A possibility theorem on aggregation over multiple interconnected propositions

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A Possibility Theorem on Aggregation Over Multiple Interconnected Propositions

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Abstract. Drawing on the so-called "doctrinal paradox", List and Pettit (2002a) have shown that, given an unrestricted domain condition, there exists no procedure for aggregating individual sets of judgments over multiple interconnected propositions into corresponding collective ones, where the procedure satisfies some minimal conditions similar to the conditions of Arrow's theorem. I prove that we can avoid the paradox and the associated impossibility result by introducing an appropriate domain restriction: a structure condition, called *unidimensional alignment*, is shown to open up a possibility result, similar in spirit to Black's median voter theorem (1948). Specifically, I prove that, given unidimensional alignment, propositionwise majority voting is the *unique* procedure for aggregating individual sets of judgments into collective ones in accordance with the above mentioned minimal conditions.

1. The Problem

A new problem of aggregation has increasingly come to the attention of scholars in law, economics and philosophy. While social choice theory classically focuses on the aggregation of preference orderings or utility functions, the new problem concerns the aggregation of sets of judgments over multiple interconnected propositions. The propositions are interconnected in that the judgments on some of the propositions logically constrain the judgments on others. Interest in the new aggregation problem was first sparked by the identification of the so-called "doctrinal paradox" (Kornhauser & Sager 1986; Kornhauser 1992; Kornhauser & Sager 1993; Chapman 1998; Brennan 2000; Pettit 2001).

Suppose a three-member court has to decide whether a defendant is liable under a charge of breach of contract. Legal doctrine requires that the court should find that the defendant is liable (proposition R) if and only if it finds, first, that the defendant did some action X (proposition P), and, second, that the defendant had a contractual obligation not to do action X (proposition Q). Thus legal doctrine demands $(R \leftrightarrow (P \wedge Q))$. Suppose the judgments of the three judges are as in table 1.

	P	Q	$(R \leftrightarrow (P \wedge Q))$	R
Judge 1	Yes	Yes	Yes	Yes
Judge 2	Yes	No	Yes	No
Judge 3	No	Yes	Yes	No
Majority	Yes	Yes	Yes	No

Table 1

All judges accept $(R \leftrightarrow (P \wedge Q))$. Judge 1 accepts both P and Q and, by implication, R. Judges 2 and 3 each accept only one of P or Q and, by implication, they both reject R. If the court

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applies majority voting on each of the four propositions P, Q, R, and $(R \leftrightarrow (P \wedge Q))$, it faces a paradoxical outcome. A majority accepts P, a majority accepts Q, a majority (unanimity) accepts $(R \leftrightarrow (P \wedge Q))$, and yet a majority rejects R. Propositionwise majority voting thus generates an inconsistent collective set of judgments, namely $\{P, Q, (R \leftrightarrow (P \wedge Q)), \neg R\}$ (corresponding to the last row of table 1). This set is inconsistent in the standard sense of propositional logic: there exists no assignment of truth-values to P, Q and R that makes all the propositions in the set simultaneously true. Note that the sets of judgments of the individual judges (corresponding to the first three rows of table 1) are all consistent. The doctrinal paradox is related to Anscombe's paradox, or Ostrogorski's paradox (Anscombe 1976; Kelly 1989; Brams, Kilgour and Zwicker 1997). Like the doctrinal paradox, these paradoxes are concerned with aggregation over multiple propositions. Unlike the doctrinal paradox, they do not explicitly incorporate logical connections between different propositions.[2]

Just as Condorcet's paradox of cyclical majority preferences can be associated with a more general impossibility result – Arrow's theorem –, so the doctrinal paradox is illustrative of a more general result. As stated in more detail below, given an unrestricted domain condition, there exists no procedure for aggregating individual sets of judgments into collective ones, where the procedure satisfies some minimal conditions similar to Arrow's conditions (List and Pettit 2002a).

In this paper, I prove that we can restrict the domain of admissible individual sets of judgments in such a way as to avoid the paradox and the associated impossibility result. If a profile of individual sets of judgments satisfies a structure condition called *unidimensional alignment*, then propositionwise majority voting will generate a consistent collective set of judgments. If the number of individuals is odd, that set will be the set of judgments of the median individual with respect to a suitably defined structuring ordering; if the number of individuals is even, the set will be the intersection of the sets of judgments of the median pair of individuals. Moreover, I provide a characterization result: given unidimensional alignment, propositionwise majority voting is the *unique* aggregation procedure satisfying the above mentioned minimal conditions.

The condition of unidimensional alignment is similar in spirit to Black's condition of single-peakedness in the context of preference aggregation (Black 1948). Like single-peakedness,

[2] "Premise-based" and "conclusion-based" decision procedures have been proposed as escape-routes from the doctrinal paradox (for example, Pettit 2001). On the premise-based procedure, majority voting is applied on each of P and Q (the "premises"), but not on R (the "conclusion"), and the rule $(R \leftrightarrow (P \wedge Q))$ is then used to determine the collective judgment on R, so that the majority verdict on R is ignored. Given table 1, this leads to the acceptance of R. On the conclusion-based procedure, majority voting is applied only on R, but not on P and Q, so that the majority verdicts on P and Q are ignored. Given table 1, this leads to the rejection of R. Thus the premise-based and conclusion-based procedures may produce divergent outcomes. It is easily seen that they each violate condition (S) below. Using the framework of the Condorcet jury theorem, Bovens and Rabinowicz (2001) have compared the performance of the two procedures in terms of "tracking the truth" on R – where there is such a truth to be tracked.

