end{aligned}
\tag{33}
$$

In a similar way, we can get

$$
\left|(\Phi\varphi)_p^{\nu}(t)\right| \le \frac{\Omega_p\kappa + u_p^{\nu^+}}{d_p^-}, \quad p \in T,\ \nu = I, J, K. \tag{34}
$$

By (H_4), we have

$$
\|\Phi\varphi\|_{\mathbb{X}} \le \kappa, \tag{35}
$$

which implies that $\Phi\varphi \in \mathbb{X}^*$, so the mapping Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Next, we shall prove that Φ is a contraction mapping. In fact, for any $\varphi, \psi \in \mathbb{X}^*$, we have

$$
\begin{aligned}
&\left|(\Phi\varphi - \Phi\psi)_p^R(t)\right| = \left|\int_{-\infty}^{t} e^{-\int_s^t d_p(u)\mathrm{d}u}\left(\sum_{q=1}^{n}\left(a_{pq}^R(s)\left(f_q^R[s,\varphi] - f_q^R[s,\psi]\right) - a_{pq}^I(s)\left(f_q^I[s,\varphi] - f_q^I[s,\psi]\right) - a_{pq}^J(s)\left(f_q^J[s,\varphi] - f_q^J[s,\psi]\right) - a_{pq}^K(s)\left(f_q^K[s,\varphi] - f_q^K[s,\psi]\right)\right)\right.\right.\\
&\quad \left.\left. + \sum_{q=1}^{n}\left(b_{pq}^R(s)\left(g_q^R\left[s-\tau_{pq}(s),\varphi\right] - g_q^R\left[s-\tau_{pq}(s),\psi\right]\right) - b_{pq}^I(s)\left(g_q^I\left[s-\tau_{pq}(s),\varphi\right] - g_q^I\left[s-\tau_{pq}(s),\psi\right]\right) - b_{pq}^J(s)\left(g_q^J\left[s-\tau_{pq}(s),\varphi\right] - g_q^J\left[s-\tau_{pq}(s),\psi\right]\right) - b_{pq}^K(s)\left(g_q^K\left[s-\tau_{pq}(s),\varphi\right] - g_q^K\left[s-\tau_{pq}(s),\psi\right]\right)\right)\right)\mathrm{d}s\right|\\
&\le \int_{-\infty}^{t} e^{-\int_s^t d_p(u)\mathrm{d}u}\left(\sum_{q=1}^{n}\left(a_{pq}^{R^+} + a_{pq}^{I^+} + a_{pq}^{J^+} + a_{pq}^{K^+}\right)\times\left(l_q^R + l_q^I + l_q^J + l_q^K\right)\|\varphi-\psi\|_{\mathbb{X}} + \sum_{q=1}^{n}\left(b_{pq}^{R^+} + b_{pq}^{I^+} + b_{pq}^{J^+} + b_{pq}^{K^+}\right)\left(m_q^R + m_q^I + m_q^J + m_q^K\right)\|\varphi-\psi\|_{\mathbb{X}}\right)\mathrm{d}s \le \frac{1}{d_p^-}\left(\Lambda_p + \Delta_p\right)\|\varphi-\psi\|_{\mathbb{X}} = \frac{\Omega_p}{d_p^-}\|\varphi-\psi\|_{\mathbb{X}}, \quad p \in T.
\end{aligned}
\tag{36}
$$

Similarly, we can get

$$
\left|(\Phi\varphi - \Phi\psi)_p^{\nu}(t)\right| \le \frac{\Omega_p}{d_p^-}\|\varphi - \psi\|_{\mathbb{X}}, \quad p \in T,\ \nu = I, J, K. \tag{37}
$$

By (H_4), we have

$$
\|\Phi\varphi - \Phi\psi\|_{\mathbb{X}} \le r\|\varphi - \psi\|_{\mathbb{X}}. \tag{38}
$$

Hence, Φ is a contraction mapping. Therefore, system (16) has a unique almost periodic solution in the region $\mathbb{X}^* = \{\varphi \in \mathbb{X} : \|\varphi\|_{\mathbb{X}} \le \kappa\}$. The proof is complete. □

Theorem 8. *Suppose that* (H_1)–(H_5) *hold. Then the drive system (16) and the response system (19) are globally exponentially synchronized based on the controller (22).*

Proof. In view of the error system (21), we have

$$
\begin{aligned}
\left|z_p^R(t)\right|' \le{}& -d_p^-\left|z_p^R(t)\right|\\
&+ \sum_{q=1}^{n}\left(a_{pq}^{R^+}\left|f_q^R[t,y] - f_q^R[t,x]\right|\right.\\
&+ a_{pq}^{I^+}\left|f_q^I[t,y] - f_q^I[t,x]\right|\\
&+ a_{pq}^{J^+}\left|f_q^J[t,y] - f_q^J[t,x]\right|\\
&\left.+ a_{pq}^{K^+}\left|f_q^K[t,y] - f_q^K[t,x]\right|\right)\\
&+ \sum_{q=1}^{n}\left(b_{pq}^{R^+}\left|g_q^R\left[t-\tau_{pq}(t),y\right] - g_q^R\left[t-\tau_{pq}(t),x\right]\right|\right.\\
&+ b_{pq}^{I^+}\left|g_q^I\left[t-\tau_{pq}(t),y\right] - g_q^I\left[t-\tau_{pq}(t),x\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{pq}^{J^{+}} \left| g_q^J \left[t - \tau_{pq}(t), y \right] - g_q^J \left[t - \tau_{pq}(t), x \right] \right| \\
&+ b_{pq}^{K^{+}} \left| g_q^K \left[t - \tau_{pq}(t), y \right] - g_q^K \left[t - \tau_{pq}(t), x \right] \right| \Big) \\
&+ \epsilon_p \left| z_p^R(t) \right| \leq \left(\epsilon_p - d_p^{-} \right) \left| z_p^R(t) \right| + \sum_{q=1}^{n} \Big(a_{pq}^{R^{+}} + a_{pq}^{I^{+}} \\
&+ a_{pq}^{J^{+}} + a_{pq}^{K^{+}} \Big) \Big(l_q^R \left| z_q^R(t) \right| + l_q^I \left| z_q^I(t) \right| + l_q^J \left| z_q^J(t) \right| \\
&+ l_q^K \left| z_q^K(t) \right| \Big) + \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \\
&\cdot \Big(m_q^R \left| z_q^R \left(t - \tau_{pq}(t) \right) \right| + m_q^I \left| z_q^I \left(t - \tau_{pq}(t) \right) \right| \\
&+ m_q^J \left| z_q^J \left(t - \tau_{pq}(t) \right) \right| + m_q^K \left| z_q^K \left(t - \tau_{pq}(t) \right) \right| \Big), \\
&\qquad p \in T.
\end{aligned} \tag{39}
$$

