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Normative systems and their revision: An algebraic approach

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Abstract. The paper discusses normative systems and their revision within an algebraic framework. If a system is logically well-formed, certain norms, called connecting norms, determine the system as a whole. It is maintained that, if the system is well-formed, a relation "at least as low as" determines a lattice or quasi-lattice of its connecting norms. The ideas are presented mainly in the form of comments on a legal example concerning acquisition of movable property by extinction of another person's previous rights.

Key words: Alchourrón, Boolean quasi-ordering, Bulygin, connecting norm, defeasibility, norm revision, normative system, organic whole

1. Introduction

Among different features of a normative system, coherence is central. Paying attention to this feature is highly important when a normative system is revised. The purpose of the present paper is to suggest algebraic requirements of coherence, with a view, in particular, to revision. The coherence of a system will be dealt with in terms of the system's being "well-formed".

To handle the problem of coherence and revision, a formal framework for representing normative systems is needed. In 1971, Alchourrón and Bulygin published their important book *Normative Systems,* which contains a logical and model-theoretical analysis of systems of norms. Partly influenced by that book, the present authors have developed an algebraic theory of normative systems based on a framework of so-called Boolean quasi-orderings and condition implication structures.[1] In the present paper, a portion of this framework will be used. Criteria will be given for testing whether the new system is logically well-formed, after a revision has taken place. These criteria take into account the role of certain norms of a well-formed system, namely such norms as are called "connecting" norms. The set of connecting norms determine a lattice or "quasi-lattice" with respect to a particular kind of relation "at least as low as".

The present paper does not intend to provide a full theory of revision. Rather it intends to present some theoretical tools, convenient for facilitating revision, and for testing whether, after a revision, the new system is well-formed. The exposition will proceed by a discussion of some legal mini-systems, solving a particular normative problem in different ways.

Before introducing the example in Section 3, some suggestions regarding the framework are needed.

2. A semi-formal framework

2.1. IMPLICATIVE RELATION AND ORDERED PAIRS

In predicate logic, a norm-sentence is (usually) expressed as a universal sentence. For example:

(n_1) For any x, y and z : if x has borrowed y's car z, then x has an obligation to return to y the car Z.

Within predicate logic, we can formalize (n_1) as follows, where "Obligation to" (or "Obligatory") is a deontic operator resulting in a new predicate when it is applied to a given predicate.

(n_2) $\forall x, y, z$: Borrowed$(x, y, z) \longrightarrow$ Obligation to Return(x, y, z).

Thus, a typical norm-sentence is a universal implication. Syntactically it consists of three parts: the sequence of universal quantifiers, the antecedent formula and the consequent formula. Note that norm (n_2) correlates open sentences: Borrowed from(x, y, z) is correlated to Obligation to Return(x, y, z).

A norm like (n_2) can be represented as a relational statement correlating a ground to a consequence:

Borrowed R Obligation to Return.

Generally, pRq represents the norm

$$(n_3)\, \forall x_1. \ldots, x_v : p(x_1, \ldots, x_v) \rightarrow q(x_1, \ldots, x_v)$$

given that p and q are v-ary predicates. It is important here that the free variables in $p(x_1, ..., x_v)$ are the same and in the same order as the free variables in $q(x_1, ..., x_v)$. R is a binary relation, and pRq is a relational statement equivalent to $\langle p, q \rangle \in R$. Thus, a norm can be represented as pRq or $\langle p, q \rangle \in R$. Note that the implicative relation R can be such that only some of the elements are norms. pRq as a representation of (n_3) does not generally presuppose that q is a normative (or deontic) predicate, so pRq can be used as a representation of any sentence which has the same form as (n_3).

In the discussion above of the representation of norms, Borrowed and Obligation to Return, as well as p and q, appear as predicates. But the term predicate is often used for syntactical entities, and, therefore, interpreting

pRq, p and q are rather to be conceived of as *conditions*. A norm is represented as a sentence pRq (or $\langle p, q \rangle \in R$) relating, or "correlating", a ground to a consequence; thus, grounds and consequences in a norm are represented as conditions. Grounds are descriptive and consequences are normative conditions. If, in a context, it is presupposed that pRq where p is descriptive and q is normative, the ordered pair $\langle p, q \rangle$ is referred to as a norm.[2]

2.2. CONDITIONS

A normative system is a structure on a set of norms. To describe the structure some preliminary notions are needed. As is easy to see, conjunctions, disjunctions and negations of conditions can be formed by the operations $\wedge, \vee, '$, namely in the following way (where $x_1, ..., x_v$ are place-holders, not individual constants).

$(p \wedge q)(x_1, \ldots, x_v)$ if and only if $p(x_1, \ldots, x_v)$ and $q(x_1, \ldots, x_v)$;
$(p \vee q)(x_1, \ldots, x_v)$ if and only if $p(x_1, \ldots, x_v)$ or $q(x_1, \ldots, x_v)$;
$(p')(x_1, \ldots, x_v)$ if and only if not $p(x_1 \ldots, x_v)$.

$\perp$ (Falsum) is the empty condition, not fulfilled by any v-tuple, and $\top$ (Verum) is the universal condition, fulfilled by all v-tuples.

As is well-known, the truth-functional connectives can be used as operations in Boolean algebras. It is therefore possible to construct Boolean algebras of conditions.

2.3. ALGEBRAS, LINKS, AND NARROWNESS

The role of the set of norms is to join two Boolean algebras (see Figure 1):

- a Boolean algebra of grounds generated by conditions $\{p_1, ..., p_k\}$,
- a Boolean algebra of consequences generated by normative conditions $\{q_1, ..., q_m\}$.

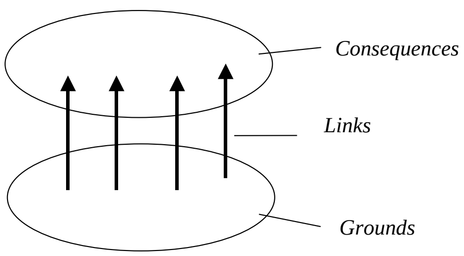

Figure 1.

In a normative system, an implicative relation R holds within each of the algebra G of grounds and the algebra C of consequences, as well as between certain elements of G and C. Let $\leq_1$ be the Boolean implicative relation within G and $\leq_2$ the Boolean implicative relation within C, and let L be the links from G to C. Then the implicative relation R of the system is such that each of $\leq_1, \leq_2$ and L is a subset of R. For example, let a_1, b_1, c_1 belong to the algebra of grounds and a_2, b_2, c_2 to the algebra of consequences. Then, for instance $a_1 \wedge b_1 \leq_1 a_1$, $a_2 \leq_2 a_2 \vee b_2$; and it may be the case that $a_1 L a_2$. In this case, each of the pairs $\langle a_1 \wedge b_1, a_1\rangle$, $\langle a_2, a_2 \vee b_2\rangle$, $\langle a_1, a_2\rangle$ is an element of the implicative relation R of the system. Among these, only the last one is a norm.

Of two norms $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$ one can be "narrower" than the other. If $\langle a_1, a_2\rangle$ is at least as narrow as $\langle b_1, b_2\rangle$, we can say alternatively that $\langle a_1, a_2\rangle$ "lies between" b_1 and b_2. Figure 2 (where the arrows denote the implicative relation) illustrates that the norm $\langle a_1, a_2\rangle$ lies between b_1 and b_2. Using another expression, as well suggested by the picture, we can say that $\langle b_1, b_2\rangle$ *encompasses* $\langle a_1, a_2\rangle$. (Thus "encompasses" is the converse of "at least as narrow as".)