it is a simple condition that lends itself to an interpretation in terms of 'onedimensionality', as I will suggest. However, the simplicity of the condition, like that of single-peakedness, comes at a price. From a logical point of view, the condition is too restrictive: it is only *sufficient but not necessary* for avoiding the relevant paradox and the associated impossibility result. But I show that a version of the presented characterization result on propositionwise majority voting holds for *any* domain restriction condition where (i) the condition is less restrictive than unidimensional alignment (i.e. it corresponds to a domain that is a superset of the unidimensional alignment domain) and (ii) the condition also provides an escape-route from the paradox and associated impossibility result.

2. The Framework

Let $N = \{1, 2, ..., n\}$ be a set of individuals ($n \geq 2$). Let X be a set of propositions from the propositional calculus, interpreted as those propositions on which judgments are to be made. The set X includes atomic propositions, such as P, Q and R, and compound propositions, such as $(R \leftrightarrow (P \wedge Q))$ or $(P \leftrightarrow \neg Q)$. To make the problem non-trivial, we assume that X contains at least two distinct atomic propositions, P and Q, and their conjunction, $(P \wedge Q)$.[3] Moreover, we assume that X contains proposition-negation pairs: we assume that, for every $\phi \in X$, we also have $\neg\phi \in X$. For simplicity, for every $\phi \in X$, we identify $\neg\neg\phi$ with ϕ.

For each $i \in N$, individual i's *set of judgments* is a subset $\Phi_i \subseteq X$. Then $\phi \in \Phi_i$ means "individual i accepts ϕ". A *profile of individual sets of judgments* is an n-tuple $\{\Phi_i\}_{i \in N} = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$. An *aggregation function* is a function F whose input is a profile of individual sets of judgments and whose output is a *collective set of judgments* $\Phi \subseteq X$. Here $\phi \in \Phi$ means "the group N accepts ϕ". We use $\mathbf{D}$ to denote the domain of F. Below we consider different domain conditions. We define *propositionwise majority voting* (as in the doctrinal paradox) to be the following aggregation function on some domain $\mathbf{D}$: for each $\{\Phi_i\}_{i \in N} \in \mathbf{D}$, $F(\{\Phi_i\}_{i \in N}) := \{\phi \in X : |\{i \in N : \phi \in \Phi_i\}| > n/2\}$.

An (individual or collective) set of judgments $\Phi \subseteq X$ is *complete* if, for all $\phi \in X$, at least one of $\phi \in \Phi$ or $\neg\phi \in \Phi$ holds. The set Φ is *consistent* if, for all $\phi \in X$, at most one of $\phi \in \Phi$ or $\neg\phi \in \Phi$ holds. The set Φ is *deductively closed* if, for all $\varphi \in X$, if Φ logically entails φ, then $\varphi \in \Phi$. Let $\mathbf{U}$ be the set of all logically possible profiles of complete, consistent and deductively closed individual sets of judgments.

[3] The use of conjunction ($\wedge$) here is not essential, and the use of other logical connectives would yield a similar result. Particularly, as the set of connectives $\{\neg, \wedge\}$ is expressively adequate, any logically possible proposition of the propositional calculus can be expressed as a proposition using $\neg$ and $\wedge$ as the only connectives.

List and Pettit (2002a) use the following minimal conditions on an aggregation function F:

UNRESTRICTED DOMAIN (U). The domain of F is $\mathbf{D} = \mathbf{U}$.

ANONYMITY (A). For any $\{\Phi_i\}_{i\in N}\in \mathbf{D}$ and any permutation $\sigma: N \rightarrow N$, $F(\{\Phi_i\}_{i\in N}) = F(\{\Phi_{\sigma(i)}\}_{i\in N})$.

SYSTEMATICITY (S). There exists a function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ such that, for any $\{\Phi_i\}_{i\in N}\in \mathbf{D}$, $F(\{\Phi_i\}_{i\in N}) = \{\phi\in X : f(\delta_1(\phi), \delta_2(\phi), ..., \delta_n(\phi)) = 1\}$, where, for each $i\in N$ and each $\phi\in X$, $\delta_i(\phi) = 1$ if $\phi\in \Phi_i$ and $\delta_i(\phi) = 0$ if $\phi\notin \Phi_i$.

Conditions (U) and (A) have direct counterparts in standard social choice theory. Condition (U) requires that F should accept as admissible input any logically possible profile of individual sets of judgments, where the individual sets of judgments satisfy completeness, consistency and deductive closure. Condition (A) requires that F be invariant under permutations of the individuals, thereby giving all individuals formally equal weight in the aggregation. Condition (S) is closest to a combination of independence of irrelevant alternatives and neutrality. It requires that (i) the collective judgment on each proposition should depend exclusively on the pattern of individual judgments on that proposition (the “independence” part) and (ii) the same pattern of dependence should hold for all propositions (the “neutrality” part).[4] Condition (S) is demanding in so far as we might sometimes wish to treat different kinds of propositions differently, e.g. to treat ‘premises’ and ‘conclusions’ differently.[5] Nonetheless, propositionwise majority voting, which has some *prima facie* plausibility as an aggregation function, satisfies all of (U), (A) and (S), but the doctrinal paradox shows that it may fail to generate complete, consistent and deductively closed collective sets of judgments.