In a similar way, we can obtain

$$
\begin{aligned}
&\left| z_p^{\nu}(t) \right|' \leq \left(\epsilon_p - d_p^{-} \right) \left| z_p^{\nu}(t) \right| + \sum_{q=1}^{n} \Big(a_{pq}^{R^{+}} + a_{pq}^{I^{+}} + a_{pq}^{J^{+}} \\
&+ a_{pq}^{K^{+}} \Big) \Big(l_q^R \left| z_q^R(t) \right| + l_q^I \left| z_q^I(t) \right| + l_q^J \left| z_q^J(t) \right| \\
&+ l_q^K \left| z_q^K(t) \right| \Big) + \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \\
&\cdot \Big(m_q^R \left| z_q^R \left(t - \tau_{pq}(t) \right) \right| + m_q^I \left| z_q^I \left(t - \tau_{pq}(t) \right) \right| \\
&+ m_q^J \left| z_q^J \left(t - \tau_{pq}(t) \right) \right| + m_q^K \left| z_q^K \left(t - \tau_{pq}(t) \right) \right| \Big), \\
&\qquad p \in T, \ \nu = I, J, K.
\end{aligned} \tag{40}
$$

Now, we consider the Lyapunov function as follows:

$$
V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{41}
$$

where

$$
\begin{aligned}
V_1(t) &= \sum_{p=1}^{n} \left(\left| z_p^R(t) \right| e^{\lambda t} + W_p \right), \\
V_2(t) &= \sum_{p=1}^{n} \left(\left| z_p^I(t) \right| e^{\lambda t} + W_p \right), \\
V_3(t) &= \sum_{p=1}^{n} \left(\left| z_p^J(t) \right| e^{\lambda t} + W_p \right), \\
V_4(t) &= \sum_{p=1}^{n} \left(\left| z_p^K(t) \right| e^{\lambda t} + W_p \right), \\
W_p &= \frac{1}{1-\beta} \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \\
&\cdot \int_{t-\tau_{pq}(t)}^{t} \Big(m_q^R \left| z_q^R(s) \right| + m_q^I \left| z_q^I(s) \right| + m_q^J \left| z_q^J(s) \right| \\
&+ m_q^K \left| z_q^K(s) \right| \Big) e^{\lambda(s+\tau)} \mathrm{d}s,
\end{aligned} \tag{42}
$$

Computing the derivative of $V(t)$ along the solutions of the error system (21), we can get

$$
\begin{aligned}
V_1'(t) &= \sum_{p=1}^{n} \Bigg\{ \left| z_p^R(t) \right|' e^{\lambda t} + \lambda e^{\lambda t} \left| z_p^R(t) \right| + \frac{1}{1-\beta} \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \left(m_q^R \left| z_q^R(t) \right| + m_q^I \left| z_q^I(t) \right| + m_q^J \left| z_q^J(t) \right| + m_q^K \left| z_q^K(t) \right| \right) \times e^{\lambda(t+\tau)} - \frac{1}{1-\beta} \\
&\cdot \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \left(m_q^R \left| z_q^R \left(t - \tau_{pq}(t) \right) \right| + m_q^I \left| z_q^I \left(t - \tau_{pq}(t) \right) \right| + m_q^J \left| z_q^J \left(t - \tau_{pq}(t) \right) \right| + m_q^K \left| z_q^K \left(t - \tau_{pq}(t) \right) \right| \right) \times e^{\lambda(t-\tau_{pq}(t)+\tau)} \left(1 - \tau_{pq}'(t) \right) \Bigg\} \leq \sum_{p=1}^{n} \Bigg\{ \left(\lambda - d_p^{-} + \epsilon_p \right) \left| z_p^R(t) \right| \\
&+ \sum_{q=1}^{n} \Big[\left(a_{pq}^{R^{+}} + a_{pq}^{I^{+}} + a_{pq}^{J^{+}} + a_{pq}^{K^{+}} \right) \times \left(l_q^R \left| z_q^R(t) \right| + l_q^I \left| z_q^I(t) \right| + l_q^J \left| z_q^J(t) \right| + l_q^K \left| z_q^K(t) \right| \right) + \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \left(m_q^R \left| z_q^R \left(t - \tau_{pq}(t) \right) \right| + m_q^I \left| z_q^I \left(t - \tau_{pq}(t) \right) \right| + m_q^J \left| z_q^J \left(t - \tau_{pq}(t) \right) \right| + m_q^K \left| z_q^K \left(t - \tau_{pq}(t) \right) \right| \right) \Big] \\
&+ \frac{1}{1-\beta} \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \left(m_q^R \left| z_q^R(t) \right| + m_q^I \left| z_q^I(t) \right| + m_q^J \left| z_q^J(t) \right| + m_q^K \left| z_q^K(t) \right| \right) e^{\lambda \tau} \\
&- \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \left(m_q^R \left| z_q^R \left(t - \tau_{pq}(t) \right) \right| + m_q^I \left| z_q^I \left(t - \tau_{pq}(t) \right) \right| + m_q^J \left| z_q^J \left(t - \tau_{pq}(t) \right) \right| + m_q^K \left| z_q^K \left(t - \tau_{pq}(t) \right) \right| \right) \Bigg\} e^{\lambda t} \leq \sum_{p=1}^{n} \Bigg\{ \left(\lambda - d_p^{-} + \epsilon_p \right) \left| z_p^R(t) \right| + \sum_{q=1}^{n} \left(a_{pq}^{R^{+}} + a_{pq}^{I^{+}} + a_{pq}^{J^{+}} + a_{pq}^{K^{+}} \right) \\
&\times \left(l_q^R \left| z_q^R(t) \right| + l_q^I \left| z_q^I(t) \right| + l_q^J \left| z_q^J(t) \right| + l_q^K \left| z_q^K(t) \right| \right) + \frac{1}{1-\beta} \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \left(m_q^R \left| z_q^R(t) \right| + m_q^I \left| z_q^I(t) \right| + m_q^J \left| z_q^J(t) \right| + m_q^K \left| z_q^K(t) \right| \right) e^{\lambda \tau} \Bigg\} e^{\lambda t} \leq \sum_{p=1}^{n} \Bigg[\lambda - d_p^{-} + \epsilon_p + \sum_{q=1}^{n} \left(a_{pq}^{R^{+}} + a_{pq}^{I^{+}} + a_{pq}^{J^{+}} + a_{pq}^{K^{+}} \right) \\
&\times \left(l_q^R + l_q^I + l_q^J + l_q^K \right) + \frac{1}{1-\beta} \sum_{q=1}^{n} \left(b_{pq}^{R^{+}} + b_{pq}^{I^{+}} + b_{pq}^{J^{+}} + b_{pq}^{K^{+}} \right) \times \left(m_q^R + m_q^I + m_q^J + m_q^K \right) e^{\lambda \tau} \Bigg] e^{\lambda t} \left\| Z(t) \right\| = \sum_{p=1}^{n} \left[\lambda - d_p^{-} + \epsilon_p + \Lambda_p + \frac{\Delta_p}{1-\beta} e^{\lambda \tau} \right] e^{\lambda t} \left\| Z(t) \right\|.
\end{aligned} \tag{43}
$$

