Hindawi
Complexity
Volume 2018, Article ID 8701219, 8 pages
https://doi.org/10.1155/2018/8701219

Research Article

Input-to-State Stability of Nonlinear Switched Systems via Lyapunov Method Involving Indefinite Derivative

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Received 30 July 2017; Revised 24 November 2017; Accepted 28 December 2017; Published 24 January 2018

Academic Editor: Peter Galambos

This paper studies the input-to-state stability (ISS) of nonlinear switched systems. By using Lyapunov method involving indefinite derivative and average dwell-time (ADT) method, some sufficient conditions for ISS are obtained. In our approach, the time-derivative of the Lyapunov function is not necessarily negative definite and that allows wider applications than existing results in the literature. Examples are provided to illustrate the applications and advantages of our general results and the proposed approach.

1. Introduction

Switched systems are a special subclass of hybrid systems which consist of two components: a family of systems and a switching signal. The systems in the family are described by a collection of indexed differential or difference equations. The switching signal selects an active mode at every instant of time, that is, the system from the family that is currently being followed. As a special class of hybrid systems, switched systems arise in a variety of applications, such as biological systems [1], automobiles and locomotives with different gears [2], DC-DC converters [3], manufacturing processes [4], and shrimp harvesting mode [5]. Many interesting results for switched systems have been reported in the literature [6–9]. Qualitative behaviour of switched systems depends not only on the behaviour of individual subsystems in the family, but also on the switching signal. For instance, divergent trajectories can be generated by switching appropriately among stable subsystems, while a proper switching signal may ensure stability of a switched system even when all the subsystems are unstable. Due to such interesting features, stability of switched systems has attracted considerable research attention over the past few decades; see [10–15].

When investigating stability of a system, it is important to characterize the effects of external inputs. The concepts of input-to-state stability (ISS) introduced by Sontag et al. in [16, 17] have been proved useful in this regard. Roughly speaking, the ISS property means that no matter what the size of the initial state is, the state will eventually approach a neighborhood of the origin whose size is proportional to the magnitude of the input. Many interesting results on ISS properties of various systems such as discrete systems, switched systems, and hybrid systems have been reported; see [18–29]. For example, [19] presented converse Lyapunov theorems for input-to-state stability and integral input-to-state stability (iISS) of switched nonlinear systems; [22, 23] studied the ISS of nonlinear systems subject to delayed impulses; [29] dealt with the ISS of discrete-time nonlinear systems. However, one may observe that most of them, such as those in [18–31], require the derivative of Lyapunov functions to be negative definite in order to derive the desired ISS property. Recently, [32] proposed a new approach for ISS property of nonlinear systems. It presents a new comparison principle for estimating an upper bound on the state of the system in which the derivative of the Lyapunov function may be indefinite, rather than negative definite, which improves the previous work on this topic greatly. The authors of [33] developed the idea to delayed systems and established a class of continuously differentiable Lyapunov-Krasovskii functionals involving indefinite derivative, which generalizes the classic Lyapunov-Krasovskii functional method. However, the approach used there only applies for systems without switched structures. Moreover, to the best of our knowledge, there are few results on ISS of switched systems based on Lyapunov method involving indefinite derivative.

Motivated by the above discussions, in this paper, we shall study the ISS property for switched systems via Lyapunov method involving indefinite derivative. Some sufficient conditions based on ADT method are derived. It is worth mentioning that, although the method used in this paper is based on [32], the results in this paper are more general than [32], even for the case of systems without switched structures. The rest of this paper is organized as follows. In Section 2 the problem is formulated and some notations and definitions are given. In Section 3, we present some new characterizations of ISS based on Lyapunov method involving indefinite derivative. Examples are given in Section 4. Finally, the paper is concluded in Section 5.

2. Preliminaries

Notations. Let $\mathbb{Z}_+$ denote the set of positive integer numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of all nonnegative real numbers, and $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the n-dimensional and $m\times n$-dimensional real spaces equipped with the Euclidean norm $|\cdot|$, respectively. $a\wedge b$ and $a\vee b$ are the minimum and maximum of a and b, respectively. $P=\{1,2,\dots,m\}$, $m\in\mathbb{Z}_+$, is an index set, $C(J,S)=\{\varphi\colon J\to S$ is continuous$\}$, $\mathscr{F}=\{\varphi\colon [t_0,\infty)\to P$, is a piecewise constant function$\}$. The notations $\mathscr{A}^T$ and $\mathscr{A}^{-1}$ denote the transpose and the inverse of $\mathscr{A}$, respectively. I denotes the identity matrix with appropriate dimensions.