Figure 2 illustrates that the norm $\langle a_1, a_2\rangle$ is "narrower" than the norm $\langle b_1, b_2\rangle$ which encompasses it. We define the relation " at least as narrow as" , expressed by $\trianglelefteq$, in the following way:

$$\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle \text{ if and only if } b_1 R a_1 \text{ and } a_2 R b_2.$$

It is easy to see that $\trianglelefteq$ is a quasi-ordering, i.e., transitive and reflexive.[3]

In a normative system, the set of norms that are *maximally* narrow play a crucial role. (A norm $\langle a_1, a_2\rangle$ is maximally narrow if there is no norm in the system that is strictly encompassed by $\langle a_1, a_2\rangle$, i.e., if $\langle a_1, a_2\rangle$ is a minimal element with respect to "at least as narrow as".) Given certain requirements for a well-formed normative system, all the other norms of the system are determined by its maximally narrow norms and, therefore, any change of such a system implies a change of some maximally narrow norm(s).[4] In a

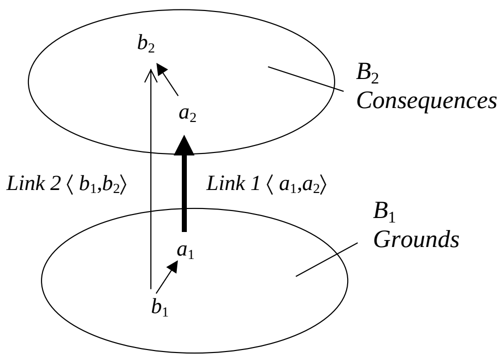

Figure 2.

well-formed normative system, the maximally narrow norms will be called the *connecting norms* of the system.

2.4. WELL-FORMED NORMATIVE SYSTEMS: A FIRST FORMULATION

Suppose that we have a set of norms that is a subset of $\{\langle p_i, q_j \rangle : 1 \le i \le k \ \& \ 1 \le j \le m\}$. If the system is well-formed, three Boolean algebras can be formed, viz.

the Boolean algebra $\mathcal{B}_1$ of grounds generated by $\{p_1, ..., p_k\}$,
the Boolean algebra $\mathcal{B}_2$ of consequences generated by $\{q_1, ..., q_m\}$,
the Boolean algebra $\mathcal{B}_0$ generated by $\{p_1, ..., p_k, q_1, ..., q_m\}$.

$\mathcal{B}_1$ and $\mathcal{B}_2$ are subalgebras of $\mathcal{B}_0$. Since it is presupposed that grounds are descriptive while consequences are normative, $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint, i.e. they contain no common element.[5] As shown in Figure 3, the norms are links (within the Boolean algebra $\mathcal{B}_0$) from the Boolean algebra $\mathcal{B}_1$ of grounds to the Boolean algebra $\mathcal{B}_2$ of consequences.

From a formal point of view, the role of the set of norms is thus to join two Boolean subalgebras of $\mathcal{B}_0$. A first proposal is that a logically well-formed normative system is a Boolean algebra of conditions and the norms are links from the subalgebra of grounds to the subalgebra of consequences.[6]

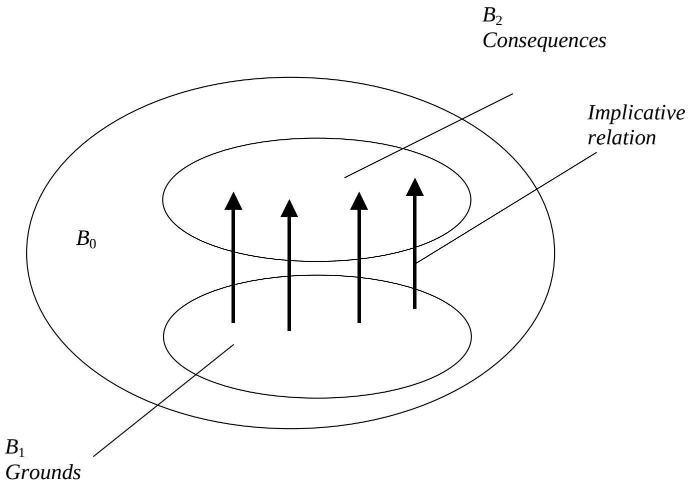

Figure 3.

3. A normative mini-system

3.1. THE PROBLEM OF ACQUISITION OF MOVABLE PROPERTY BY EXTINCTION OF PREVIOUS RIGHTS

In this Section, a normative problem concerning acquisition of movable property by extinction of previous rights will be discussed.[7] This will be done by devising different normative mini-systems resulting from promulgation and derogation of norms.

The exposition of the legal mini-systems will be complex since it aims at representing grounds and consequences in an actual legal system. The complexity, however, is the price to be paid for making the examples fairly realistic.

The point of the discussion is twofold:

- to show the crucial role of connecting norms in a well-formed system;
- to introduce a "lattice feature" that is a necessary condition for a normative system's being well-formed.

The "lattice feature" is given in terms of how connecting norms are ordered by a relation "lower than" that will be defined below.

The representation of different mini-systems will make use of the semi-formal framework (with conditions and implication from grounds to consequences), introduced in the preceding section. When the mini-systems are discussed, the aim of simplicity of presentation motivates disregarding a couple of special problems. Therefore, the framework used in the present main section and the next, is a simplified version of the framework of "Boolean quasi-orderings", developed by the authors in a number of papers.[8] In Section 5, two complications will be introduced. It will be shown how these complications can be handled by some changes of the simplified framework.

For normative mini-systems of the kind in view here a distinction will be made between a background situation K which is held constant throughout, and different legal implicative relations which can vary between different legal mini-systems, solving the problem in different ways. Five kinds of individuals are involved throughout:

x: a transferor;
y: a piece of movable property;
z: a transferee;
w: a (previous) owner;
t: a time.

Let situation K be the set of all quintuples $\langle x, y, z, w, t \rangle$ such that the following holds:

1. at $t-1$, x (transferor) makes a contract with z (transferee) that z buys y from x;
2. y is movable property ("chattel");
3. at $t-1$, x (transferor) has y in possession;
4. at $t-1$, w is the owner of y, a fact which is known by x (transferor);
5. at t, the possession of y is with x or with z (but not with both of x and z).

The normative problem for situation K is:

– Does x (the transferor) have an obligation at t to deliver y to w?
– Does z (the transferee) have an obligation at t to deliver y to w?

In this first mini-system (others are to follow), we suppose that the answer depends on:

1. Is z (the transferee) at $t-1$ in good faith regarding x's ownership to y? (At $t-1$, is it the case that z neither suspects or has reason to suspect that x, the transferor, is not owner of y?)
2. Does z (transferee) at t have y in possession?

Suppose that in the normative system presently under review the solution of the problem is as follows:

– if z has not possession of y at t (i.e., if possession is still with x), then, regardless of z's good or bad faith, x has an obligation at t to deliver y to w;
– if z has possession of y at t, but is not in good faith at $t-1$, then z has an obligation at t to deliver y to w;
– if z has possession of y at t and is in good faith at $t-1$, then neither x nor z has an obligation to deliver y to w.

Let us express grounds and consequences as conditions, namely as follows:

Conditions F ("faith") and P ("possession") of *legal grounds* on situation K are defined by

1. $\mathrm{F}(x, y, z, w, t)$ if and only if at $t-1$, z is in good faith regarding x's ownership to y;
2. $\mathrm{P}(x, y, z, w, t)$ if and only if at t, z has y in possession.

Conditions O1 and O2 of *legal consequences* on situation K are defined by

1. $\mathrm{O1}(x, y, z, w, t)$ if and only if at t, x has a duty to deliver y to w;
2. $\mathrm{O2}(x, y, z, w, t)$ if and only if at t, z has a duty to deliver y to w.[9]

If norms are conceived of as ordered pairs, the legislator's three pronouncements expressed above can be represented by saying that the following ordered pairs belong to an implicative relation that we denote by ρ, established by the legislator in a legal mini-system $\mathcal{J}$ for situation K. Thus,

$$\rho = \{\langle \mathrm{P}', \mathrm{O1}\rangle, \langle \mathrm{F}' \wedge \mathrm{P}, \mathrm{O2}\rangle, \langle \mathrm{F} \wedge \mathrm{P}, \mathrm{O1}' \wedge \mathrm{O2}'\rangle\}.$$

(Recall that $'$ denotes negation.) Whether the relation ρ containing these three pairs is part of a logically well-formed normative system depends on logical regimentation.