Similarly, we can get

$$
\begin{aligned}
&V_2'(t) \\
&\quad \leq \sum_{p=1}^{n} \left[\lambda - d_p^{-} + \varepsilon_p + \Lambda_p + \frac{\Delta_p}{1-\beta} e^{\lambda \tau} \right] e^{\lambda t} \left\| Z(t) \right\|, \\
&V_3'(t) \\
&\quad \leq \sum_{p=1}^{n} \left[\lambda - d_p^{-} + \rho_p + \Lambda_p + \frac{\Delta_p}{1-\beta} e^{\lambda \tau} \right] e^{\lambda t} \left\| Z(t) \right\|, \\
&V_4'(t)
\end{aligned}
$$

$$\leq \sum_{p=1}^{n}\left[\lambda - d_p^- + \varrho_p + \Lambda_p + \frac{\Delta_p}{1-\beta}e^{\lambda\tau}\right]e^{\lambda t}\|Z(t)\|. \tag{44}$$

From (H_5), we obtain

$$V'(t) \leq 0, \tag{45}$$

which implies that $V(t) \leq V(0)$ for all $t \geq 0$.

On the other hand, we have

$$\begin{aligned} V_1(0) \leq \sum_{p=1}^{n}&\left\{\left|z_p^R(0)\right| + \frac{1}{1-\beta}\sum_{q=1}^{n}\left(b_{pq}^{R^+} + b_{pq}^{I^+} + b_{pq}^{J^+} + b_{pq}^{K^+}\right)\right. \\ &\times \left.\int_{0-\tau_{pq}(0)}^{0}\left(m_q^R\left|z_q^R(s)\right| + m_q^I\left|z_q^I(s)\right| + m_q^J\left|z_q^J(s)\right| + m_q^K\left|z_q^K(s)\right|\right)e^{\lambda(s+\tau)}ds\right\} \leq \left\{1 + \frac{1}{1-\beta}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{e^{\lambda\tau}}{\lambda}\Delta_p\right\}\|\psi - \phi\|. \end{aligned} \tag{46}$$

Similarly, we can get

$$\begin{aligned} V_2(0) &\leq \left\{1 + \frac{1}{1-\beta}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{e^{\lambda\tau}}{\lambda}\Delta_p\right\}\|\psi - \phi\|, \\ V_3(0) &\leq \left\{1 + \frac{1}{1-\beta}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{e^{\lambda\tau}}{\lambda}\Delta_p\right\}\|\psi - \phi\|, \\ V_4(0) &\leq \left\{1 + \frac{1}{1-\beta}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{e^{\lambda\tau}}{\lambda}\Delta_p\right\}\|\psi - \phi\|. \end{aligned} \tag{47}$$

It is obvious that $\|Z(t)\|e^{\lambda t} \leq V(t)$; thus, for $t \geq 0$, we have

$$\|Z(t)\| \leq V(t)e^{-\lambda t} \leq V(0)e^{-\lambda t} \leq M\|\psi - \phi\|e^{-\lambda t}, \tag{48}$$

where

$$M = \left\{1 + \frac{1}{1-\beta}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{e^{\lambda\tau}}{\lambda}\Delta_p\right\} > 1. \tag{49}$$

Therefore, the drive system (16) and the response system (19) are globally exponentially synchronized based on the controller (22). The proof is complete. □

4. An Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Section 3.

Example 1. Consider the following quaternion-valued neural networks with time-varying delay:

$$\begin{aligned} x_p'(t) = -d_p(t)x_p(t) &+ \sum_{q=1}^{2}a_{pq}(t)f_q\left(x_q(t)\right) \\ &+ \sum_{q=1}^{2}b_{pq}(t)g_q\left(x_q\left(t-\tau_{pq}(t)\right)\right) + u_p(t), \end{aligned} \tag{50}$$

and its response system

$$\begin{aligned} y_p'(t) = -d_p(t)y_p(t) &+ \sum_{q=1}^{2}a_{pq}(t)f_q\left(y_q(t)\right) \\ &+ \sum_{q=1}^{2}b_{pq}(t)g_q\left(y_q\left(t-\tau_{pq}(t)\right)\right) + u_p(t) \\ &+ \theta_p(t). \end{aligned} \tag{51}$$