Consider the following switched system:

$$\dot{x}=f_{\sigma(t)}(t,x,u), \tag{1}$$

where $x\in\mathbb{R}^n$ is the system state, $u\in\mathbb{R}^m$ is a measurable locally bounded disturbance input, $\dot{x}$ denotes the right-hand derivative of x, and $\sigma\in\mathscr{F}$ denotes the switching function, which is assumed to be a piecewise constant function continuous from the right. When $\sigma(t)=i$, $1\le i\le m$, we say that the mode $\dot{x}=f_i$ is activated. A sequence of discrete times $\{t_n\}$, $n\in\mathbb{Z}_+$, called the switching times, determines when the switching occurs. Throughout this paper, we assume that it satisfies $0\le t_0<t_1<\cdots<t_k\to+\infty$ as $k\to+\infty$ (t_1 is the first switching time). In particular, we exclude the possibility of the $\{t_k\}$ having a finite accumulation point, often referred to as chattering. It indicates that a switching signal $(\{t_n\},\sigma)$ has at most finite switching times over a finite time interval. $f_{\sigma(t)}\in C(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R}^n)$ with local Lipschitz, and $f_{\sigma(t)}(t,0,0)\equiv 0$, $t\in\mathbb{R}_+$. In order to study the ISS, in the following we assume that the solution of system (1) with an initial condition $x(t_0)=x_0$ exists on $[t_0,+\infty)$ uniquely.

By the ideas proposed by Hespanha and Morse [34] for switched systems, we say that a switching signal $(\{t_n\},\sigma)$ has average dwell-time (ADT) τ if there exist numbers $N_0\ge 0$ and $\tau>0$ such that

$$N_\sigma(T,t)\le N_0+\frac{T-t}{\tau},\quad \forall T\ge t\ge t_0, \tag{2}$$

where N_0 is called the "chatter bound" and $N_\sigma(T,t)$ is the number of switches occurring in the interval $[t,T)$. We denote such kind of switching signals by set $\mathscr{F}_\tau$. Denote the switching times in the interval $(t,T]$ by $t_1,t_2,\dots,t_{N_\sigma(t,t_0)}$ and the index of the system that is active in the interval $[t_n,t_{n+1})$ by p_n.

A function $\alpha:[0,\infty)\to[0,\infty)$ is of class $\mathscr{K}$ if α is continuous and strictly increasing and $\alpha(0)=0$. If α is also unbounded, it is of class $\mathscr{K}_\infty$. A function $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is of class $\mathscr{KL}$ if $\beta(\cdot,t)$ is of class $\mathscr{K}$ for each fixed $t\ge 0$, and $\beta(r,t)$ decreases to 0 as $t\to\infty$ for each fixed $r\ge 0$.

Definition 1 (see [16]). Suppose that a switching signal $(\{t_n\},\sigma)$ is given. The system (1) is said to be ISS if there exist functions $\gamma\in\mathscr{K}_\infty$ and $\beta\in\mathscr{KL}$ such that for each $t_0\ge 0$, $x_0\in\mathbb{R}^n$ and for each input u, the solution satisfies

$$|x(t)|\le\beta(|x_0|,t-t_0)+\gamma\left(|u|_{[t_0,t]}\right), \tag{3}$$

for all $t\ge t_0$, where $|\cdot|_J$ denotes the supremum norm on the interval J. This definition depends on the choice of the switching signal; however, it is often of interest to characterize ISS over classes of switching signals. We say that system (1) is uniformly input-to-state stable (UISS) over the class $\mathscr{F}_\tau$ (of switching signal) if for any $(\{t_n\},\sigma)\in\mathscr{F}_\tau$ condition (3) is satisfied with the same γ and β for every $(\{t_n\},\sigma)\in\mathscr{F}_\tau$.

3. ISS Theorems

In this section, we shall present some ADT results for ISS of switched system (1) based on Lyapunov method involving indefinite derivative.

Theorem 2. *Assume that there exist functions $\alpha_1,\alpha_2\in\mathscr{K}_\infty$, $\rho\in\mathscr{K}$, a continuous function $\phi:\mathbb{R}_+\to\mathbb{R}$, continuous differentiable functions $V_p:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, and constants $\eta>0$, $\mu\ge 1$ such that, for all $t\in\mathbb{R}_+$, $x\in\mathbb{R}^n$, and all $p,q\in P$,*

$$\alpha_1(|x|)\le V_p(t,x)\le\alpha_2(|x|); \tag{4}$$

$$\frac{\partial V_p}{\partial x}f_p(t,x,u)+\frac{\partial V_p}{\partial t}\le\phi(t)V_p(t,x) \quad \text{whenever } V_p\ge\rho(|u|)\exp\left(\int_{t_0}^{t}\phi(s)\,ds\right); \tag{5}$$

$$V_p(t,x)\le\mu V_q(t,x); \tag{6}$$

$$\int_{t_0}^{t}(\phi(s)+\eta)\,ds\le 0. \tag{7}$$

Then the switched system (1) is UISS over the class $\mathscr{F}_\tau$, where ADT constant $\tau>0$ satisfies

$$\tau>\frac{\ln\mu}{\eta}. \tag{8}$$

Proof. Let $x(t)$ be a solution of system (1). Define $V_{\sigma(t)}(t)=V_{\sigma(t)}(t,x(t))$. If

$$V_p(t)\ge\rho(|u|)\exp\left(\int_{t_0}^{t}\phi(s)\,ds\right), \tag{9}$$

during some interval $[t', t'')$, in this case, suppose that there exists switching signal $\{t_n\}$ such that $t' < t_n < t_{n+1} < \cdots < t_{n+m} < t''$. For $t \in [t', t_n)$, it follows from (5) that

$$V_{p_{n-1}}(t) \le V_{p_{n-1}}(t') \exp\left(\int_{t'}^{t} \phi(s)\, ds\right). \tag{10}$$