3.2. THE LOGICALLY WELL-FORMED SYSTEM

The system $\mathcal{J}$, incorporating ρ, can be extended into a *logically well-formed* normative system $\mathcal{N}[R]$ in the following way. Let

1. $\mathcal{G} = \langle G, \wedge, ' \rangle$ be a Boolean algebra of grounds, where conditions F, P are among the elements of $\mathcal{G}$.
2. $\mathcal{C} = \langle C, \wedge, ' \rangle$ be a Boolean algebra of consequences, generated by conditions O1, O2.[10]
3. $\mathcal{N} = \langle N, \wedge, ' \rangle$ be a Boolean algebra with a domain N of conditions such that $\mathcal{G}$ and $\mathcal{C}$ are subalgebras of $\mathcal{N}$.
4. Let $\top$ be the unit element and $\bot$ the zero element of $\mathcal{N}$.[11]
5. R be a reflexive and transitive implicative relation on N satisfying some requirements of classical implication, namely:

   (i) aRb and aRc implies $aR(b \wedge c)$.

   (ii) aRb implies $b'Ra'$.[12]

   (iii) $(a \wedge b)Ra$.

   (iv) not $\top R \bot$.
6. $\mathcal{N}[R]$ be $\langle N, \wedge, ', R\rangle$, i.e., $\mathcal{N}[R]$ is the relational system obtained by extending the Boolean algebra $\mathcal{N}$ with relation R.
7. $\rho \subseteq R$.

(The relation Q of similarity with respect to R is defined by aQb iff aRb and bRa.)

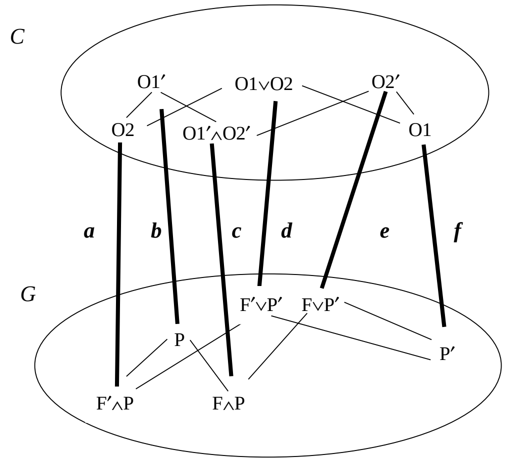

Figure 4.

The (maximally narrow) norms connecting $\mathcal{G}$ to $\mathcal{C}$ in the logically well-formed structure $\mathcal{N}[R]$ of our example are shown in the Figure 4.[13]

Spelling out some of the connecting norms, **a** says that transferee's possession at t (i.e., P) conjoined with transferee's lack of good faith at $t-1$ (i.e., F′) implies transferee's duty to deliver back at t (i.e., O2), **f** says that transferee's lack of possession at t (i.e., P′) implies transferor's duty to deliver back (i.e., O1). And so forth for the other connecting norms.

As the example is constructed, the norms $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ are the only connecting norms in $\mathcal{N}[R]$ from $\mathcal{G}$ to $\mathcal{C}$. Of these connecting norms, **a**, **c** and **f** are those mentioned in Section 3.1 as established by the legislator.

As stated in the introduction, a central issue in the present paper is the role of connecting norms in a normative system. This role can be exhibited at different levels of generality and complexity. In the subsequent development of the example (introduced in Section 3.1), two simplifying assumptions are made:

1. In the example, the implicative relation R is assumed to be not only reflexive and transitive but anti-symmetric as well; in other words it is assumed that $\langle N, R\rangle$ is a partial ordering rather than only a quasi-ordering.[14]
2. The example is made in such a way that no "organic unities" are assumed.

What these assumptions amount to will be made clear at a later stage. Stating the assumptions now amounts to a caveat that our general theory for representation of normative systems uses a framework that is more general and complex than what appears from the example (see Section 5).

3.3. THE LATTICE FEATURE OF CONNECTIONS IN THE EXAMPLE

As a first step we introduce the operations of conjunction ($\bar{\wedge}$) and disjunction ($\underline{\vee}$) of two norms $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$:

$$\langle a_1, a_2\rangle \bar{\wedge} \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, a_2 \wedge b_2\rangle,$$
$$\langle a_1, a_2\rangle \underline{\vee} \langle b_1, b_2\rangle = \langle a_1 \vee b_1, a_2 \vee b_2\rangle.$$

In the example, it can be verified, for example that $\mathbf{a} = \mathbf{b}\bar{\wedge}\mathbf{d}$, that $\mathbf{e} = \mathbf{c}\underline{\vee}\mathbf{f}$, and that $\mathbf{a}\bar{\wedge}\mathbf{b} = \mathbf{a}$, $\mathbf{f}\bar{\wedge}\mathbf{e} = \mathbf{f}$, etc. Thus, for instance, $\mathbf{a} = \mathbf{b}\bar{\wedge}\mathbf{d}$, since

$$\mathbf{b}\bar{\wedge}\mathbf{d} = \langle \mathrm{P} \wedge (\mathrm{F}' \vee \mathrm{P}'), \mathrm{O1}' \wedge (\mathrm{O1} \vee \mathrm{O2})\rangle = \langle \mathrm{F}' \wedge \mathrm{P}, \mathrm{O2}\rangle = \mathbf{a}.$$

We recall from Section 2 that **a**, **c**, and **f** are the connecting norms established by the legislator. It is of special interest to note that for the remaining connecting norms **b**, **d** and **e** it holds that

$$\mathbf{b} = \mathbf{a}\underline{\vee}\mathbf{c},$$
$$\mathbf{d} = \mathbf{a}\underline{\vee}\mathbf{f},$$
$$\mathbf{e} = \mathbf{c}\underline{\vee}\mathbf{f}.$$

A second step is the introduction of a relation "at least as low as" between norms. This relation is denoted $\lessapprox$, and is defined by,

(D) $\langle a_1, a_2\rangle \lessapprox \langle b_1, b_2\rangle$ iff $a_1 R b_1$ and $a_2 R b_2$.

This definition is closely related to the assumption,

(S) $\langle a_1, a_2\rangle \lessapprox \langle b_1, b_2\rangle$ iff $\langle a_1, a_2\rangle \bar{\wedge} \langle b_1, b_2\rangle = \langle a_1, a_2\rangle$.

It holds generally that (S) implies (D). Moreover, if R is a partial ordering, then (S) and (D) are equivalent. In the development of the example in this and the subsequent section, since R is assumed to be a partial ordering, for simplicity, we will use (S).

Thus, in our example, since, for instance, $\mathbf{a}\bar{\wedge}\mathbf{b} = \mathbf{a}$ and $\mathbf{f}\bar{\wedge}\mathbf{e} = \mathbf{f}$, it follows that $\mathbf{a}\lessapprox\mathbf{b}$ and that $\mathbf{f}\lessapprox\mathbf{e}$.

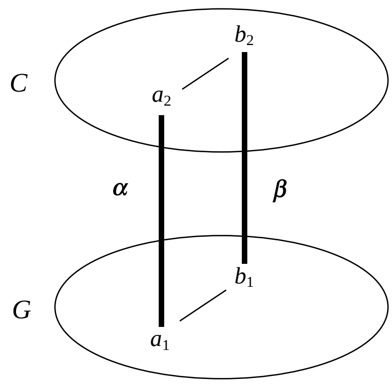

Figure 5.

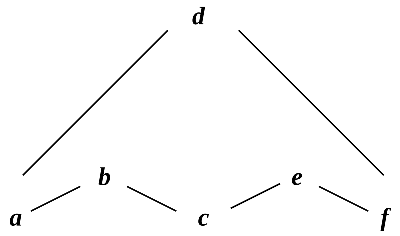

Figure 6.

In a picture exhibiting the norms of $\mathcal{N}[R]$, the assumption that a norm $\alpha = \langle a_1, a_2 \rangle$ is lower than a norm $\beta = \langle b_1, b_2 \rangle$ appears as a parallelogram, see Figure 5.