The coefficients are taken as follows:

$$\begin{aligned} d_1(t) &= 4 + |\cos(2t)|, \\ d_2(t) &= 6 - \left|\sin\left(\sqrt{3}t\right)\right|, \\ a_{11}(t) &= 0.1\sin\left(\sqrt{2}t\right) + i0.3\cos t + j0.2\sin t \\ &\quad + k0.4\cos\left(\sqrt{2}t\right), \\ a_{12}(t) &= 0.3\cos\left(\sqrt{3}t\right) + i0.1\sin(2t) \\ &\quad + j0.5\sin\left(\sqrt{2}t\right) + k0.2\sin t, \\ a_{21}(t) &= 0.4\sin(2t) + i0.5\cos t + j0.1\cos\left(\sqrt{3}t\right) \\ &\quad + k0.2\sin t, \\ a_{22}(t) &= 0.3\sin t + i0.4\sin\left(\sqrt{3}t\right) + j0.1\cos t \\ &\quad - k0.2\cos(3t), \\ b_{11}(t) &= 0.2\cos t + i0.4\sin t - j0.3\sin\left(\sqrt{2}t\right) \\ &\quad + k0.1\sin\left(\sqrt{2}t\right), \\ b_{12}(t) &= 0.5\sin t - i0.2\cos t - j0.3\cos\left(\sqrt{2}t\right) \\ &\quad - k0.1\cos t, \\ b_{21}(t) &= 0.2\cos(2t) - i0.5\sin(2t) + j0.3\sin(3t) \\ &\quad - k0.4\cos\left(\sqrt{3}t\right), \end{aligned}$$

$$\begin{aligned}
b_{22}(t) &= -0.6\cos t - i0.1\cos\left(\sqrt{3}t\right) + j0.2\cos t \\
&\quad - k0.5\sin(3t), \\
f_q\left(x_q\right) &= \frac{1}{32}\tan\left(x_q^R + x_q^I\right) + i\frac{1}{32}\sin x_q^I + j\frac{1}{32}\left|x_q^J\right| \\
&\quad + k\frac{1}{32}\sin x_q^K, \\
g_q\left(x_q\right) &= \frac{1}{24}\sin x_q^R + i\frac{1}{24}\left|x_q^I + x_q^K\right| + j\frac{1}{24}\sin x_q^J \\
&\quad + k\frac{1}{24}\left|x_q^K\right|, \\
\tau_{11}(t) &= \frac{1}{5}|\sin 2t|, \\
\tau_{12}(t) &= \tau_{21}(t) = \frac{1}{6}|\cos t|, \\
\tau_{22}(t) &= \frac{1}{4}|\sin t|, \\
u_1(t) &= 2\sin\left(\sqrt{2}t\right) - i\cos t + j\sin t + k3\cos t, \\
u_2(t) &= 2\cos t + i3\sin\left(\sqrt{3}t\right) - 5j\cos t - k4\sin t.
\end{aligned} \tag{52}$$

By a simple calculation, we have

$$\begin{aligned}
d_1^- &= 4, \\
d_2^- &= 5, \\
a_{11}^{R^+} &= 0.1, \\
a_{11}^{I^+} &= 0.3, \\
a_{11}^{J^+} &= 0.2, \\
a_{11}^{K^+} &= 0.4, \\
a_{12}^{R^+} &= 0.3, \\
a_{12}^{I^+} &= 0.1, \\
a_{12}^{J^+} &= 0.5, \\
a_{12}^{K^+} &= 0.2, \\
a_{21}^{R^+} &= 0.4, \\
a_{21}^{I^+} &= 0.5, \\
a_{21}^{J^+} &= 0.1, \\
a_{21}^{K^+} &= 0.2, \\
a_{22}^{R^+} &= 0.3, \\
a_{22}^{I^+} &= 0.4, \\
a_{22}^{J^+} &= 0.1, \\
a_{22}^{K^+} &= 0.2, \\
b_{11}^{R^+} &= 0.2, \\
b_{11}^{I^+} &= 0.4, \\
b_{11}^{J^+} &= 0.3, \\
b_{11}^{K^+} &= 0.1, \\
b_{12}^{R^+} &= 0.5, \\
b_{12}^{I^+} &= 0.2, \\
b_{12}^{J^+} &= 0.3, \\
b_{12}^{K^+} &= 0.1, \\
b_{21}^{R^+} &= 0.2, \\
b_{21}^{I^+} &= 0.5, \\
b_{21}^{J^+} &= 0.3, \\
b_{21}^{K^+} &= 0.4, \\
b_{22}^{R^+} &= 0.6, \\
b_{22}^{I^+} &= 0.1 \\
b_{22}^{J^+} &= 0.2, \\
b_{22}^{K^+} &= 0.5, \\
u_1^{R^+} &= 2, \\
u_1^{I^+} &= 1, \\
u_1^{J^+} &= 1, \\
u_1^{K^+} &= 3, \\
u_2^{R^+} &= 2, \\
u_2^{I^+} &= 3, \\
u_2^{J^+} &= 5, \\
u_2^{K^+} &= 4, \\
\tau &= \frac{1}{4}, \\
l_q^\nu &= \frac{1}{32}, \\
m_q^\nu &= \frac{1}{24}, \\
&\qquad q = 1, 2,\ \nu \in E, \\
\Lambda_1 &= 0.525,
\end{aligned}$$