For $t \in [t_n, t_{n+1})$, it follows from (5), (6), and (10) that

$$\begin{aligned} V_{p_n}(t) &\le V_{p_n}(t_n) \exp\left(\int_{t_n}^{t} \phi(s)\, ds\right) \\ &\le \mu V_{p_{n-1}}(t') \exp\left(\int_{t'}^{t} \phi(s)\, ds\right). \end{aligned} \tag{11}$$

Then it can be deduced that

$$V_{\sigma(t)}(t) \le \mu^{N_\sigma(t',t)} \exp\left(\int_{t'}^{t} \phi(s)\, ds\right) V_{\sigma(t')}(t'), \quad \forall t \in [t', t''). \tag{12}$$

Since $\mu \ge 1$, it follows from the ADT condition (2) that

$$\begin{aligned} V_{\sigma(t)}(t) &\le \mu^{N_0 + (t-t')/\tau} \exp\left(\int_{t'}^{t} \phi(s)\, ds\right) V_{\sigma(t')}(t') \\ &= \mu^{N_0} \exp\left(\int_{t'}^{t} \frac{\ln \mu}{\tau} + \phi(s)\, ds\right) V_{\sigma(t')}(t'). \end{aligned} \tag{13}$$

We denote the first time when

$$V_{\sigma(t)}(t) \le \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right) \tag{14}$$

by $\check{t}$; that is,

$$\check{t}_1 = \inf\left\{t \ge t_0 : V_{\sigma(t)}(t) \le \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right)\right\}. \tag{15}$$

If $\check{t}_1 = \infty$, then it holds that

$$V_{\sigma(t)}(t) \le \mu^{N_0} \exp\left(\int_{t_0}^{t} \frac{\ln \mu}{\tau} + \phi(s)\, ds\right) V_{\sigma(t_0)}(t_0), \quad \forall t \ge t_0. \tag{16}$$

It follows from (8) that there exists $\varepsilon > 0$ small enough such that

$$\frac{\ln \mu}{\tau} + \varepsilon \le \eta, \tag{17}$$

which together with (7) yields that

$$V_{\sigma(t)}(t) \le \mu^{N_0} \exp(-\varepsilon(t - t_0)) V_{\sigma(t_0)}(t_0), \quad \forall t \ge t_0. \tag{18}$$

It then follows from (4) that

$$|x(t)| \le \alpha_1^{-1}\left(\mu^{N_0} \exp(-\varepsilon(t - t_0))\, \alpha_2(x(t_0))\right). \tag{19}$$

Thus $x(t)$ is bounded by a $\mathscr{KL}$-class function, which implies that system (1) is ISS. Hence we only need to consider the case that $\check{t}_1 < \infty$. It follows from (18) that

$$V_{\sigma(t)}(t) \le \mu^{N_0} \exp(-\varepsilon(t - t_0)) V_{\sigma(t_0)}(t_0), \quad t \in [t_0, \check{t}_1). \tag{20}$$

For $t \ge \check{t}_1$, we denote the first time when

$$V_{\sigma(t)}(t) > \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right) \tag{21}$$

by $\hat{t}_1$; that is,

$$\hat{t}_1 = \inf\left\{t \ge \check{t}_1 : V_{\sigma(t)}(t) > \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right)\right\}. \tag{22}$$

If $\hat{t}_1 = \infty$, then it is obvious that system (1) is ISS. Assuming that $\hat{t}_1 < \infty$, then

$$\begin{aligned} V_{\sigma(t)}(t) &\le \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right) \\ &\le \mu^{N_0} \rho\left(|u|_{[t_0,t]}\right) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right), \quad t \in [\check{t}_1, \hat{t}_1). \end{aligned} \tag{23}$$

Then we further denote the second time when

$$V_{\sigma(t)}(t) \le \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right) \tag{24}$$

by $\check{t}_2$; that is,

$$\check{t}_2 = \inf\left\{t \ge \hat{t}_1 : V_{\sigma(t)}(t) \le \rho(|u|) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right)\right\}. \tag{25}$$