Theorem 1. (List and Pettit 2002a) There exists no aggregation function F (generating complete, consistent and deductively closed collective sets of judgments) which satisfies (U), (A) and (S).

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[4] It can be shown that, when the present framework is extended in such a way that X contains pairwise ranking propositions of the form xPy (from the predicate calculus), then condition (S) entails that (i) each collective pairwise ranking judgment depends exclusively on the pattern of individual ranking judgments over the same pair of alternatives and (ii) the same pattern of dependence holds for all pairwise ranking judgments. Here part (i) is precisely Arrow’s condition of independence of irrelevant alternatives. But while independence of irrelevant alternatives still allows different pairs of alternatives to be treated differently (so long as their collective pairwise rankings depend only on the corresponding individual pairwise rankings and not on rankings involving third alternatives), part (ii) here disallows such differential treatment. Thus condition (S) entails the conjunction of independence of irrelevant alternatives and neutrality, a condition that is sometimes called independence of non-welfare characteristics in Arrowian social choice theory. Independence of non-welfare characteristics entails, but is not entailed by, independence of irrelevant alternatives. See List and Pettit (2002b).

[5] See note 2 above.

3. The Result

To define unidimensional alignment, a few preliminary definitions are due. Fix a profile of individual sets of judgments $\{\Phi_i\}_{i\in N}$. For each $\phi\in X$, define $N_{accept\text{-}\phi} := \{i\in N : \phi\in \Phi_i\}$ and $N_{reject\text{-}\phi} := \{i\in N : \phi\notin \Phi_i\}$. Further, given any linear ordering Ω on N and any $N_1, N_2 \subseteq N$, we write $N_1\, \Omega\, N_2$ as an abbreviation for [for all $i\in N_1$ and all $j\in N_2$, $i\Omega j$].

A profile of individual sets of judgments, $\{\Phi_i\}_{i\in N}$, satisfies *unidimensional alignment* if there exists a linear ordering Ω on N such that

for every $\phi\in X$, either $N_{accept\text{-}\phi}\, \Omega\, N_{reject\text{-}\phi}$ or $N_{reject\text{-}\phi}\, \Omega\, N_{accept\text{-}\phi}$.[6]

An ordering Ω with this property will be called a *structuring ordering* of N for $\{\Phi_i\}_{i\in N}$. If n is odd, then we say that individual $m\in N$ is the *median individual* with respect to Ω if $|\{i\in N : i\Omega m\}| = |\{i\in N : m\Omega i\}|$. If n is even, there exists no median individual under this definition. We then say that individuals $m_1, m_2\in N$ are the *median pair of individuals* with respect to Ω if (i) $m_1\Omega m_2$, (ii) there exists no $i\in N$ such that $m_1\Omega i$ and $i\Omega m_2$, and (iii) $|\{i\in N : i\Omega m_1\}| = |\{i\in N : m_2\Omega i\}|$.

Informally, a profile of individual sets of judgments satisfies unidimensional alignment if the individuals can be ordered from left to right such that, for *every* proposition $\phi\in X$, the individuals accepting ϕ are either all to the left, or all the right, of those rejecting ϕ.

	Individual 3	Individual 2	Individual 5	Individual 4	Individual 1
P	No	No	No	Yes	Yes
Q	Yes	Yes	No	No	No
R	Yes	No	No	No	No
$(R \leftrightarrow (P \wedge Q))$	No	Yes	Yes	Yes	Yes

Table 2

The profile in table 2 satisfies unidimensional alignment, whereas the profile in table 1 violates the condition. Let **UAD** be the set of all logically possible profiles of complete, consistent and deductively closed individual sets of judgments satisfying unidimensional alignment.

UNIDIMENSIONAL ALIGNMENT DOMAIN (UAD). The domain of F is **D** = **UAD**.

Proposition 1. Let F be propositionwise majority voting. For any $\{\Phi_i\}_{i\in N}\in$ **UAD**, the following holds:

- If n is odd, $F(\{\Phi_i\}_{i\in N}) = \Phi_m$, where m is the median individual with respect to a structuring ordering of N for $\{\Phi_i\}_{i\in N}$.
- If n is even, $F(\{\Phi_i\}_{i\in N}) = \Phi_{m_1}\cap\Phi_{m_2}$, where m_1 and m_2 are the median pair of individuals with respect to a structuring ordering of N for $\{\Phi_i\}_{i\in N}$.

[6] Note that this permits $N_{accept\text{-}\phi} = \emptyset$ or $N_{reject\text{-}\phi} = \emptyset$.