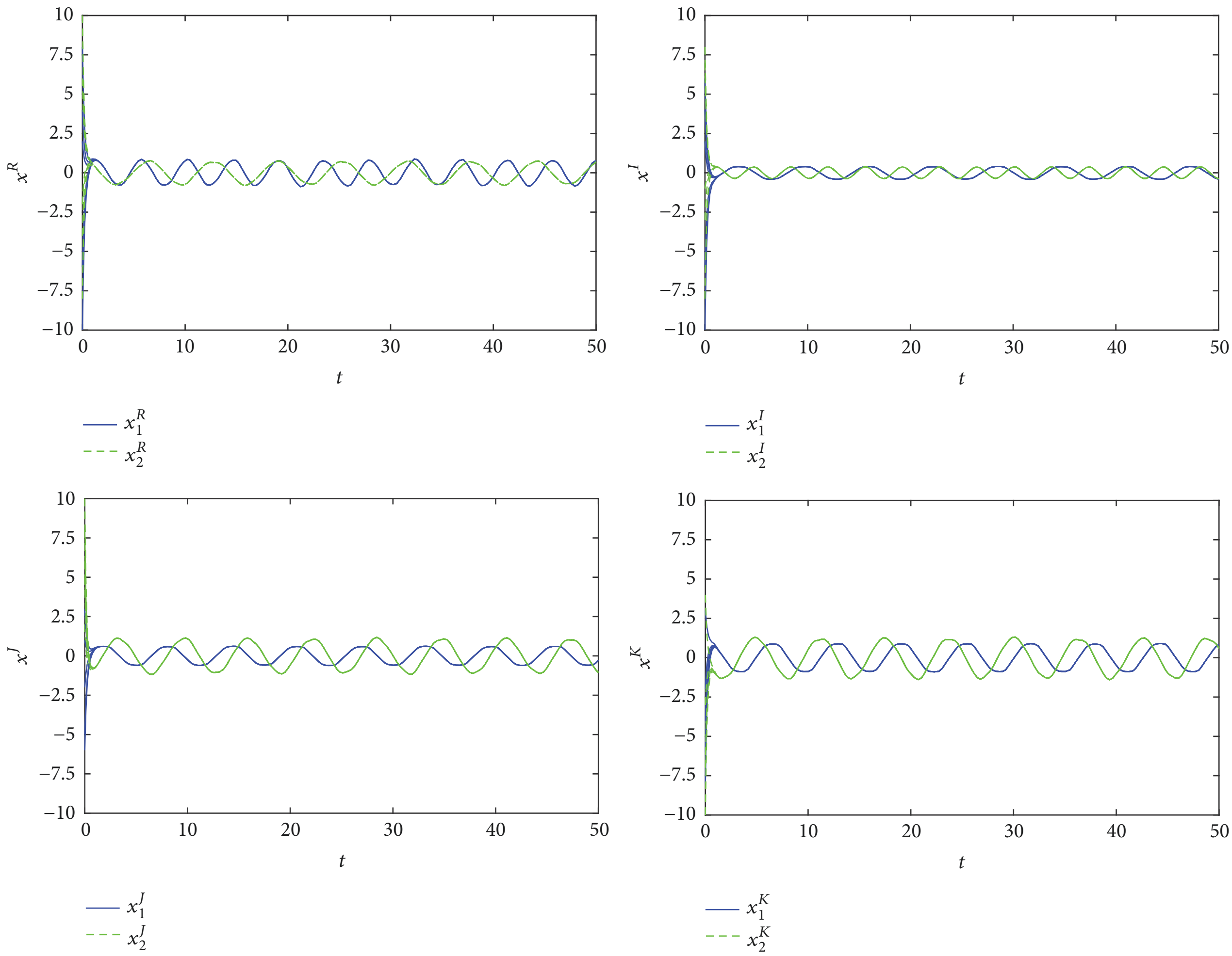


FIGURE 1: The states of four parts of x_1 and x_2.

$$\Lambda_2 = 0.55,$$
$$\Delta_1 = 0.7,$$
$$\Delta_2 \approx 0.9333. \tag{53}$$

Take $\kappa = 2$; then we have $\Omega_1 = 1.225$, $\Omega_2 = 1.4833$,

$$\max_{\nu \in E} \left\{ \frac{\Omega_1 \kappa + u_1^{\nu^+}}{d_1^-}, \frac{\Omega_2 \kappa + u_2^{\nu^+}}{d_2^-} \right\} \approx \{1.3625, 1.5933\}$$
$$= 1.5933 < \kappa = 2, \tag{54}$$
$$\max \left\{ \frac{\Omega_1}{d_1^-}, \frac{\Omega_2}{d_2^-} \right\} \approx \{0.30625, 0.29666\} = 0.30625 < 1.$$

Moreover, take $\epsilon_1 = 1$, $\epsilon_2 = 1.5$, $\varepsilon_1 = 0.8$, $\varepsilon_2 = 1.8$, $\rho_1 = 1.2$, $\rho_2 = 1.6$, $\varrho_1 = 0.5$, $\varrho_2 = 1.9$, $\lambda = 1$, $\beta = 1/5$,

$$\lambda - d_1^- + \epsilon_1 + \Lambda_1 + \frac{\Delta_1}{1-\beta} e^{\lambda\tau} \approx -0.3515 < 0,$$
$$\lambda - d_2^- + \epsilon_2 + \Lambda_2 + \frac{\Delta_2}{1-\beta} e^{\lambda\tau} \approx -0.4520 < 0,$$
$$\lambda - d_1^- + \varepsilon_1 + \Lambda_1 + \frac{\Delta_1}{1-\beta} e^{\lambda\tau} \approx -0.5515 < 0,$$
$$\lambda - d_2^- + \varepsilon_2 + \Lambda_2 + \frac{\Delta_2}{1-\beta} e^{\lambda\tau} \approx -0.1520 < 0,$$
$$\lambda - d_1^- + \rho_1 + \Lambda_1 + \frac{\Delta_1}{1-\beta} e^{\lambda\tau} \approx -0.1515 < 0,$$
$$\lambda - d_2^- + \rho_2 + \Lambda_2 + \frac{\Delta_2}{1-\beta} e^{\lambda\tau} \approx -0.3520 < 0,$$
$$\lambda - d_1^- + \varrho_1 + \Lambda_1 + \frac{\Delta_1}{1-\beta} e^{\lambda\tau} \approx -0.8515 < 0,$$
$$\lambda - d_2^- + \varrho_2 + \Lambda_2 + \frac{\Delta_2}{1-\beta} e^{\lambda\tau} \approx -0.0520 < 0. \tag{55}$$

We can verify that all the assumptions of Theorems 7 and 8 are satisfied. Therefore, the drive system (50) and its response system (51) are globally exponential synchronized (see Figures 1–3).