The partial ordering $\lessapprox$ of maximally narrow norms **a**–**f** in $\mathcal{N}[R]$ can be observed from Figure 4.[15] In a more perspicuous version, the ordering $\lessapprox$ obtained is as shown in Figure 6.

The set ordered is the set of connecting norms in the example. We now introduce a "lattice test" for this set. The point is that if the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ is not ordered in a certain way as a lattice, then it is not the set of connecting norms of a well-formed normative system.

For convenience, let us denote the set of connecting norms of the system $\mathcal{N}[R]$ by $Conn\mathcal{N}[R]$ (*"Conn"* for "connections"). Thus, in our example we have assumed that $Conn\mathcal{N}[R] = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$. The ordering of $Conn\mathcal{N}[R]$ is a *lattice* iff $\alpha \overline{\wedge} \beta$ and $\alpha \veebar \beta$ exist in $Conn\mathcal{N}[R]$ for every pair $\{\alpha, \beta\}$ such that $\alpha, \beta \in Conn\mathcal{N}[R]$.

We see immediately that, as it is represented above, $Conn\mathcal{N}[R]$ is not a lattice. $\alpha \overline{\wedge} \beta$ does not exist in $Conn\mathcal{N}[R]$ if $\{\alpha, \beta\}$ is any of the pairs $\{\mathbf{a}, \mathbf{c}\}$, $\{\mathbf{a}, \mathbf{f}\}$ or $\{\mathbf{c}, \mathbf{f}\}$, and $\alpha \veebar \beta$ does not exist if $\{\alpha, \beta\}$ is any of the pairs $\{\mathbf{a}, \mathbf{e}\}$, $\{\mathbf{b}, \mathbf{d}\}$, $\{\mathbf{b}, \mathbf{e}\}$, $\{\mathbf{b}, \mathbf{f}\}$, $\{\mathbf{c}, \mathbf{d}\}$, $\{\mathbf{c}, \mathbf{f}\}$ or $\{\mathbf{e}, \mathbf{d}\}$. An innocuous remedy can be found, however, if we supplement $Conn\mathcal{N}[R]$ by the two dummy norms $\langle \bot, \bot \rangle$ and $\langle \top, \top \rangle$. Let $Conn\mathcal{N}[R]$ supplemented by $\{\langle \bot, \bot \rangle, \langle \top, \top \rangle\}$ be denoted by $\mathcal{C}$, i.e., let $\mathcal{C} = Conn\mathcal{N}[R] \cup \{\langle \bot, \bot \rangle, \langle \top, \top \rangle\}$. (On the constants $\bot$ and $\top$, see Section 3.2.) Furthermore, let $\overline{\wedge}, \veebar$ be extended to the set $\mathcal{C}$ and, similarly, for $\lessapprox$. Using these tools, the ordering $\lessapprox$ over $\mathcal{C}$ is shown in Figure 7. This ordering is a lattice. For example, $\mathbf{a} \overline{\wedge} \mathbf{c} = \langle \bot, \bot \rangle$ since

$$\bot = (\mathrm{F}' \wedge \mathrm{P}) \wedge (\mathrm{F} \wedge \mathrm{P}),$$

$$\bot = \mathrm{O2} \wedge (\mathrm{O1}' \wedge \mathrm{O2}'),$$

and therefore,

$$\langle \mathrm{F}' \wedge \mathrm{P}, \mathrm{O2} \rangle \overline{\wedge} \langle \mathrm{F} \wedge \mathrm{P}, \mathrm{O1}' \wedge \mathrm{O2}' \rangle = \langle \bot, \bot \rangle.$$

Similarly, for example, $\mathbf{b} \veebar \mathbf{f} = \langle \top, \top \rangle$, since

$$\mathbf{b} \veebar \mathbf{f} = \{\langle \mathrm{P} \vee \mathrm{P}', \mathrm{O1}' \vee \mathrm{O1} \rangle\},$$

and

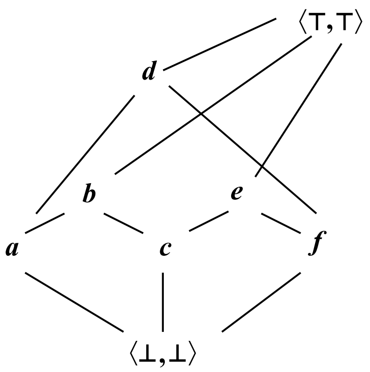

Figure 7.

$$\top = \mathrm{P} \vee \mathrm{P}',$$
$$\top = \mathrm{O1}' \vee \mathrm{O1}.$$

Thus the set $\mathcal{C} = \mathcal{N}[R] \cup \{\langle\bot,\bot\rangle, \langle\top,\top\rangle\}$ is ordered as a lattice by the relation $\lessapprox$. This is a special case of a more general principle:

If $\mathcal{N}[R]$ is any well-formed system (in the sense of Section 3.2), where R is a partial ordering and there are no organic unities, then the structure $\langle\mathcal{C}, \bar{\wedge}, \veebar\rangle$ is a lattice. Under the assumption that $\alpha \lessapprox \beta$ if and only if $\alpha \bar{\wedge} \beta = \alpha$ (see above), saying that $\langle\mathcal{C}, \bar{\wedge}, \veebar\rangle$ is a lattice is equivalent to saying that $\lessapprox$ orders $\mathcal{C}$ as a lattice.

4. Changes of the mini-system

4.1. SUBTRACTION OF NORMS

In this Section the example is developed with a view to subtraction of norms. Perhaps one or more of the norms in $\mathcal{N}[R]$ appears as unreasonable from a legal point of view? We first consider the case of merely subtracting norm **c**= ⟨F∧P,O1′∧O2′⟩.

A legal argument for the elimination of **c** might go as follows. The norm system $\mathcal{N}[R]$ may be thought to be unreasonable since it does not attach relevance to the possibility that the previous right owner w can be willing to *pay a ransom* to z for getting y back. We now take this consideration into account. Let condition R ("R" for ransom) on situation K be defined by:

(R) R(x, y, z, w, t) if and only if at t, w offers to pay ransom for y to z.[16]

If condition R and its Boolean combinations belong to the domain $\mathcal{G}$ of grounds in system $\mathcal{N}[R]$, the connecting norm **c** = ⟨F∧P,O1′∧O2′⟩ is encompassed in $\mathcal{N}[R]$ by the norm ⟨F∧P∧R,O2′⟩, the latter norm thus belonging to $\mathcal{N}[R]$ as well.[17] It might be held that $\langle$F∧P∧R,O2′$\rangle$, negating the obligation of z to give back y to w even in case a ransom is offered by w, is unreasonable. If $\langle$F∧P∧R,O2′$\rangle$ is unacceptable, the connecting norm $\langle$F∧P,O1′∧O2′$\rangle$, i.e., **c**, is unacceptable as well. Moreover, in a system that is logically well-formed (in the sense of Section 3.2), a norm $\langle a_1, a_2\rangle$ is eliminated from the system only if all connecting norms that lie between a_1 and a_2 are eliminated. Therefore, if $\langle$F∧P∧R,O2′$\rangle$ is to be eliminated, the encompassed connection $\langle$F∧P,O1′∧O2′$\rangle$, i.e., **c**, must be eliminated as well.

Now, as suggested above, suppose that the only stipulation made by the legislator is that **c** is derogated, so that, rather than R we have $R\backslash\{\mathbf{c}\}$. We are looking for a well-formed system not containing **c** but containing as much as possible of the rest of R. To this aim, let us see what happens with the set of connecting norms of R if **c** is eliminated (see Figure 8).

The set $M = \{\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ is not the set of connecting norms of a well-formed system. This is shown by the fact that if $\mathcal{C} = M \cup \{\langle\bot,\bot\rangle, \langle\top,\top\rangle\}$ then $\langle\mathcal{C}, \bar{\wedge}, \underline{\vee}\rangle$ is not a lattice.