The proof is given in the appendix. Proposition 1 can be summarized as follows. Given a profile of individual sets of judgments that satisfies unidimensional alignment, order the individuals along a structuring ordering. Consider first the case in which n is odd. Then the set of judgments of the median individual with respect to the structuring ordering will be accepted in propositionwise majority voting (in the case of table 2, the judgments of individual 5). The reason is that, by unidimensional alignment, the median individual shares the majority judgment on each proposition. More precisely, the condition of unidimensional alignment implies that, for each proposition $\phi \in X$, a majority of individuals (i.e. at least $(n+1)/2$) accepts ϕ if and only if the median individual accepts ϕ. It follows immediately that, provided that the set of judgments of the median individual satisfies completeness, consistency and deductive closure, so will the resulting collective set. Consider next the case in which n is even. The situation is very similar to the previous case, although there exists no single median individual. This time the intersection of the sets of judgments of the median *pair* of individuals will be accepted in propositionwise majority voting. Here a majority of individuals (i.e. at least ${}^{n}/_{2}+1$) accepts a proposition ϕ if and only if *both* members of the median pair accept ϕ. It is easy to check that the intersection of two consistent and deductively closed individual sets of judgments is also consistent and deductively closed. Hence, provided that the sets of judgments of the median pair of individuals satisfy consistency and deductive closure, so will the collective set. But there is one complication. The median pair of individuals may not agree on ϕ, and as a result there may be some propositions $\phi \in X$ such that neither ϕ nor $\neg\phi$ is contained in the intersection of the sets of judgments of the median pair. The collective set of judgments may thus violate completeness.

However, the next theorem implies that such violations of completeness occur only in one special situation – and, moreover, in the intuitively 'right' kind of situation, namely when ϕ and $\neg\phi$ are supported by an equal number of individuals, i.e. when ϕ and $\neg\phi$ are tied in majority voting.

We say that a set of judgments $\Phi \subseteq X$ is *almost complete* if, for all $\phi \in X$, $|N_{accept\text{-}\phi}| \neq |N_{reject\text{-}\phi}|$ implies that at least one of $\phi \in \Phi$ or $\neg\phi \in \Phi$ holds. When n is odd, we can never have $|N_{accept\text{-}\phi}| = |N_{reject\text{-}\phi}|$, and hence the notions of *completeness* and *almost completeness* coincide. When n is even, on the other hand, the two notions are distinct: completeness implies almost completeness, but not vice-versa.[7]

[7] Suppose n is even. An almost complete set of judgments that satisfies consistency and deductively closure can always be extended to a complete set of judgments that also satisfies consistency and deductive closure. For example, if the collective set of judgments Φ is $\Phi_{m_1} \cap \Phi_{m_2}$, where Φ_{m_1} and Φ_{m_2} are the sets of judgments of the median pair of individuals, then each of Φ_{m_1} and Φ_{m_2} is a complete, consistent and deductively closed extension of Φ. So, when there are ties as a result of a disagreement between the median pair of individuals, the ties can be broken by consistently taking sides with either individual m_1 or individual m_2. However, if we add such a tie-breaking rule to

Theorem 2. Let F be an aggregation function satisfying (UAD). Then

(i) F generates almost complete, consistent and deductively closed collective sets of judgments and satisfies (A) and (S)

<u>if and only if</u>

(ii) F is propositionwise majority voting.

The proof is also given in the appendix. Theorem 2 implies not only that propositionwise majority voting is an aggregation function (generating almost complete, consistent and deductively closed collective sets of judgments) which satisfies (UAD), (A) and (S), but also that propositionwise majority voting is the *unique* such aggregation function.[8]

Moreover, the proof of theorem 2 in the appendix (specifically, the proof of "(i) implies (ii)") establishes a more general result:

Proposition 2. Let F be an aggregation function on some domain **D** satisfying $\mathbf{UAD} \subseteq \mathbf{D} \subseteq \mathbf{U}$. If, on the domain **D**, F generates almost complete, consistent and deductively closed collective sets of judgments and satisfies (A) and (S), then F is propositionwise majority voting.

Proposition 2 has some useful implications. For instance, if **D** is the *maximal* subset of **U** on which propositionwise majority voting generates almost complete, consistent and deductively closed collective sets of judgments, then propositionwise majority voting is also the *unique* such aggregation function on **D** satisfying (A) and (S).[9] Further, although we cannot currently give an easily definable less demanding domain restriction condition than (UAD), there is already

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our definition of propositionwise majority voting, the resulting aggregation function F violates condition (S). But F still satisfies a slightly weakened version of systematicity. We say that F is *almost systematic* if there exists a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ such that, for any $\{\Phi_i\}_{i\in N}$ in the domain of F, for all $\phi \in X$ with $|N_{accept\text{-}\phi}| \neq |N_{reject\text{-}\phi}|$, $\phi \in F(\{\Phi_i\}_{i\in N})$ if and only if $f(\delta_1(\phi), \delta_2(\phi), ..., \delta_n(\phi)) = 1$, where, for each $i \in N$ and each $\phi \in X$, $\delta_i(\phi) = 1$ if $\phi \in \Phi_i$ and $\delta_i(\phi) = 0$ if $\phi \notin \Phi_i$.