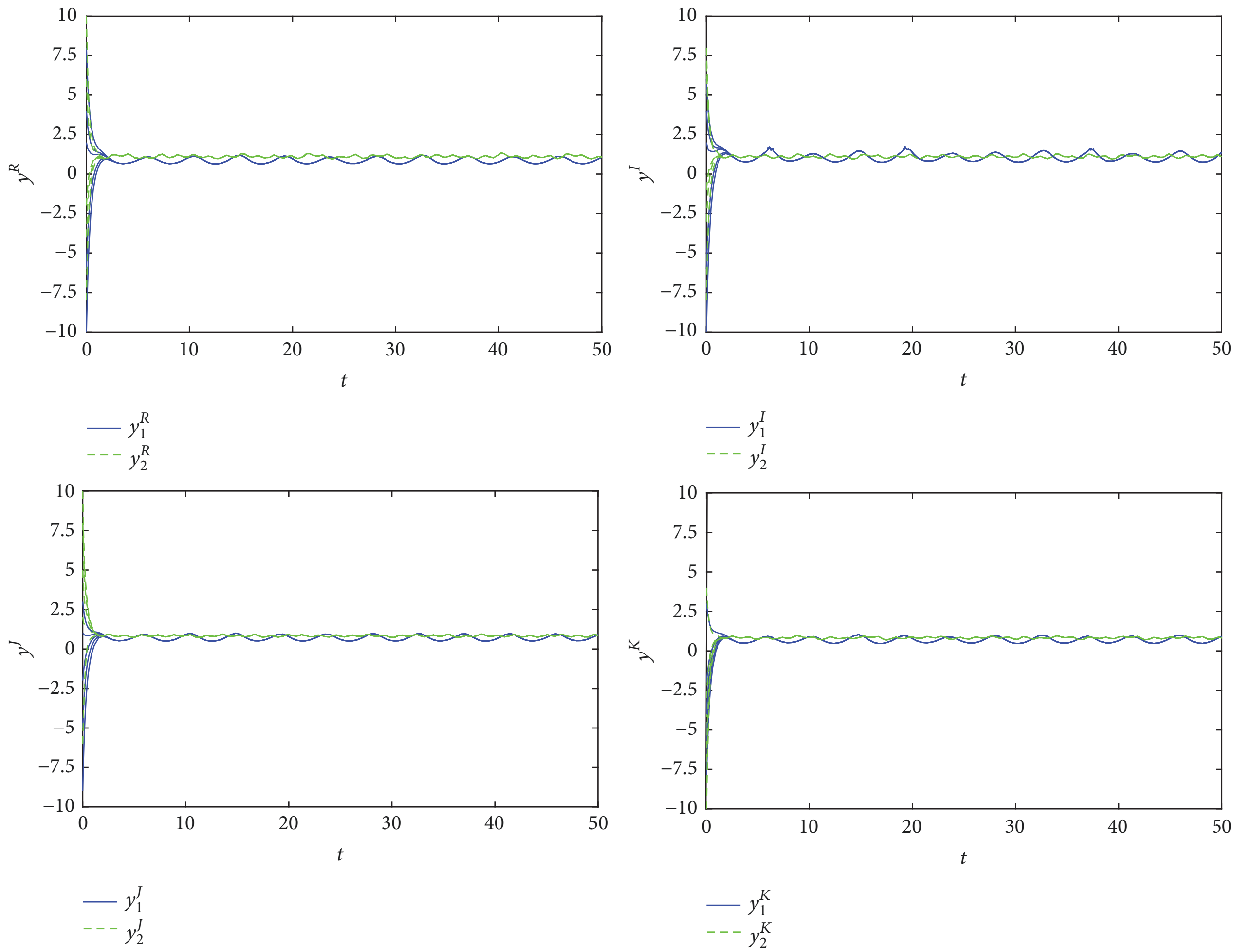


FIGURE 2: The states of four parts of y_1 and y_2.

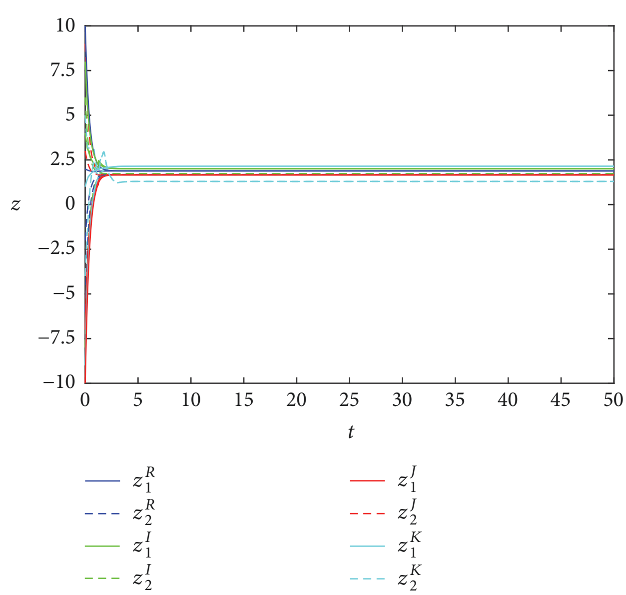


FIGURE 3: Synchronization.

5. Conclusion

In this paper, we consider the global exponential almost periodic synchronization of quaternion-valued neural networks with time-varying delays. By means of the exponential dichotomy of linear differential equations, Banach's fixed point theorem, Lyapunov functional method, and differential inequality technique, we establish the existence and global exponential synchronization of almost periodic solutions for system (9). To the best of our knowledge, this is the first time to study the almost periodic synchronization problem for neural networks. Furthermore, the method of this paper can be used to study other types of neural networks.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 11361072.

References

[1] T. Isokawa, T. Kusakabe, N. Matsui, and F. Peper, "Quaternion neural network and its application," *Lecture Notes in Computer Science*, vol. 2774, pp. 318–324, 2003.

[2] N. Matsui, T. Isokawa, H. Kusamichi, F. Peper, and H. Nishimura, "Quaternion neural network with geometrical operators," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, vol. 15, no. 3-4, pp. 149–164, 2004.

[3] R. Kristiansen and P. J. Nicklasson, "Satellite attitude control by quaternion-based backstepping," in *Proceedings of the 2005 American Control Conference*, pp. 907–912, USA, June 2005.

[4] X.-Y. Wang, C.-P. Wang, H.-Y. Yang, and P.-P. Niu, "A robust blind color image watermarking in quaternion Fourier transform domain," *The Journal of Systems and Software*, vol. 86, no. 2, pp. 255–277, 2013.

[5] V. R. Dubey, "Quaternion fourier transform for colour images," *International Journal of Computer Science and Information Technologies*, vol. 5, no. 3, pp. 4411–4416, 2014.

[6] Z. Wang, X. Wang, Y. Li, and X. Huang, "Stability and Hopf Bifurcation of Fractional-Order Complex-Valued Single Neuron Model with Time Delay," *International Journal of Bifurcation and Chaos*, vol. 27, no. 13, 1750209, 13 pages, 2017.

[7] C. Huang, J. Cao, M. Xiao, A. Alsaedi, and T. Hayat, "Bifurcations in a delayed fractional complex-valued neural network," *Applied Mathematics and Computation*, vol. 292, pp. 210–227, 2017.

[8] Y. Liu, D. Zhang, and J. Lu, "Global exponential stability for quaternion-valued recurrent neural networks with time-varying delays," *Nonlinear Dynamics*, vol. 87, no. 1, pp. 553–565, 2017.