Due to the continuity of $V_{\sigma(t)}(t)$ and the monotonicity of $\rho(|u(t)|)$, when $\check{t}_2 < \infty$, it holds that

$$\begin{aligned} V_{\sigma(\hat{t}_1)}(\hat{t}_1) &= \rho(|u(\hat{t}_1)|) \exp\left(\int_{t_0}^{\hat{t}_1} \phi(s)\, ds\right), \\ V_{\sigma(t)}(t) &\le \mu^{N_0} \exp\left(\int_{\hat{t}_1}^{t} \left(\frac{\ln \mu}{\tau} + \phi(s)\right) ds\right) V_{\sigma(\hat{t}_1)}(\hat{t}_1) \\ &\le \mu^{N_0} \exp\left(\int_{\hat{t}_1}^{t} \left(\frac{\ln \mu}{\tau} + \phi(s)\right) ds\right) \rho(|u(\hat{t}_1)|) \\ &\quad \cdot \exp\left(\int_{t_0}^{\hat{t}_1} \phi(s)\, ds\right) \le \mu^{N_0} \exp\left(\int_{\hat{t}_1}^{t} \frac{\ln \mu}{\tau} ds\right) \\ &\quad \cdot \rho\left(|u|_{[\hat{t}_1,t]}\right) \exp\left(\int_{t_0}^{t} \phi(s)\, ds\right) \le \mu^{N_0} \end{aligned}$$

$$\cdot \exp\left(\int_{t_0}^{t}\left(\frac{\ln\mu}{\tau}+\phi(s)\right)ds\right)\rho\left(|u|_{[t_0,t]}\right)\le\mu^{N_0}$$
$$\cdot \exp\left(-\varepsilon\left(t-t_0\right)\right)\rho\left(|u|_{[t_0,t]}\right)\le\mu^{N_0}\rho\left(|u|_{[t_0,t]}\right),$$
$$\forall t\in\left[\hat{t}_1,\check{t}_2\right). \tag{26}$$

By this way, it can be deduced that, for every $t \ge \check{t}_1$, it holds that

$$V_{\sigma(t)}(t)\le\mu^{N_0}\lambda\rho\left(|u|_{[t_0,t]}\right), \tag{27}$$

where

$$\lambda=\exp\left(\int_{t_0}^{t}\phi(s)\,ds\right)\vee 1. \tag{28}$$

It follows from (20) and (27) that

$$\begin{aligned}V_{\sigma(t)}(t)&\le\mu^{N_0}\exp\left(-\varepsilon\left(t-t_0\right)\right)V_{\sigma(t_0)}\left(t_0\right)\\&\quad+\mu^{N_0}\lambda\rho\left(|u|_{[t_0,t]}\right),\end{aligned} \tag{29}$$

for all $t \ge t_0$, which together with (4) yields that

$$\begin{aligned}|x(t)|&\le\alpha_1^{-1}\left(2\mu^{N_0}\exp\left(-\varepsilon\left(t-t_0\right)\right)\alpha_2\left(\left|x\left(t_0\right)\right|\right)\right)\\&\quad+\alpha_1^{-1}\left(2\mu^{N_0}\lambda\rho\left(|u|_{[t_0,t]}\right)\right)\\&\coloneqq\beta\left(\left|x\left(t_0\right)\right|,t-t_0\right)+\gamma\left(|u|_{[t,t_0]}\right),\end{aligned} \tag{30}$$

for all $t \ge t_0$. This indicates that system (1) is UISS over the class $\mathcal{F}_\tau$. The proof is completed. □

In particular, if system (1) is given in the form of

$$\dot{x}=f(t,x,u), \tag{31}$$

which is a general case without switched structure, by Theorem 2, one may derive the following corollary.

Corollary 3. *Assume that there exist functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, $\rho\in\mathcal{K}$, a continuous function $\phi:\mathbb{R}_+\to\mathbb{R}$, a continuous differentiable function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, and a constant $\eta>0$ such that, for all $t\in\mathbb{R}_+$, $x\in\mathbb{R}^n$, (7) and the following conditions hold:*

$$\begin{gathered}\alpha_1(|x|)\le V(t,x)\le\alpha_2(|x|);\\ \frac{\partial V}{\partial x}f(t,x,u)+\frac{\partial V}{\partial t}\le\phi(t)V(t,x)\\ \text{whenever } V\ge\rho(|u|)\exp\left(\int_{t_0}^{t}\phi(s)\,ds\right).\end{gathered} \tag{32}$$

Then system (31) is ISS.

Remark 4. Recently, [32] has presented some sufficient conditions for ISS property of system (31) based on Lyapunov method involving indefinite derivative under the assumption that

$$\begin{gathered}\int_{t_0}^{\infty}\phi^{+}(s)\,ds<\infty,\\ \int_{t_0}^{t}\phi^{-}(s)\,ds\ge\epsilon\left(t-t_0\right),\end{gathered} \tag{33}$$

where $\phi^+(s)=\phi(s)\vee 0$, $\phi^-(s)=[-\phi(s)]\vee 0$, and $\epsilon>0$ is a constant, while our ISS result in Corollary 3 only requires that (7) holds, which has wider applications. For example, $\phi=\sin t-0.9$ and $t_0=0$, and it is easy to see that ϕ is a sign reversal function. In this case, one may choose $\eta=0.1$ such that

$$\int_{t_0}^{t}(\phi(s)+\eta)\,ds=\int_0^t(\sin s-0.8)\,ds<0,\quad\forall t>0, \tag{34}$$

which implies that (7) holds. However, it is easy to see that

$$\int_0^{\infty}\phi^{+}(s)\,ds=\infty. \tag{35}$$

Next we consider the time-varying linear switched system in the form of

$$\dot{x}=A_{\sigma(t)}(t)\,x(t)+B_{\sigma(t)}(t)\,u(t),\quad t\ge 0, \tag{36}$$

where $x(t)\in\mathbb{R}^n$ is the system state, $u\in\mathbb{R}^m$ is locally bounded input, and $A_{\sigma(t)}(t)\in\mathbb{R}^{n\times n}$ and $B_{\sigma(t)}(t)\in\mathbb{R}^{m\times n}$ are time-varying functions. To ensure the ISS property of (36), we present the following result.