That $\langle\mathcal{C}, \bar{\wedge}, \underline{\vee}\rangle$ is no lattice is verified by the fact that $\mathbf{b}\bar{\wedge}\mathbf{e} \notin \mathcal{C}$. The norm-conjunction $\mathbf{b}\bar{\wedge}\mathbf{e}$ would be

$$\mathbf{b}\bar{\wedge}\mathbf{e} = \langle \mathrm{P}, \mathrm{O1}'\rangle \bar{\wedge} \langle \mathrm{F} \vee \mathrm{P}', \mathrm{O2}'\rangle,$$

i.e.,

$$\mathbf{b}\bar{\wedge}\mathbf{e} = \langle \mathrm{P} \wedge (\mathrm{F} \vee \mathrm{P}'), \mathrm{O1}'\wedge\mathrm{O2}'\rangle,$$

where

$$\mathrm{P} \wedge (\mathrm{F} \vee \mathrm{P}') = \mathrm{F} \wedge \mathrm{P} \neq \bot \quad \text{and } \mathrm{O1}'\wedge\mathrm{O2}' \neq \bot.$$

Hence, $\mathbf{b}\bar{\wedge}\mathbf{e}$ is not an element of $\mathcal{C}$, since $\langle$F∧P,O1′∧O2′$\rangle$ is neither a member of M nor equal to $\langle\bot,\bot\rangle$ or $\langle\top,\top\rangle$. This implies that M is not the set of connecting norms of a well-formed system.

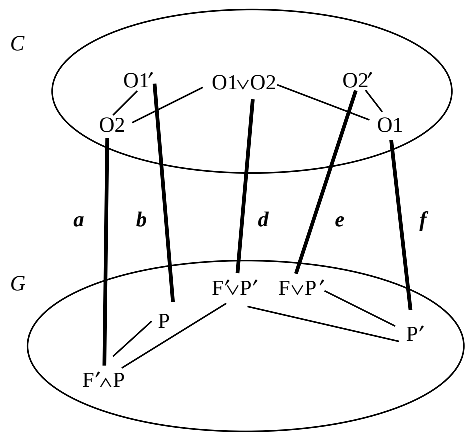

Figure 8.

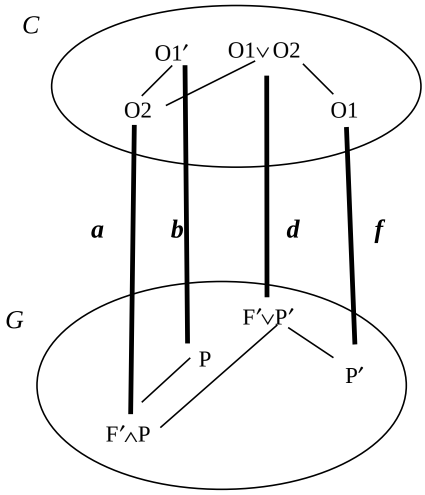

Figure 9.

In order to obtain the set of connecting norms of a well-formed system by subtraction, the legislator must eliminate either **b** or **e**, or both, as well. Since elimination of **b**, i.e., ⟨P,O1′⟩ is unreasonable from a legal point of view, the appropriate choice would be to eliminate **e**, i.e., ⟨F∨ P′,O2′⟩. If this is done we obtain the following maximally narrow norms (Figure 9).

The set $\{\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{f}\}$ is the set of connecting norms of a well-formed system. Let the new system of norms be called $\mathcal{N}[R^{(2)}]$, and let

$$\mathcal{C}^{(2)} = Conn\mathcal{N}[R^{(2)}] \cup \{\langle\bot, \bot\rangle, \langle\top, \top\rangle\}.$$

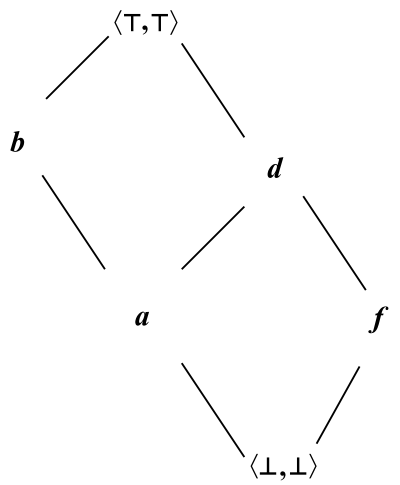

Figure 10.

An essential characteristic of $\mathcal{N}[R^{(2)}]$, required for its being well-formed, is that $\langle \mathcal{C}^{(2)}, \overline{\wedge}, \underline{\vee} \rangle$ is a lattice. The way $\lessapprox^{(2)}$orders $\mathcal{C}^{(2)}$ is shown in Figure 10. It can be verified that $\alpha \overline{\wedge} \beta$ and $\alpha \underline{\vee} \beta$ exists for every pair.

We note that, in a sense, $\mathcal{N}[R^{(2)}]$ is a "conservative" transformation of the original system $\mathcal{N}[R]$, since the lattice of *Conn*$\mathcal{N}[R^{(2)}]$ is a sub-lattice of the lattice for *Conn*$\mathcal{N}[R]$ earlier shown above in Section 3.2.[18]

4.2. ADDITION OF NORMS

The system $\mathcal{N}[R^{(2)}]$, in turn, may be considered normatively unsatisfactory due to its incompleteness. Plausibly, the legislator will consider addition of the following two norms, taking the "ransom" condition R into account:

1. $\langle$P∧R,O2$\rangle$: If transferee *z* has *y* in possession and *w* pays ransom for *y*, then *z* has the obligation to deliver *y* to *w*.
2. $\langle$F∧P∧R′,O2′$\rangle$: If transferee *z* has *y* in possession and fulfills the good faith condition, and *w* does not pay ransom, then *z* has no obligation to deliver *y* back to *w*.[19]

If the new system of norms is formulated in an appropriate way the resulting system will be logically well-formed, and we may denote it $\mathcal{N}[R^{(3)}]$. (Thus $\langle$P∧R,O2$\rangle$ and $\langle$F∧ P∧R′,O2′$\rangle$ belong to $R^{(3)}$.)

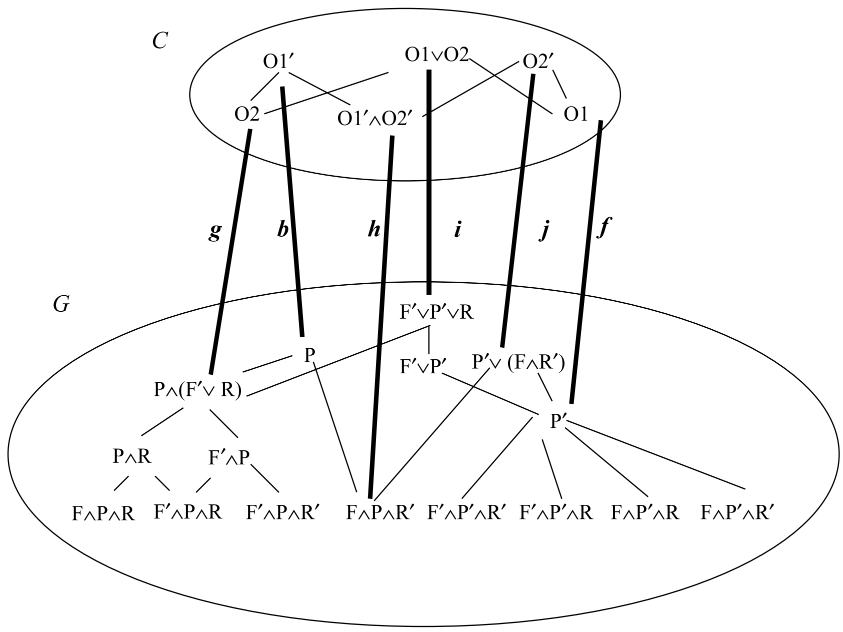

Figure 11.