[9] Z. Tu, J. Cao, A. Alsaedi, and T. Hayat, "Global dissipativity analysis for delayed quaternion-valued neural networks," *Neural Networks*, vol. 89, pp. 97–104, 2017.

[10] X. Chen, Z. Li, Q. Song, J. Hu, and Y. Tan, "Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties," *Neural Networks*, vol. 91, pp. 55–65, 2017.

[11] H. Shu, Q. Song, Y. Liu, Z. Zhao, and F. E. Alsaadi, "Global μ–stability of quaternion-valued neural networks with non-differentiable time-varying delays," *Neurocomputing*, vol. 247, pp. 202–212, 2017.

[12] Y. Liu, D. Zhang, J. Lu, and J. Cao, "Global μ-stability criteria for quaternion-valued neural networks with unbounded time-varying delays," *Information Sciences*, vol. 360, pp. 273–288, 2016.

[13] X. Chen, Q. Song, and Z. Li, "Design and Analysis of Quaternion-Valued Neural Networks for Associative Memories," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2017.

[14] Y. Liu, D. Zhang, J. Lou, J. Lu, and J. Cao, "Stability Analysis of Quaternion-Valued Neural Networks: Decomposition and Direct Approaches," *IEEE Transactions on Neural Networks and Learning Systems*, 2017.

[15] D. Zhang, K. I. Kou, Y. Liu, and J. Cao, "Decomposition approach to the stability of recurrent neural networks with asynchronous time delays in quaternion field," *Neural Networks*, vol. 94, pp. 55–66, 2017.

[16] Y. Liu, P. Xu, J. Lu, and J. Liang, "Global stability of Clifford-valued recurrent neural networks with time delays," *Nonlinear Dynamics*, vol. 84, no. 2, pp. 767–777, 2016.

[17] Y. Li and X. Meng, "Existence and Global Exponential Stability of Pseudo Almost Periodic Solutions for Neutral Type Quaternion-Valued Neural Networks with Delays in The Leakage Term on Time Scales," *Complexity*, Article ID 9878369, 15 pages, 2017.

[18] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[19] X. Huang and J. Cao, "Generalized synchronization for delayed chaotic neural networks: a novel coupling scheme," *Nonlinearity*, vol. 19, no. 12, pp. 2797–2811, 2006.

[20] W. He and J. Cao, "Exponential synchronization of chaotic neural networks: a matrix measure approach," *Nonlinear Dynamics*, vol. 55, no. 1-2, pp. 55–65, 2009.

[21] X. Li and M. Bohner, "Exponential synchronization of chaotic neural networks with mixed delays and impulsive effects via output coupling with delay feedback," *Mathematical and Computer Modelling*, vol. 52, no. 5-6, pp. 643–653, 2010.

[22] Y. Liu, Z. Wang, J. Liang, and X. Liu, "Synchronization of coupled neutral-type neural networks with jumping-mode-dependent discrete and unbounded distributed delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 43, pp. 102–114, 2013.

[23] L. Pan, J. Cao, and J. Hu, "Synchronization for complex networks with Markov switching via matrix measure approach," *Applied Mathematical Modelling*, vol. 39, no. 18, pp. 5636–5649, 2015.

[24] Y. Li and C. Li, "Matrix measure strategies for stabilization and synchronization of delayed BAM neural networks," *Nonlinear Dynamics*, vol. 84, no. 3, pp. 1759–1770, 2016.

[25] L. Duan, L. Huang, and X. Fang, "Finite-time synchronization for recurrent neural networks with discontinuous activations and time-varying delays," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 27, no. 1, Article ID 013101, 2017.

[26] Y. Liu, L. Sun, J. Lu, and J. Liang, "Feedback controller design for the synchronization of Boolean control networks," *IEEE*

Transactions on Neural Networks and Learning Systems, vol. 27, no. 9, pp. 1991–1996, 2016.

[27] A. Chen, L. Huang, and J. Cao, "Existence and stability of almost periodic solution for BAM neural networks with delays," *Applied Mathematics and Computation*, vol. 137, no. 1, pp. 177–193, 2003.

[28] T. Zhou, M. Wang, and C. Li, "Almost periodic solution for multidirectional associative memory neural network with distributed delays," *Mathematics and Computers in Simulation*, vol. 107, pp. 52–60, 2015.

[29] X. Huang and J. Cao, "Almost periodic solution of shunting inhibitory cellular neural networks with time-varying delay," *Physics Letters A*, vol. 314, no. 3, pp. 222–231, 2003.

[30] X. Huang, J. Cao, and D. W. Ho, "Existence and attractivity of almost periodic solution for recurrent neural networks with unbounded delays and variable coefficients," *Nonlinear Dynamics*, vol. 45, no. 3-4, pp. 337–351, 2006.

[31] M. Yan, J. Qiu, X. Chen, X. Chen, C. Yang, and A. Zhang, "Almost periodic dynamics of the delayed complex-valued recurrent neural networks with discontinuous activation functions," *Neural Computing and Applications*, pp. 1–14, 2017.

[32] M. Y. Choi, Y. W. Kim, and D. C. Hong, "Periodic synchronization in a driven system of coupled oscillators," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 49, no. 5, pp. 3825–3832, 1994.

[33] H. Hong, M. Y. Choi, J. Yi, and K.-S. Soh, "Inertia effects on periodic synchronization in a system of coupled oscillators," *Physical Review E*, vol. 59, no. 1, pp. 353–363, 1999.

[34] F. Cummins, "Periodic and Aperiodic Synchronization in Skilled Action," *Frontiers in Human Neuroscience*, vol. 5, 2011.

[35] W.-K. Lee, Y. Kim, and S.-Y. Kim, "Loss of periodic synchronization in unidirectionally coupled nonlinear systems," *Journal of the Korean Physical Society*, vol. 40, no. 5, pp. 788–792, 2002.

[36] X. L. Deng and H. B. Huang, "Spatial periodic synchronization of chaos in coupled ring and linear arrays of chaotic systems," *Physical Review E*, vol. 65, no. 5, Article ID 055202, 2002.

[37] W. Zou and M. Zhan, "Complete periodic synchronization in coupled systems," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 18, no. 4, Article ID 043115, 2008.