[8] Since propositionwise majority voting on the domain **UAD** selects the set of judgments of the median individual (or the intersection of the sets of judgments of the median pair), a natural generalization of this aggregation rule on **UAD** seems to be a generalized median voter rule (with parameter $k \in N$) which selects, for each $\{\Phi_i\}_{i\in N} \in \mathbf{UAD}$, the set of judgments of the k^{th} individual with respect to a structuring ordering of N for $\{\Phi_i\}_{i\in N}$. However, there are at least two problems with the class of generalized median voter rules. The first problem is that, for each $\{\Phi_i\}_{i\in N} \in \mathbf{UAD}$, the corresponding structuring ordering is unique at most up to reversal of the ordering: if Ω is a structuring ordering for $\{\Phi_i\}_{i\in N}$, then so is Ω^*, where, for all $i, j \in N$, $i\Omega^* j$ if and only if $j\Omega i$. To make a generalized median voter rule well-defined, we must select one of Ω or Ω^*. But if we make this selection on the basis of, for instance, which of the two orderings ranks a lower-numbered individual in N left-most (or right-most), then the resulting rule will violate condition (A). The second problem is that, in conjunction with condition (A), generalized median voter rules may violate condition (S). For example, in table 2 above, if we consider a generalized median voter rule which selects the set of judgments of the 2[nd] individual (from left) with respect to the structuring ordering, then Q will be accepted and P will be rejected. By conditions (S) and (A), however, the collective judgments on P and Q must coincide. I am grateful to an anonymous reviewer for drawing my attention to the class of generalized median voter rules.

[9] Formally, $\mathbf{D} := \{\{\Phi_i\}_{i\in N} \in \mathbf{U} : F(\{\Phi_i\}_{i\in N})$ is almost complete, consistent and deductively closed$\}$, where F is propositionwise majority voting. We know that $\mathbf{UAD} \subseteq \mathbf{D}$. In the general case, $\mathbf{UAD} \neq \mathbf{D}$.

something we know about such a condition: whatever the condition is, so long as it permits the existence of aggregation functions (generating almost complete, consistent and deductively closed collective sets of judgments) which satisfy (A) and (S), then propositionwise majority voting is the *unique* such aggregation function.

One simple generalization of unidimensional alignment is this. Suppose the set of propositions X can be partitioned into disjoint non-empty subsets X_1, X_2, …, X_r such that the propositions in distinct subsets have no atomic propositions in common.[10] This implies that the propositions in distinct subsets are logically independent from each other. We now say that a profile $\{\Phi_i\}_{i\in N}$ satisfies *generalized unidimensional alignment* if, for each $s\in\{1, 2, \ldots, r\}$, the profile $\{\Phi_i\}_{i\in N}$ restricted to X_s – i.e. $\{\Phi_i\cap X_s\}_{i\in N}$ – satisfies unidimensional alignment. It can easily be checked that propositionwise majority voting generates almost complete, consistent and deductively closed collective sets of judgments for profiles satisfying generalized unidimensional alignment.

4. Concluding Remarks

The condition of unidimensional alignment is similar in spirit to Black's condition of single-peakedness (Black 1948). Each of these two conditions is *sufficient but not necessary* for avoiding the relevant paradox and associated impossibility result. From a logical perspective, both may therefore seem unnecessarily restrictive, but, in compensation, they are both simple and easily interpretable. An attractive feature of Black's condition is that it has a natural interpretation in terms of 'onedimensionality': if the individuals agree on a left/right dimension along which policy options are aligned, then their preference orderings may satisfy single-peakedness with respect to that dimension.

Is unidimensional alignment just an artificial condition, or can we also interpret it plausibly in terms of onedimensionality? Suppose (i) the individuals disagree on what set of judgments to endorse, but they agree on a single left/right dimension (such as from "most liberal" to "most conservative") that characterizes the range of their disagreement; in particular, suppose that each individual takes a certain position on that dimension. And suppose (ii), for each proposition, the extreme positions on the left/right dimension correspond to either clear acceptance or clear rejection of the proposition and there exists an 'acceptance threshold' on the dimension (possibly different for different propositions) such that all the individuals to the left of the threshold accept the proposition and all the individuals to its right reject it (or vice-versa). For

[10] Such partitions may not exist for all sets of propositions X. For example, if X contains a finite number of atomic propositions and there exists a compound proposition in X that includes all these atomic propositions, then the required partition will not exist. Also, if X is a complete Boolean algebra, the required partition will not exist.

example, in a political context, all individuals to the left of some threshold might be in favour of a certain binary "free trade" proposition while all individuals to its right might be against it. Conditions (i) and (ii) entail unidimensional alignment. In other words, agreement on a dimension can induce unidimensional alignment.

This suggests that, if the individuals reach agreement on a dimension, the doctrinal paradox and the associated impossibility result can be avoided; and, in analogy with Black's famous result, the output of the aggregation will then be the median individual's input (if the number of individuals is odd) or the intersection of the median pair's inputs (if the number of individuals is even). A challenge for further work will be to find a domain restriction condition that is less restrictive than (UAD) but still sufficient for the avoidance of the "doctrinal paradox", and yet easy to define and interpret. But, as we have seen, whatever domain such a condition will describe, propositionwise majority will be the *unique* aggregation procedure on that domain which generates almost complete, consistent and deductively closed collective sets of judgments and satisfies conditions (A) and (S).

Appendix: Proofs

Proof of proposition 1.