Theorem 5. *Assume that there exist constants $\eta>0,\mu\ge 1$, $\omega_p>0$, and continuous functions $\phi:\mathbb{R}_+\to\mathbb{R}$ and $\overline{\phi}:\mathbb{R}_+\to\mathbb{R}$ such that, for all $t\in\mathbb{R}_+$ and all $p,q\in P$, $\omega_p\le\mu\omega_q$, (7), and the following hold:*

$$\begin{aligned}&A_p(t)+A_p^T(t)+B_p(t)B_p^T(t)\exp\left(-\int_0^t\overline{\phi}(s)\,ds\right)\\&\quad+\omega_p\exp\left(\int_0^t\overline{\phi}(s)-\phi(s)\,ds\right)\cdot I_{n\times n}\\&\le\phi(t)\cdot I_{n\times n}.\end{aligned} \tag{37}$$

Then system (36) is UISS over the class $\mathcal{F}_\tau$, where ADT constant $\tau>0$ satisfies (8).

Proof. Let $x(t)$ be a solution of system (36) and define $V_p(t)=V_p(t,x(t))$. Then the proof of Theorem 5 is similar to Theorem 2. We only need to notice that the following are chosen: $V_p(t)=\omega_p x(t)^T x(t)$ and $\rho(|u|)=u^T(t)u(t)$. It then follows from Theorem 2 that when $V_p(t)\ge\rho(|u|)\exp(\int_0^t\phi(s)ds)$, it holds that $\omega_p x^T(t)x(t)\ge u^T(t)u(t)\exp(\int_0^t\phi(s)ds)$, which, together with (37), leads to the following:

$$\begin{aligned}\dot{V}_p(t)&=2\omega_p x^T(t)\,\dot{x}(t)=2\omega_p x^T(t)\left(A_p(t)\,x(t)\right.\\&\quad\left.+B_p(t)\,u(t)\right)=2\omega_p x^T(t)\,A_p(t)\,x(t)\\&\quad+2\omega_p x^T(t)\,B_p(t)\,u(t)\le 2\omega_p x^T(t)\,A_p(t)\,x(t)\end{aligned}$$

$$+\exp\left(-\int_0^t \overline{\phi}(s)\,ds\right)\omega_p x^T(t)\times B_p(t)B_p^T(t)x(t)$$
$$+\omega_p u^T(t)u(t)\exp\left(\int_0^t \overline{\phi}(s)\,ds\right)\le \omega_p x^T(t)$$
$$\cdot\left[A_p(t)+A_p^T(t)+B_p(t)B_p^T(t)\right.$$
$$\times\exp\left(-\int_0^t \overline{\phi}(s)\,ds\right)$$
$$\left.+\omega_p\exp\left(\int_0^t \overline{\phi}(s)-\phi(s)\,ds\right)\cdot I_{n\times n}\right]x(t)\le\phi(t)$$
$$\cdot V_p(t), \tag{38}$$

which implies that condition (5) holds. Then it is easy to check that all conditions in Theorem 2 hold and thus Theorem 5 can be derived. The proof is completed. □

In particular, if we choose $\overline{\phi}(t)=\phi(t)$, then the following corollary can be derived directly.

Corollary 6. *Assume that there exist constants $\eta>0$, $\mu\ge 1$, $\omega_p>0$, and continuous function $\phi:\mathbb{R}_+\to\mathbb{R}$ such that, for all $t\in\mathbb{R}_+$ and all $p,q\in P$, $\omega_p\le\mu\omega_q$, (7), and the following hold:*

$$A_p(t)+A_p^T(t)+B_p(t)B_p^T(t)\exp\left(-\int_0^t\phi(s)\,ds\right)\le\left(\phi(t)-\omega_p\right)\cdot I_{n\times n}. \tag{39}$$

Then the system (36) is UISS over the class $\mathcal{F}_\tau$, where ADT constant $\tau>0$ satisfies (8).

In addition, note that the ISS property guarantees the uniform asymptotic stability (UAS) of a system with a zero input. Consider the nonlinear switched system

$$\dot{x}=f_{\sigma(t)}(t,x), \tag{40}$$

where $\sigma\in\mathcal{F}$ is the switching function, $f_\sigma\in C(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n)$ is local Lipschitz and $f_\sigma(t,0)=0$, $x(t,t_0,x(t_0))$ is the solution for system (40) with the initial value $x(t_0)\in\mathbb{R}^n$ and an initial time $t_0\ge 0$. Then we have the following result for system (40).