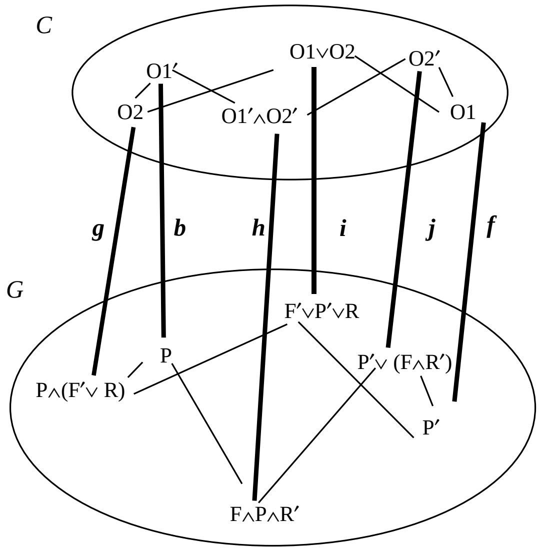

Figure 12.

In $\mathcal{N}[R^{(3)}]$, the earlier connecting norms **b** and **f** remain as connecting. The norms **a** and **d**, previously connecting, remain as norms in $\mathcal{N}[R^{(3)}]$ but are not connecting norms any more. New connecting norms **g,h,i,j**, however, are present. A rather comprehensive part of the system is shown in Figure 11.

A compressed picture of the connecting norms in $\mathcal{N}[R^{(3)}]$ is shown in Figure 12.

The structure $\langle \mathcal{C}^{(3)}, \overline{\wedge}, \underline{\vee} \rangle$ is a lattice. The partial ordering of

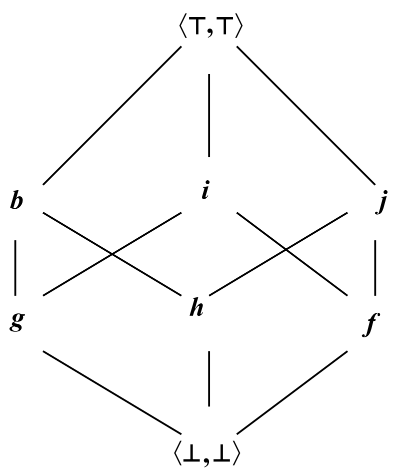

Figure 13.

$$\mathcal{C}^{(3)} = Conn\mathcal{N}[R^{(3)}] \cup \{\langle\bot,\bot\rangle,\langle\top,\top\rangle\}$$

by $\lessapprox$ is shown in Figure 13.

This a lattice, and it can be verified that it so ordered by $\lessapprox$. This shows that $\{\mathbf{b},\mathbf{f},\mathbf{g},\mathbf{h},\mathbf{i},\mathbf{j}\}$ fulfills an essential requirement for being the set of connections of a logically well-formed normative system.

4.3. A REMARK ON "DEFEASIBILITY"

In the transition from $\mathcal{N}[R]$ to $\mathcal{N}[R^{(3)}]$ via $\mathcal{N}[R^{(2)}]$ the connecting norm ⟨F∧P,O1′∧O2′⟩ in $\mathcal{N}[R]$ was subtracted from $\mathcal{N}[R]$ for being encompassed both by ⟨F∧P∧ R,O2′⟩ and by ⟨F∧P∧R′,O2′⟩; the argument was that the "ransom condition" R should be taken into account by different normative consequences for the cases where R and R′, respectively, is a conjunct of a ground. The final system $\mathcal{N}[R^{(3)}]$ takes R into account by having, among its norms ⟨F∧P∧R,O2⟩, ⟨F∧P∧R′,O2′⟩.[20] From a normative point of view, the transition from $\mathcal{N}[R]$ to $\mathcal{N}[R^{(3)}]$ is due to the consideration that the original norm ⟨F∧P,O1′∧O2′⟩ was too "narrow" in the sense defined earlier (see Section 2.3). In $\mathcal{N}[R^{(3)}]$ it has been replaced by norms that are "wider".

The transition to $\mathcal{N}[R^{(3)}]$ does not mean, however, that F∧P loses its relevance. Given R′, F∧P still has O2′ as consequence. Thus, though in $\mathcal{N}[R^{(3)}]$, F∧P is denied the strong normative import it had in $\mathcal{N}[R]$, it still leads to O2′ if R' is fulfilled.

Thus, the change from $\mathcal{N}[R]$ to $\mathcal{N}[R^{(3)}]$ as now described is different from a change to the effect that F∧P (or one of F and P) is made irrelevant for O2′. The transition described rather bears some similarity to those situations where a norm is called "defeasible".

In classical logic of implication, the rule permitting a thickening of the antecedent allows that a sentence $p \,\&\, q \rightarrow r$ is derived from $p \rightarrow r$. In accord with this, if a normative system $\mathcal{S}[R]$ is well-formed in the sense of Section 3.2, and $\mathcal{S}[R]$ contains a_1, b_1, and a_2, then a norm-sentence $a_1 \wedge b_1 R a_2$ can be derived from a norm-sentence $a_1 R a_2$. (This follows from rule (iii) for R, Section 3.2, combined with the transitivity of R.) Often, however, an implicative sentence $p \rightarrow r$ is said to be defeasible if a case can be made for $\neg(p \,\&\, q \rightarrow r)$. One way of dealing with this problem is to reject the rule permitting a thickening of the antecedent. The corresponding way of dealing with defeasibility within our framework is to deviate from the rules for a well-formed system introduced previously by rejecting the rule permitting derivation of $a_1 \wedge b_1 R a_2$ from $a_1 R a_2$. Another approach, in accord with classical logic of "implies", is to stick to the rules introduced for R, but to think of the problem in terms of two different, but interrelated, systems $\mathcal{S}[R]$ and $\mathcal{S}^*[R^*]$. It is assumed that $\mathcal{S}[R]$ contains a_1

and a_2 but does not contain b_1; also, it is assumed that R contains the norm $\langle a_1, a_2\rangle$. For $\mathcal{S}^*[R^*]$, on the other hand, it is assumed that it contains both a_1 and b_1 as well as a_2. Moreover, it is assumed that R^* contains the norm $\langle a_1 \wedge b_1', a_2\rangle$, but that R^* contains neither $\langle a_1 \wedge b_1, a_2\rangle$ nor $\langle a_1, a_2\rangle$. In this approach, the defeasibility of $a_1 R a_2$ means that, due to normative considerations, one system $\mathcal{S}[R]$, containing $\langle a_1, a_2\rangle$ is replaced by a "finer" (i.e., more discriminating), system $\mathcal{S}^*[R^*]$, not containing $\langle a_1, a_2\rangle$ but containing a related norm $\langle a_1 \wedge b_1', a_2\rangle$ instead. As will have emerged from previous sections, the second line of thought is more in accordance with the approach in the present paper.

5. Two reasons for an extended theory

Leaving the illustrative examples of the three normative mini-systems, we now turn to a more general perspective. This is done by the introduction of a couple of complications and by suggestions on how these complications are handled with in a general framework for representing normative systems.

Two complications, to be handled in a general theory for the representation of normative systems, will be discussed. The first is due to the possible occurrence of what will be called "organic wholes"; the second comes, inter alia, from the fact that sometimes it is appropriate to distinguish between two conditions a, b because they have different meaning, even though, in a sense, a and b are equivalent.

5.1. ORGANIC WHOLES AS COMPONENTS OF CONNECTING NORMS

Attention should be drawn to the possible occurrence in normative systems of a phenomenon analogous to what G.E. Moore in *Principia Ethica* called an "organic unity" or "organic whole".[21] Characteristic of an organic unity, according to Moore, is "that the value of such a whole bears no regular proportion to the sum of the values of its parts." (Moore 1971, p. 27.)[22] In the present context, the issue is not about value in Moore's sense but about what might be called the "normative force" of grounds within a normative system of grounds and consequences.

Of particular interest is the question whether an organic unity can be the ground, i.e., the first component, in a *connecting* norm. If $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$ are two norms from G to C the norm

$$\langle a_1, a_2\rangle \bar{\wedge} \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, a_2 \wedge b_2\rangle$$

is the result of applying the operation $\wedge$ to the grounds a_1, and b_1 as well as to the consequences a_2 and b_2. The following question now arises: If

$\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$ are connecting norms from G to C, is $\langle a_1, a_2\rangle \overline{\wedge} \langle b_1, b_2\rangle$ a connecting norm from G to C as well?