Let F be propositionwise majority voting. Let $\{\Phi_i\}_{i\in N}\in \mathbf{UAD}$. Then $F(\{\Phi_i\}_{i\in N}) = \{\phi\in X : |N_{accept\text{-}\phi}|>n/2\}$. Let Ω be a structuring ordering of N for $\{\Phi_i\}_{i\in N}$. If n is even, let m_1 and m_2 be the median pair of individuals with respect to Ω. If n is odd, let $m_1=m_2:=m$ be the median individual with respect to Ω. We show that, for any $\phi\in X$, $\phi\in F(\{\Phi_i\}_{i\in N})$ if and only if $\phi\in \Phi_{m_1}\cap\Phi_{m_2}$, which implies that $F(\{\Phi_i\}_{i\in N})=\Phi_{m_1}\cap\Phi_{m_2}$. Note that this implies the desired result both for n even and for n odd, because if n is odd we have $\Phi_{m_1}\cap\Phi_{m_2}=\Phi_m$ under the present definitions.

Take any $\phi\in X$. Suppose $\phi\in \Phi_{m_1}\cap\Phi_{m_2}$. Then $m_1, m_2\in N_{accept\text{-}\phi}$. But since $[N_{accept\text{-}\phi}\,\Omega\, N_{reject\text{-}\phi}$ or $N_{reject\text{-}\phi}\,\Omega\, N_{accept\text{-}\phi}]$, we must have either [for every $i\in N$, if $i\Omega m_1$ then $i\in N_{accept\text{-}\phi}$] or [for every $i\in N$, if $m_2\Omega i$ then $i\in N_{accept\text{-}\phi}$]. Hence either $|N_{accept\text{-}\phi}| \geq |\{m_1, m_2\}|+|\{i\in N : i\Omega m_1\}|$ or $|N_{accept\text{-}\phi}| \geq |\{m_1, m_2\}|+|\{i\in N : m_2\Omega i\}|$. Note that $|\{m_1, m_2\}| = 1$ if n is odd and $|\{m_1, m_2\}| = 2$ if n is even. But, by the definition of m_1 and m_2 (as the median individual or pair with respect to Ω), we have

$$|\{i\in N : i\Omega m_1\}| = |\{i\in N : m_2\Omega i\}| = \begin{cases} (n\text{-}1)/2 & \text{if } n \text{ is odd} \\ (n\text{-}2)/2 & \text{if } n \text{ is even,} \end{cases}$$

and hence $|N_{accept\text{-}\phi}| > n/2$. Therefore $\phi\in F(\{\Phi_i\}_{i\in N})$.

Conversely, suppose $\phi\in F(\{\Phi_i\}_{i\in N})$. Then $|N_{accept\text{-}\phi}|>n/2$. Assume, for a contradiction, that $\phi\notin \Phi_{m_1}\cap\Phi_{m_2}$. Then $m_1\notin N_{accept\text{-}\phi}$ or $m_2\notin N_{accept\text{-}\phi}$ (or both). Without loss of generality, assume $m_1\notin N_{accept\text{-}\phi}$ (the case $m_2\notin N_{accept\text{-}\phi}$ is perfectly analogous). Then $m_1\in N_{reject\text{-}\phi}$. We must then have either [for every $i\in N$, if $m_1\Omega i$ then $i\in N_{reject\text{-}\phi}$] or [for every $i\in N$, if $i\Omega m_1$ then $i\in N_{reject\text{-}\phi}$], since [$N_{accept\text{-}\phi}\,\Omega\, N_{reject\text{-}\phi}$ or $N_{reject\text{-}\phi}\,\Omega\, N_{accept\text{-}\phi}$] by unidimensional alignment. But then either $|N_{reject\text{-}\phi}| \geq 1+|\{i\in N : m_1\Omega i\}| \geq 1+|\{i\in N : m_2\Omega i\}|$ or $|N_{reject\text{-}\phi}| \geq 1+|\{i\in N : i\Omega m_1\}|$. If n is odd, then this means $|N_{reject\text{-}\phi}| \geq 1+(n\text{-}1)/2 > n/2$. If n is even, then this means $|N_{reject\text{-}\phi}| \geq 1+(n\text{-}2)/2 = n/2$. In both cases, $|N_{reject\text{-}\phi}| \geq n/2$, which contradicts $|N_{accept\text{-}\phi}|>n/2$. Therefore $\phi\in \Phi_{m_1}\cap\Phi_{m_2}$. We conclude that $F(\{\Phi_i\}_{i\in N}) = \Phi_{m_1}\cap\Phi_{m_2}$, as required. ■

Proof of theorem 2.