Corollary 7. *Assume that there exist functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, a continuous function $\phi:\mathbb{R}_+\to\mathbb{R}$, continuous differentiable functions $V_p:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, and constants $\eta\in\mathbb{R}_+$, $\mu\ge 1$ such that, for all $t\in\mathbb{R}_+$, $x\in\mathbb{R}^n$, and all $p,q\in P$, (4), (6), (7), and the following condition hold:*

$$\frac{\partial V_p}{\partial x}f_p(t,x)+\frac{\partial V_p}{\partial t}\le\phi(t)V_p(t,x). \tag{41}$$

Then system (40) is UAS in Lyapunov sense, where ADT constant τ satisfies (8).

4. Applications

In this section, we present two examples to illustrate our main results.

Example 8. Consider the switched system (1) with $P=\{1,2\}$, $t_0=0$, and

$$f_1(t,x,u)=a(t)x+b(t)u,$$
$$f_2(t,x,u)=c(t)x+d(t)u, \tag{42}$$

where $a(t)=-\cos t-5/6$, $b(t)=(1/6)\exp(-\sin t-(2/3)t)$, $c(t)=-\cos t-(1/2)\exp(\cos t-2/3)$, and $d(t)=\exp(\cos t-\sin t-(2/3)t)$.

Note that $a(t)$ and $c(t)$ are sign reversal functions. Most of existing results, such as those in [18–23, 26–29], are inapplicable to switched system (1). Choose $V_1(t,x,u)=|x|$ and $V_2(t,x,u)=(1/2)|x|$ as ISS-Lyapunov functions. It is easy to see that condition (6) holds with $\mu=2$. Let $\rho(t)=t$ and $\phi(t)=-\cos t-2/3$, and then when

$$V_1(t,x,u)\ge\rho(|u|)\exp\left(\int_0^t\phi(s)\,ds\right), \tag{43}$$

that is,

$$|x|\ge|u|\exp\left(-\sin t-\frac{2}{3}t\right), \tag{44}$$

it leads to

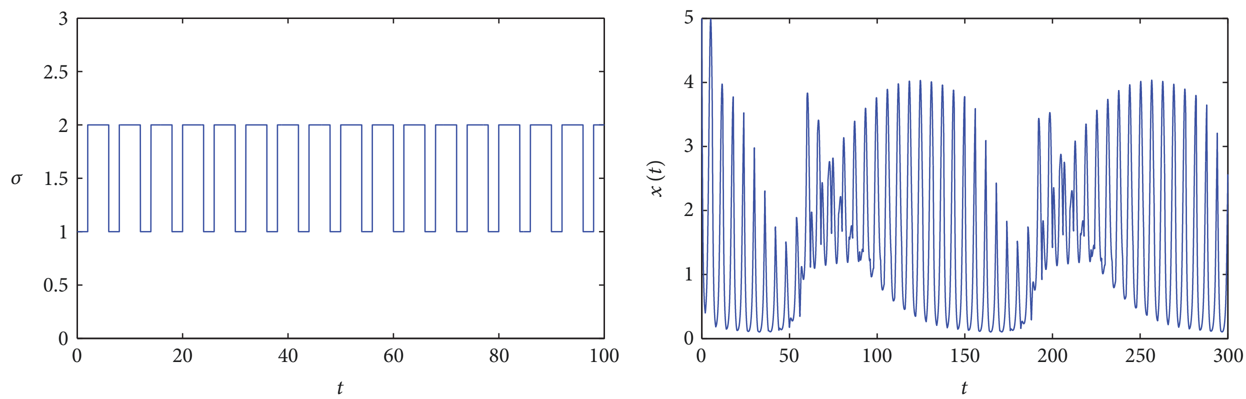
$$\begin{aligned}\dot{V}_1(t,x,u)&\le\left(-\cos t-\frac{5}{6}\right)|x|+\frac{1}{6}|x|\\&\le\left(-\cos t-\frac{2}{3}\right)V_1(t,x,u)\\&=\phi(t)V_1(t,x,u).\end{aligned} \tag{45}$$

Similarly, it can be deduced that $\dot{V}_2(t,x,u)\le(-\cos t-2/3)V_2(t,x,u)$ when $V_2(t,x,u)\ge\rho(|u|)\exp(\int_0^t\phi(s)ds)$. Thus condition (5) is satisfied. Choose $\eta=0.45$ such that

$$\int_{t_0}^t\left(\phi(v)+\eta\right)dv=\int_0^t(-\cos v-0.2167)\,dv<-0.02<0,\quad\forall t>0, \tag{46}$$

which implies that (7) holds. Note that $\ln\mu/\eta\approx 1.5403$. Hence, the switched system (1) is UISS over the class $\mathcal{F}_\tau$ with $\tau>1.5403$. In particular, if we choose the switching sequence $t_{2n-1}=6n-4$, $t_{2n}=6n$, $n\in\mathbb{Z}_+$ and let $x(0)=5$, $\tau=3, u=\text{sat}(x)$, then Figures 1(a) and 1(b) illustrate the switching signal and the state trajectory of system (1), respectively.