[38] M. Krupa, A. Vidal, and F. Clément, "A network model of the periodic synchronization process in the dynamics of calcium concentration in GnRH neurons," *Journal of Mathematical Neuroscience*, vol. 3, Art. 4, 24 pages, 2013.

[39] H. Hong, "Periodic synchronization and chimera in conformist and contrarian oscillators," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 89, no. 6, Article ID 062924, 2014.

[40] Y. Chai and L.-Q. Chen, "Periodic synchronization of community networks with non-identical nodes uncertain parameters and adaptive coupling strength," *Chinese Physics B*, vol. 23, no. 3, Article ID 030504, 2014.

[41] Z. Cai, L. Huang, Z. Guo, L. Zhang, and X. Wan, "Periodic synchronization control of discontinuous delayed networks by using extended Filippov-framework," *Neural Networks*, vol. 68, pp. 96–110, 2015.

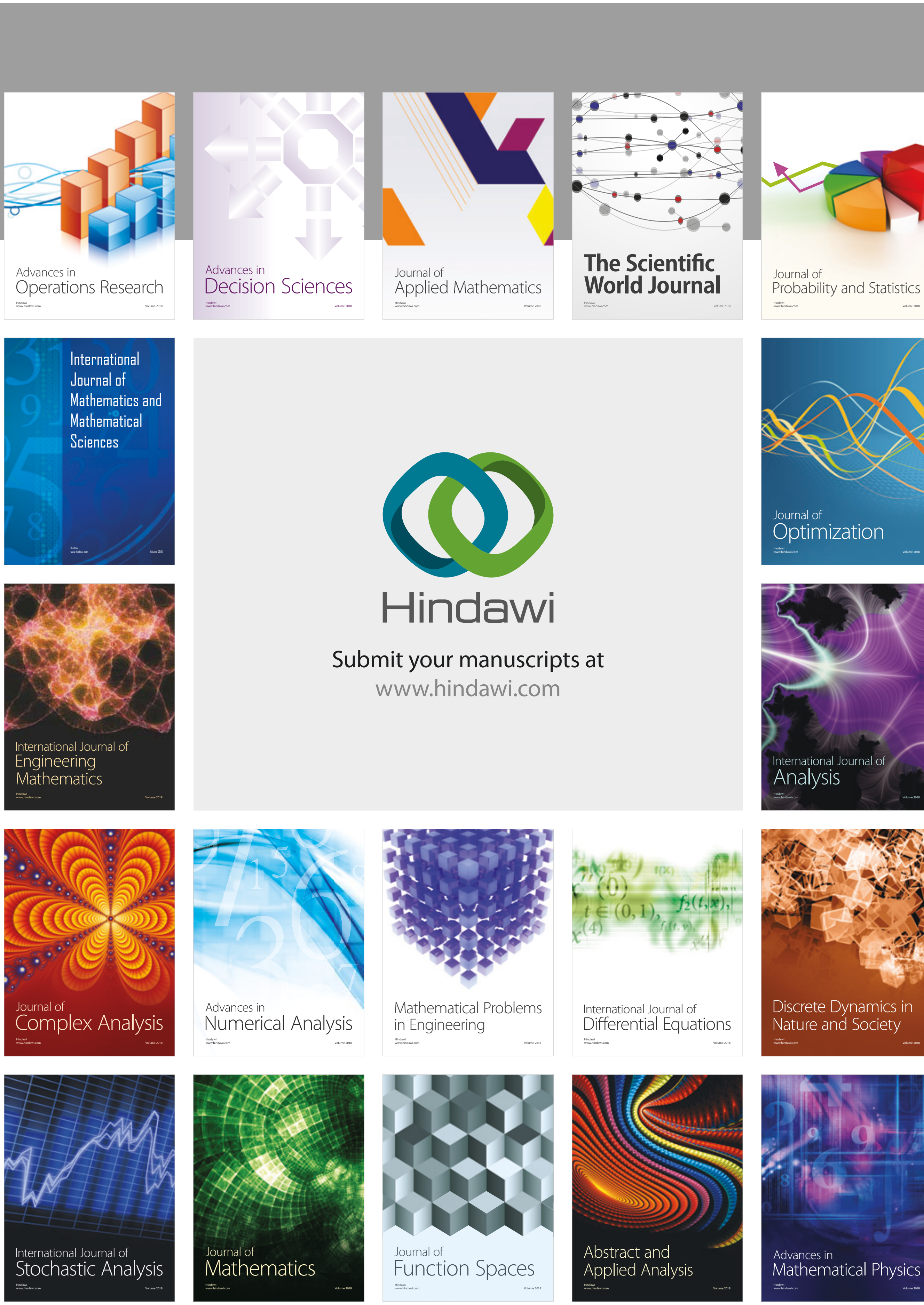
[42] H. Wu, R. Li, X. Zhang, and R. Yao, "Adaptive Finite-Time Complete Periodic Synchronization of Memristive Neural Networks with Time Delays," *Neural Processing Letters*, vol. 42, no. 3, pp. 563–583, 2015.

[43] Z. Cai, L. Huang, D. Wang, and L. Zhang, "Periodic synchronization in delayed memristive neural networks based on Filippov systems," *Journal of The Franklin Institute*, vol. 352, no. 10, pp. 4638–4663, 2015.

[44] H. Bao, J. H. Park, and J. Cao, "Synchronization of fractional-order complex-valued neural networks with time delay," *Neural Networks*, vol. 81, pp. 16–28, 2016.

[45] J. Hu and C. Zeng, "Adaptive exponential synchronization of complex-valued Cohen–Grossberg neural networks with known and unknown parameters," *Neural Networks*, vol. 86, pp. 90–101, 2017.

[46] A. M. Fink, *Almost Periodic Differential Equations*, Springer, New York, NY, USA, 1974.



Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
The Scientific
World Journal
Journal of
Probability and Statistics
International
Journal of
Mathematics and
Mathematical
Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of
Optimization
International Journal of
Engineering
Mathematics
International Journal of
Analysis
Journal of
Complex Analysis
Advances in
Numerical Analysis
Mathematical Problems
in Engineering
International Journal of
Differential Equations
Discrete Dynamics in
Nature and Society
International Journal of
Stochastic Analysis
Journal of
Mathematics
Journal of
Function Spaces
Abstract and
Applied Analysis
Advances in
Mathematical Physics