(ii) implies (i). Suppose we have (ii), i.e. F is propositionwise majority voting, defined on $\mathbf{D} = \mathbf{UAD}$. It is easy to check that F satisfies (A) and (S). We need to show that, for every $\{\Phi_i\}_{i\in N}\in \mathbf{D}$, F generates an almost complete, consistent and deductively closed collective set of judgments. Let $\{\Phi_i\}_{i\in N}\in \mathbf{D} = \mathbf{UAD}$. Let Ω be a structuring ordering of N for $\{\Phi_i\}_{i\in N}$. Consider two cases: n is odd, and n is even. If n is odd, by proposition 1, $F(\{\Phi_i\}_{i\in N}) = \Phi_m$, where m is the median individual with respect to Ω. But, by assumption, Φ_m is complete, consistent and deductively closed. It follows that $F(\{\Phi_i\}_{i\in N})$ is also complete (and by implication almost complete), consistent and deductively closed. If n is even, by proposition 1, $F(\{\Phi_i\}_{i\in N}) = \Phi_{m_1}\cap\Phi_{m_2}$, where m_1 and m_2 are the median pair of individuals with respect to Ω. Again, each of Φ_{m_1} and Φ_{m_2} is complete, consistent and deductively closed. It is easy to check that the intersection of two consistent and deductively closed individual sets of judgments is also consistent and deductively closed, and hence it follows that $F(\{\Phi_i\}_{i\in N})$ is consistent and deductively closed. To show that $F(\{\Phi_i\}_{i\in N})$ is almost complete, assume, for a contradiction, that there exists $\phi\in X$ with $|N_{accept\text{-}\phi}| \neq |N_{reject\text{-}\phi}|$ such that $\phi\notin F(\{\Phi_i\}_{i\in N})$ and $\neg\phi\notin F(\{\Phi_i\}_{i\in N})$. This means that $\phi\notin \Phi_{m_1}\cap\Phi_{m_2}$ and $\neg\phi\notin \Phi_{m_1}\cap\Phi_{m_2}$. Since each of Φ_{m_1} and Φ_{m_2} is complete and consistent, must have either [$\phi\in \Phi_{m_1}$ and $\phi\notin \Phi_{m_2}$] or [$\phi\notin \Phi_{m_1}$ and $\phi\in \Phi_{m_2}$], i.e. either [$m_1\in N_{accept\text{-}\phi}$ and $m_2\in N_{reject\text{-}\phi}$] or [$m_1\in N_{reject\text{-}\phi}$ and $m_2\in N_{accept\text{-}\phi}$]. But since m_1 and m_2 are the median pair of individuals with respect to Ω and we have [$N_{accept\text{-}\phi}\,\Omega\, N_{reject\text{-}\phi}$ or $N_{reject\text{-}\phi}\,\Omega\, N_{accept\text{-}\phi}$] (by unidimensional alignment), we must have $|N_{accept\text{-}\phi}| = |N_{reject\text{-}\phi}| = n/2$, a contradiction. Hence $F(\{\Phi_i\}_{i\in N})$ is almost complete. We conclude that (i) holds, as required.

(i) implies (ii). For this part of the proof, it is sufficient to assume that the domain $\mathbf{D}$ of F satisfies $\mathbf{UAD} \subseteq \mathbf{D} \subseteq \mathbf{U}$. This assumption holds in particular when $\mathbf{D} = \mathbf{UAD}$, as assumed in theorem 2. Suppose we have (i), i.e. F generates almost complete, consistent and deductively closed collective sets of judgments and satisfies (A) and (S). The structure of the proof is as follows. We prove the following three claims.

Claim 1. There exists a function $g : \{0, 1, \ldots, n\} \to \{0, 1\}$ such that, for any $\{\Phi_i\}_{i\in N}\in \mathbf{D}$, $F(\{\Phi_i\}_{i\in N}) = \{\phi\in X : g(|N_{accept\text{-}\phi}|) = 1\}$.

Claim 2. For any $k \in \{0, 1, \ldots, n\}$,

$$g(k) + g(n\text{-}k) = \begin{cases} 1 & \text{if } k \neq n/2 \\ 0 & \text{if } k = n/2. \end{cases}$$

Claim 3. For any $k, l \in \{0, 1, \ldots, n\}$, $k < l$ implies $g(k) \leq g(l)$.

To prove that F is propositionwise majority voting, by claim 1 it is sufficient to prove that, for all $\{\Phi_i\}_{i \in N} \in \mathbf{D}$ and all $\phi \in X$, $g(|N_{accept\text{-}\phi}|) = 1$ if and only if $|N_{accept\text{-}\phi}| > n/2$, i.e. if and only if, for all $k \in \{0, 1, \ldots, n\}$, $f(k) = 1$ if and only if $k > n/2$. Suppose $k > n/2$. Assume, for a contradiction, that $g(k) = 0$. By claim 2, $f(n\text{-}k) = 1$. But since $k > n/2$, we have $n\text{-}k < k$, and then $g(n\text{-}k) = 1 > g(k) = 0$ contradicts claim 3. Hence $g(k) = 1$. Suppose $k \leq n/2$. If $k=n/2$, then by claim 2, $g(k)=0$. If $k < n/2$, then $n\text{-}k > n/2$. We have already shown that $n\text{-}k > n/2$ implies $g(n\text{-}k) = 1$. Since $g(n\text{-}k) + g(k) = 1$, by claim 2, we must have $g(k)=0$. Hence (ii) holds, as required.