Remark 9. Note that, if we choose $V_2=|x|$ in the above example, then it can be deduced that $\dot{V}_2\le(-\cos t+(1/2)\exp(\cos t)-2/3)V_2$, which goes against condition (5). It indicates that sometimes it is necessary and important to consider multiple Lyapunov functions for switched systems. In addition, it is easy to check that the ISS Theorems in [32] are invalid for the above example due to the stronger restriction on ϕ^+.



(a) (b)

Figure 1: Simulation results for Example 8.

Example 10. Consider the time-varying switched system (36) with $P = \{1, 2\}$ and

$$\begin{aligned}\dot{x}(t) &= [A_1 + \Delta A_1(t)]\, x(t) + [B_1 + \Delta B_1(t)]\, u_1(t), \\ \dot{x}(t) &= [A_2 + \Delta A_2(t)]\, x(t) + [B_2 + \Delta B_2(t)]\, u_2(t),\end{aligned} \tag{47}$$

where

$$\begin{aligned} A_1 &= \begin{pmatrix} -0.105 & 0 \\ 0 & -0.11 \end{pmatrix}, \\ \Delta A_1(t) &= \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} -0.115 & -0.005 \\ -0.005 & -0.115 \end{pmatrix}, \\ \Delta A_2(t) &= \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \\ B_1 = B_2 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \Delta B_1(t) &= \begin{pmatrix} 0 \\ c \end{pmatrix}, \\ \Delta B_2(t) &= \begin{pmatrix} c \\ c \end{pmatrix}, \\ a_1 = a_2 = b_1 &= -\frac{1}{2}\sin t, \\ b_2 &= \frac{1}{2}(-\sin t - \exp(-2t)), \\ c &= \frac{1}{10}\exp\left(\frac{1}{2}\left(\cos t - \frac{1}{5}t - 1\right)\right). \end{aligned} \tag{48}$$

In this case, choose $\omega_1 = 0.01$, $\omega_2 = 0.02$, $\eta = 0.1$, $\mu = 2$, and $\phi(t) = -\sin t - 0.2$. Then it is easy to see that

$$\int_0^t (\phi(s) + \eta)\, ds = \int_0^t (-\sin s - 0.1)\, ds < 0, \quad \forall t > 0, \tag{49}$$

which implies that (7) holds. Moreover, note that

$$\begin{aligned} &A_1(t) + A_1^T(t) + B_1(t) B_1^T(t) \exp\left(-\int_0^t \phi(s)\, ds\right) \\ &\quad = \begin{pmatrix} -\sin t - 0.21 & 0 \\ 0 & -\sin t - 0.21 \end{pmatrix}, \\ &A_2(t) + A_2^T(t) + B_2(t) B_2^T(t) \exp\left(-\int_0^t \phi(s)\, ds\right) \\ &\quad = \begin{pmatrix} -\sin t - 0.22 & 0 \\ 0 & -\sin t - \exp(-2t) - 0.22 \end{pmatrix}. \end{aligned} \tag{50}$$

Note that $\ln \mu/\eta \approx 6.931$. Thus it follows from Corollary 6 that the switched system (36) is UISS over the class $\mathcal{F}_\tau$ with $\tau > 6.931$. In particular, if we choose the switching sequence $t_{2n-1} = 14n - 1$, $t_{2n} = 14n$, $n \in \mathbb{Z}_+$ and let $x^T(0) = (3, 3)$, $\tau = 7$, $u_1 = \sin x$, $u_2 = \text{sat}(x)$, then Figures 2(a) and 2(b) illustrate the switching signal and the 2-norm of the state trajectory of system (36), respectively.

5. Conclusion

In this paper, we presented some new ADT-based sufficient conditions for ISS of switched systems via Lyapunov method involving indefinite derivative. The ISS property of the switched system can be guaranteed under the designed ADT scheme. Our results improved some recent work in the literature. Two examples were given to show the effectiveness and advantage of the obtained results. It should be pointed out that the main results of this paper are based on multiple Lyapunov functions, which are more general than existing results in some cases. Since complex factors such as nonlinearities, impulsive perturbations, and delays exist widely in