In our previous example and in other many specific cases the answer to the question is affirmative. For example, in the picture of $\mathcal{N}[R]$ (Section 3.2) we see that **b**,**e,** as well as **c**, obtained by applying $\wedge$ to grounds and to consequences of **b** and **e**, are connecting norms. In this case, we can say, figuratively, that the normative force of the conjunction of the grounds in **b,e** equals the "sum" of the normative force of the ground in **b** and the normative force of the ground in **c**. This kind of case can be illustrated in Figure 14.

Examples can however be constructed where the answer to the question posed is negative. There are well-formed normative systems where two norms $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$ are connecting norms from G to C, but where the consequence of $a_1 \wedge b_1$ is "stronger" than $a_2 \wedge b_2$. If this is the case, $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle$ is not a connecting norm from G to C. This situation can be depicted in Figure 15 (where c_2 "is stronger" than $a_2 \wedge b_2$).

In this situation, we might say, figuratively, that $a_1 \wedge b_1$ is an organic unity in the sense that its normative force is greater than the "sum" of the force of a_1 and the force of b_1.

An example might be taken from legislation regarding the right to vote in parliamentary elections, according to Swedish law.[23] Let $\langle c_1, c_2\rangle$, $\langle d_1, d_2\rangle$, $\langle e_1, e_2\rangle$, be connecting norms where c_1, d_1, e_1 are legal grounds as follows:

c_1 : x is at least 18 years old;
d_1 : x is a Swedish citizen;
e_1 : x is or has been domiciled in Sweden:

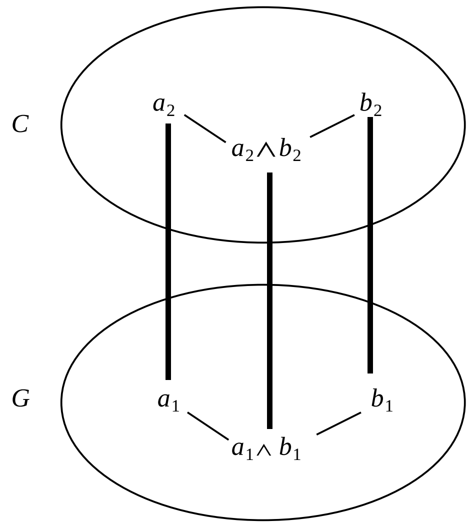

Figure 14.

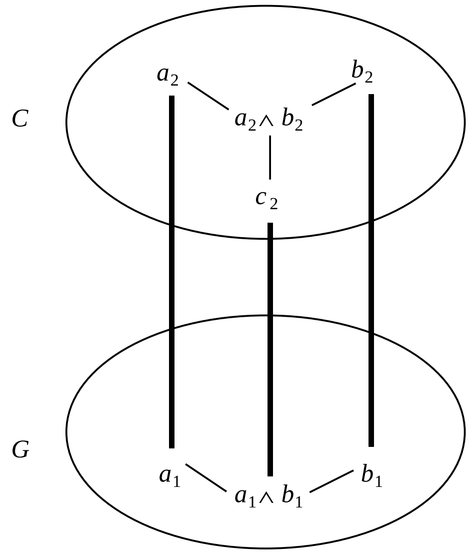

Figure 15.

In this case the normative force of the conjunction $c_1 \wedge d_1 \wedge e_1$ is greater than the "sum" of the normative forces of c_1, d_1, e_1. Let c_2, d_2, e_2 be the legal positions attached to grounds c_1, d_1, e_1 respectively, when considered separately. For example:

c_2: x is entitled to make legal contracts, x is liable to punishment if x commits a crime, etc.;
d_2: x is not being liable to expulsion from the country, x entitled to enter the country, etc.
e_2: x has the duty to pay taxes for periods when domiciled in the country, etc.

Thus, the conjunction of c_1, d_1, e_1 implies the conjunction of c_2, d_2, e_2, i.e., $c_1 \wedge d_1 \wedge e_1 R c_2 \wedge d_2 \wedge e_2$. However the conjunction of grounds implies an "extra" consequence. Let f_2 be the following position:

f_2 : x is entitled to vote in an election to the Swedish Parliament.

According to the law, the conjunction of c_1, d_1, e_1 implies f_2, i.e., $c_1 \wedge d_1 \wedge e_1 R f_2$. Therefore, if g_2 is the conjunction of the "separate" consequences *and* the "extra" one, i.e., if $g_2 = c_2 \wedge d_2 \wedge e_2 \wedge f_2$, it holds that $c_1 \wedge d_1 \wedge e_1 R g_2$. It is reasonable to assume that g_2 is a stronger consequence than $c_2 \wedge d_2 \wedge e_2$, in the sense that $g_2 R c_2 \wedge d_2 \wedge e_2$, but not conversely. And so it appears that the very conjunction of grounds c_1, d_1, e_1 is endowed with an extra normative force; in this sense, $c_1 \wedge d_1 \wedge e_1$ is an "organic unity", or has a feature of "synergy", in the system.

The example of organic unity now presented is fairly complicated. A simpler, though more banal, case is the following one, regarding two competitions T1 and T2:

c_1: winning in T1;
d_1: winning in T2;
c_2: claim to receive prize W1;
d_2: claim to receive prize W2;
e_2: claim to receive (bonus) prize W3.

We can easily conceive of a normative system such that

$$c_1 R c_2, d_1 R d_2, c_1 \wedge d_1 R c_2 \wedge d_2 \wedge e_2.$$

That is, winning both competitions entitles the winner to an extra prize apart from the two prizes attached to each of the two competitions.

The idea of "organic unity" now introduced and illustrated concerns the extra normative force of a specific conjunction of grounds. An analogous (or "dual") phenomenon concerns disjunctions of grounds. There are normative systems where two norms $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle$ are connecting norms from G to C, but where the ground for $a_2 \vee b_2$ is "weaker" than $a_1 \vee b_1$. If this is the case, $\langle a_1 \vee b_1, a_2 \vee b_2 \rangle$ is not a connecting norm from G to C. This situation can be depicted as shown in Figure 16, (where c_1 is "weaker" than $a_1 \vee b_1$).

In actual legislation, however, "organic wholes" are not too frequent. Our previous development of the example introduced in Section 3.1. was such that there were no organic unities to pay attention to. In a general theory for the representation of normative systems, however, the possible existence of "organic wholes" as grounds in connections should be taken into account.

In the algebraic theory developed by the authors in previous papers, the problem is taken into account, with respect to the respective cases of conjuctions and disjunctions of grounds. It is proved that, given reasonable requirements on an well-formed normative system, the following holds.

If $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle$ are connecting norms in the system, then

1. if $a_1 \wedge b_1 \neq \bot$, then there is a consequence c_2 in the system such that $\langle a_1 \wedge b_1, c_2 \rangle$ is a connecting norm in the system, and $c_2 R(a_2 \wedge b_2)$,
2. if $a_2 \vee b_2 \neq \top$, then there is a ground c_1 in the system such that $\langle c_1, a_2 \vee b_2 \rangle$ is a connecting norm in the system and $(a_1 \vee b_1) R c_1$.[24]

5.2. NON-ANTISYMMETRIC IMPLICATION

In the example of Sections 3 and 4, for the sake of simplicity, it has been presupposed that the relation "at least as low as" is a partial ordering of connecting norms. This assumption implies that the relation "at least as low

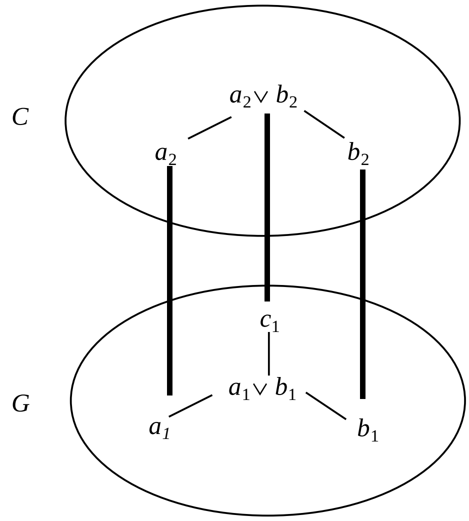

Figure 16.

as" is antisymmetric, in the sense that there are no two norms $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ that are equally low.