Proof of claim 1. Since F satisfies (S), there exists a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ such that, for any $\{\Phi_i\}_{i \in N} \in \mathbf{D}$, $F(\{\Phi_i\}_{i \in N}) = \{\phi \in X : f(\delta_1(\phi), \delta_2(\phi), ..., \delta_n(\phi)) = 1\}$, where, for each $i \in N$ and each $\phi \in X$, $\delta_i(\phi) = 1$ if $\phi \in \Phi_i$ and $\delta_i(\phi) = 0$ if $\phi \notin \Phi_i$. Now define $g : \{0, 1, \ldots, n\} \rightarrow \{0, 1\}$ as follows. For each $k \in \{0, 1, \ldots, n\}$, let $g(k) := f(d_1, d_2, ..., d_n)$ where $d_i := 1$ for $i \leq k$ and $d_i := 0$ for $i > k$. Since F satisfies (A) in addition to (S), for any $(d_1, d_2, ..., d_n) \in \{0, 1\}^n$ and any permutation $\sigma: N \rightarrow N$, we have $f(d_1, d_2, ..., d_n) = f(d_{\sigma(1)}, d_{\sigma(2)}, ..., d_{\sigma(n)})$. Therefore, for any $(d_1, d_2, ..., d_n), (e_1, e_2, ..., e_n) \in \{0, 1\}^n$, if $|\{i \in N : d_i = 1\}| = |\{i \in N : e_i = 1\}|$ then $f(d_1, d_2, ..., d_n) = f(e_1, e_2, ..., e_n)$. This implies that, for any $(e_1, e_2, ..., e_n) \in \{0, 1\}^n$, $f(e_1, e_2, ..., e_n) = f(d_1, d_2, ..., d_n)$ where $d_i := 1$ for $i \leq |\{i \in N : e_i = 1\}|$ and $d_i := 0$ for $i > |\{i \in N : e_i = 1\}|$, and thus $f(e_1, e_2, ..., e_n) = g(|\{i \in N : e_i = 1\}|)$. Hence, for any $\{\Phi_i\}_{i \in N} \in \mathbf{D}$, $F(\{\Phi_i\}_{i \in N}) = \{\phi \in X : f(\delta_1(\phi), \delta_2(\phi), ..., \delta_n(\phi)) = 1\} = \{\phi \in X : g(|N_{accept\text{-}\phi}|) = 1\}$, as required.

Proof of claim 2. Let $k \in \{0, 1, \ldots, n\}$. Consider $\{\Phi_i\}_{i \in N} \in \mathbf{D}$ and $\phi \in X$ such that $|N_{accept\text{-}\phi}| = k$. Since we can construct a profile $\{\Phi_i\}_{i \in N}$ with this property in **UAD**, the required $\{\Phi_i\}_{i \in N}$ exists in **D** ($\supseteq$**UAD**). Since, by assumption, each Φ_i in $\{\Phi_i\}_{i \in N}$ is complete and consistent, we have $N_{accept\text{-}\neg\phi} = N \setminus N_{accept\text{-}\phi}$, and thus $|N_{accept\text{-}\neg\phi}| = n\text{-}k$. By claim 1, $\phi \in F(\{\Phi_i\}_{i \in N})$ if and only if $g(|N_{accept\text{-}\phi}|) = 1$; and $\neg\phi \in F(\{\Phi_i\}_{i \in N})$ if and only if $g(|N_{accept\text{-}\neg\phi}|) = 1$. Since F generates consistent collective sets of judgments, we have *not* both $\phi \in F(\{\Phi_i\}_{i \in N})$ and $\neg\phi \in F(\{\Phi_i\}_{i \in N})$, and thus $g(k) + g(n\text{-}k) \leq 1$. First consider the case $k \neq n/2$. Since F generates almost complete collective sets of judgments and $k \neq n/2$, at least one of $\phi \in F(\{\Phi_i\}_{i \in N})$ or $\neg\phi \in F(\{\Phi_i\}_{i \in N})$ holds, and thus $g(k)+g(n\text{-}k) > 0$. Therefore $g(k) + g(n\text{-}k) = 1$. Second consider the case $k=n/2$. Then $n\text{-}k=n/2$, and $g(k) = g(n\text{-}k)$. Since $g(k) + g(n\text{-}k) \leq 1$, we have $2g(k) \leq 1$. Therefore $g(k) = 0$, as required.

Proof of claim 3. Assume, for a contradiction, that there exist $k, l \in \{0, 1, \ldots, n\}$ such $k < l$ and $g(k) > g(l)$, i.e. $g(k) = 1$ and $g(l) = 0$. Construct a profile of complete, consistent and deductively closed individual sets of judgments, $\{\Phi_i\}_{i \in N}$, as shown in table 3.

	Ind. 1	Ind. 2	...	Ind. k	Ind. k+1	...	Ind. l	Ind. l+1	...	Ind. n
P	Yes	Yes	...	Yes	No	...	No	No	No	No
Q	Yes	Yes	...	Yes	Yes	...	Yes	No	No	No
$(P \wedge Q)$	Yes	Yes	...	Yes	No	...	No	No	No	No

Table 3

Note that $\{\Phi_i\}_{i\in N}$ satisfies unidimensional alignment and thus $\{\Phi_i\}_{i\in N}\in \mathbf{UAD} \subseteq \mathbf{D}$. By claim 1, for every $\phi\in X$, $\phi\in F(\{\Phi_i\}_{i\in N})$ if and only if $g(|N_{accept\text{-}\phi}|) = 1$. Since $|N_{accept\text{-}P}| = |N_{accept\text{-}(P\wedge Q)}| = k$ and $g(k) = 1$, we have P, $(P \wedge Q)\in F(\{\Phi_i\}_{i\in N})$. Since, by assumption, $F(\{\Phi_i\}_{i\in N})$ is deductively closed, we must also have $Q\in F(\{\Phi_i\}_{i\in N})$. However, $|N_{accept\text{-}Q}| = l$ and $g(l) = 0$, which implies $Q\notin F(\{\Phi_i\}_{i\in N})$, a contradiction. Therefore, for any k, $l\in \{0, 1, \ldots, n\}$, $k < l$ implies $g(k) \leq g(l)$, as required. ■

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