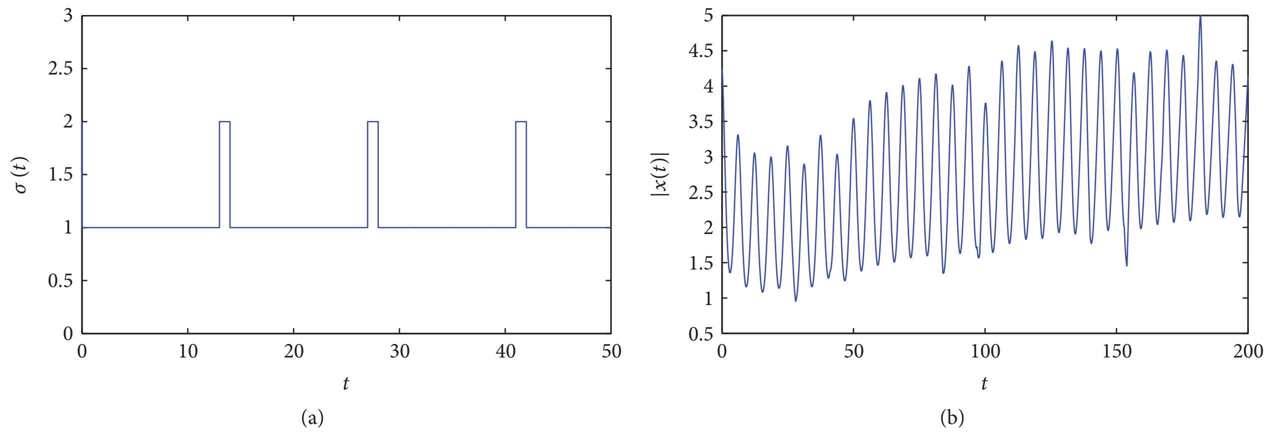


FIGURE 2: Simulation results for Example 10.

various engineering systems [35], future work can be done to develop the Lyapunov method involving indefinite derivative to switched systems subject to complex factors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by National Natural Science Foundation of China (11301308, 61673247) and the Research Fund for Excellent Youth Scholars of Shandong Province (ZR2016JL024, JQ201719).

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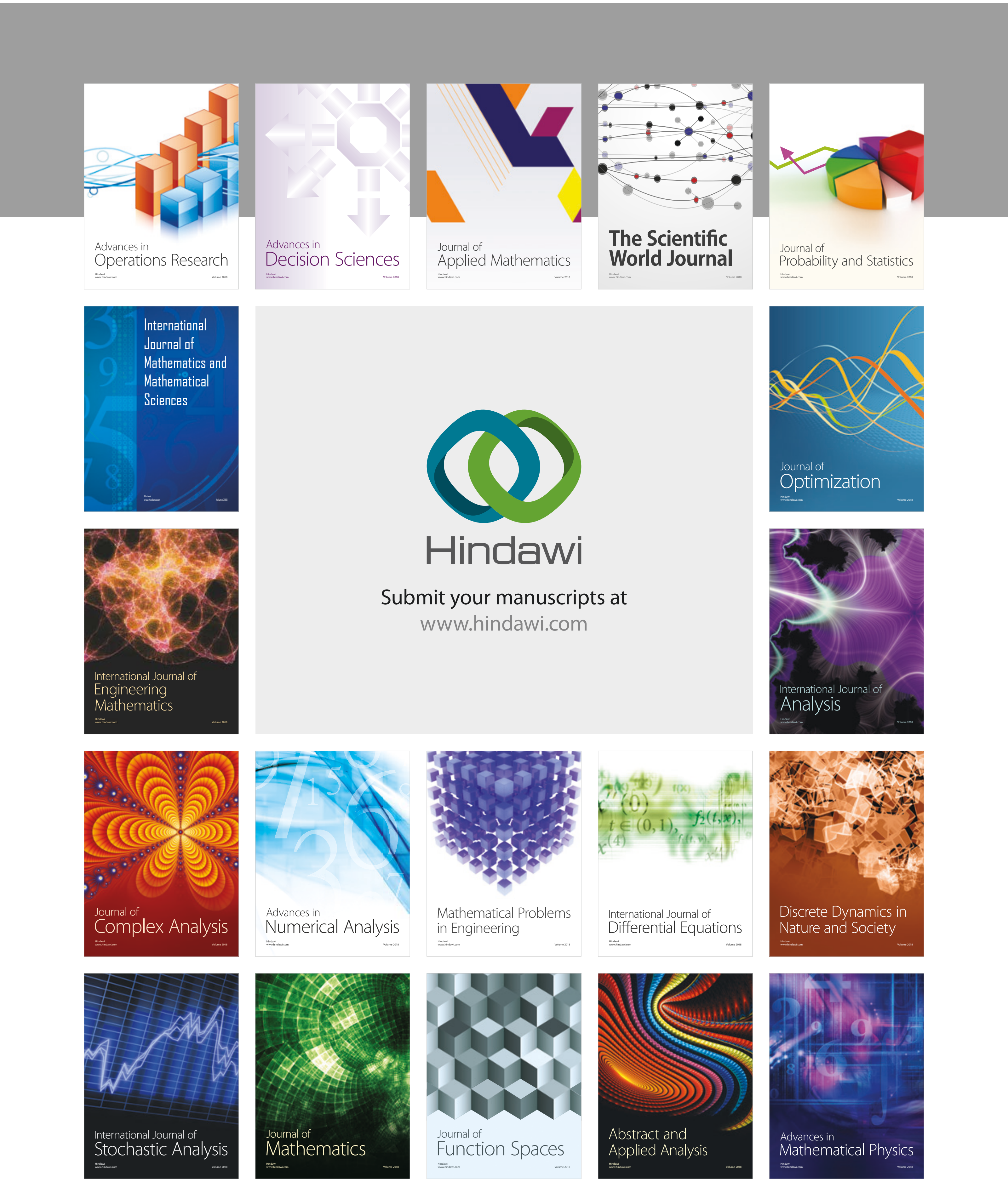
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