Whether two norms can be "equally low" depends on the nature of the implicative relation R of the Boolean quasi-ordering $\langle B, \wedge, ', R \rangle$. By definition, $\langle a_1, a_2 \rangle$ is equally low as $\langle b_1, b_2 \rangle$ if $a_1 Q b_1$ and $a_2 Q b_2$ (where aQb means that aRb and bRa). If R is a partial ordering (and, therefore, antisymmetric), aQb implies $a = b$. In this case there are no two different norms such that $\langle a_1, a_2 \rangle$ is equally low as $\langle b_1, b_2 \rangle$; hence "at least as low as" is a partial ordering. On the other hand, if R is a mere quasi-ordering, $\alpha Q \beta$ is compatible with $\alpha \neq \beta$, and there can exist different norms $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ that are equally low.

As appears from section 2, the relation Q is such that aQb represents

$$\forall x_1, \ldots, x_v : a(x_1, \ldots, x_v) \leftrightarrow b(x_1, \ldots, x_v).$$

In predicate logic, this sentence does not imply that for predicates a, b it holds that $a = b$. If predicates a, b have different meaning, $a \neq b$ may be the case, even though the sentence holds.

As a consequence of the argument now stated, if a, b are conditions, aQb should not be assumed to imply $a = b$, i.e., the relation R should not, in a general theory, be assumed to be anti-symmetric.

If, in a well-formed normative system $\mathcal{N}[R]$, the relation R is not anti-symmetric, then $Conn\mathcal{N}[R] \cup \{\langle \bot, \bot \rangle, \langle \top, \top \rangle\}$ is not a lattice (as in the example of Sections 3 and 4). Rather, in this case, it is a structure that can be called a *quasi-lattice*, i.e., a structure analogous to a lattice but based on a quasi-ordering.

6. Conclusion

In our previous papers, the logical theory is developed in a more strict way as a theory of Boolean quasi-orderings; some of the results are at a high level of generality and abstraction. One aim of this paper has been, by means of a legal application, to illustrate part of the framework and results of our joint work. A specific aim has been to apply the theory to problems of revision, indicating how the notions of connecting norms, and lattice, are useful tools for assessing the coherence of a system.

Revising a normative system means replacing the system by a different, though related, one. Therefore, the phenomenon of revision, like that of "defeasibility", is part of a larger field of legal phenomena where the interrelation between two or several normative systems is in view. For some problems within this field it may be fruitful to see a legal system as a class of normative systems, interrelated in specific ways. This perspective opens up a new range of questions. One example is the distinction between what amounts to a change of the legal system and what is a mere "amplification" ; another is whether different "sources of law" (legislation at different levels, judicial decisions at different levels etc.) should be described as a class of interrelated normative systems, all of which are components of the complex structure that is called "the legal system".

Acknowledgements

The authors are grateful for support from the Swedish Council for Research in the Humanities and Social Sciences (Project F1113/97), and from Harald and Louise Ekmans Forskningsstiftelse.

Notes

1 See Odelstad and Lindahl (2002), Lindahl and Odelstad (2004) with further references to other papers. It can be added that our joint papers (the earlier ones as well as the present paper), are the result of wholly joint work; the order of appearance of our author names has no significance.

2 Cf. Odelstad and Lindahl (2002, pp. 32 ff) and Lindahl and Odelstad (2004, Section 3.2).

3 Note that if $\langle b_1, b_2 \rangle$ encompasses $\langle a_1, a_2 \rangle$, then, from $a_1 R a_2$ it follows $b_1 R b_2$.

4 See Odelstad and Lindahl (2002, p. 36).

5 In our general theory, we do not make this presupposition, neither do we presuppose that $\mathcal{B}_1$ is descriptive and $\mathcal{B}_2$ normative. See for example, Lindahl and Odelstad (2004, Section 5).

6 In our formal theory of Boolean quasi-orderings, the links are called "joinings".

7 An example concerning recovery of property from a transferee when the transferor was not owner is used for illustration as well in Alchourrón and Bulygin (1971, pp. 9 ff). Their example relates to Argentinian legislation concerning real estate. The illustration used here relates to Swedish legislation concerning movable property.

[8] See the list of references in Lindahl and Odelstad (2004).
[9] In our example, conditions O1 and O2 are defined only in a semi-formal way. In Lindahl and Odelstad (2004) it is shown how, within our framework for well-formed systems, various consequences can be defined in terms of so-called normative positions. These positions are constructed by different combinations of a deontic operator "Shall" (or "May") and an action operator "sees to it that".
[10] For reasons that will appear later on (relating to development of the example by subtraction and addition of norms), we choose to make the example such that the domain C of consequences consists only of O1, O2, and their Boolean combinations, while the domain G of grounds may contain other conditions apart from F, P, and their combinations.
[11] In the example, we can conceive of $\top$ as the trivial condition fulfilled by all quintuples and of $\bot$ as the absurd condition, fulfilled by no quintuple.
[12] If aRb expresses a norm in the sense adopted here, where a is descriptive and b is normative, then its contraposition $b'Ra'$ is no norm. Counterexamples purporting to construct new norms by contraposition of norms, are not relevant in the present context.
[13] Since $\mathrm{O1}Q\mathrm{O1}\wedge\mathrm{O2}'$ and $\mathrm{O2}Q\mathrm{O2}\wedge\mathrm{O1}'$, for convenience, in what follows, we write O1 instead of $\mathrm{O1}\wedge\mathrm{O2}'$, and we write O2 instead of $\mathrm{O2}\wedge\mathrm{O1}'$. These abbreviations seem harmless.
[14] That R is assumed to be anti-symmetric means that if aRb and bRa, then $a = b$.
[15] We note that if the relation R partially orders the set N of conditions, then the relation "at least as low as" partially orders the set of ordered pairs in R.
[16] Observe the different use of R and R, where R (in italics) is the implicative relation of the system while R is the "ransom condition".
[17] By the rules for system $\mathcal{N}[R]$, from $\langle \mathrm{F}\wedge\mathrm{P},\mathrm{O1}'\wedge\mathrm{O2}'\rangle \in R$ it follows $\langle \mathrm{F}\wedge\mathrm{P}\wedge\mathrm{R},\mathrm{O1}'\wedge\mathrm{O2}'\rangle \in R$, and from this, in turn, $\langle \mathrm{F}\wedge\mathrm{P}\wedge\mathrm{R},\mathrm{O2}'\rangle \in R$.
[18] That the lattice for $Conn\mathcal{N}[R^{(2)}]$ is a sublattice of the lattice for $Conn\mathcal{N}[R]$ means that $Conn\mathcal{N}[R^{(2)}]$ is a non-empty subset of $Conn\mathcal{N}[R]$ and that if $\langle a,b\rangle,\langle c,d\rangle$ are members of $Conn\mathcal{N}[R^{(2)}]$, then $\langle a,b\rangle\barwedge\langle c,d\rangle$ and $\langle a,b\rangle\veebar\langle c,d\rangle$ are members of $Conn\mathcal{N}[R^{(2)}]$.
[19] Basically, this was the system adopted in Swedish legislation, *Lag (1986:796) om godtrosförvärv av lösöre,* before 2003.
[20] We observe that in $\mathcal{N}[R^{(3)}]$, $\langle \mathrm{F}\wedge\mathrm{P}\wedge\mathrm{R},\mathrm{O2}\rangle$ encompasses **g**, while $\langle \mathrm{F}\wedge\mathrm{P}\wedge\mathrm{R}',\mathrm{O2}'\rangle$ encompasses **h**.
[21] *Principia Ethica* was first published in 1903. The edition used here is Moore (1971).
[22] Using another terminology, the phenomenon can be called "synergy".
[23] See *Vallag* (1997, p. 157, Art. 2).
[24] See Theorem 18 in Lindahl and Odelstad (2004).

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