EXPLAINING QUANTUM SPONTANEOUS SYMMETRY BREAKING*

Chuang Liu[a]** and Gérard G. Emch[b]

[a] *Department of Philosophy, University of Florida, Gainesville, FL 32611, U.S.A*

[b] *All Souls College, University of Oxford, UK, & Department of Mathematics, University of Florida, Gainesville, FL 32611, U.S.A*

ABSTRACT

Two alternative accounts of quantum spontaneous symmetry breaking (SSB) are compared and one of them, the decompositional account in the algebraic approach, is argued to be superior for understanding quantum SSB. Two exactly solvable models are given as applications of our account -- the Weiss-Heisenberg model for ferromagnetism and the BCS model for superconductivity. Finally, the decompositional account is shown to be more conducive to the causal explanation of quantum SSB.

1. Introduction

The best known examples of spontaneous symmetry breaking (SSB) are found, many may think, in relativistic quantum field theory (QFT). For an up-to-date account of the mathematical state of the art see, for instance, Ojima (2003) and the references therein. Then why, one may ask, do we choose to discuss SSB in non-relativistic quantum statistical mechanics (QSM)? While the problems of SSB in QFT may challenge physicists or mathematicians and fascinate philosophers of physics, they are not, as we shall argue, the best problems (i.e. offering the best models) for understanding the nature of quantum SSB. The tentative and controversial nature of some of the famous results of SSB in QFT is often an impediment to such an understanding; and the paucity of exactly solvable and experimentally realizable models does not help either. In contrast, SSBs in infinite quantum thermo-systems are well understood and have simple and realizable models. It is not at all accidental that when it comes to the discussion of SSB per se, authors in the QFT literature often resort to the analogous examples in QSM, examples such as ferromagnetism.

If so, one may wonder, why can we not search for the meaning of SSB in classical models, which would be an even simpler task? As shown in a detailed study (cf. Liu 2003), classical SSB lacks several features that characterize *quantum* SSB. Therefore, part of the aim of this paper is to show how the essential features of SSB in QFT qua SSB manifest themselves clearly in systems of QSM, assuming that it is always preferable to study the simpler model, provided that all the essential features are captured, and none is left out.

Hence, in this paper we address the interpretative problems of explaining SSB in infinite quantum thermo-systems -- the proper subjects of QSM. The main interpretative questions that are relevant to understanding how quantum SSB is understood include:

1. Why do quantum SSBs occur only in infinite systems? What justifies the use of such systems?
2. Why do quantum SSBs occur if, and only if, the symmetries in question are not unitarily implementable? What does this mean physically?
3. Why must the degenerate fundamental states of a SSB system belong to unitarily inequivalent representations? What does this mean physically?[1]
4. What is the physics of quantum SSB, in contrast to that of classical SSB?

To those readers who may worry that little philosophical interest can be generated from the above questions, we respond briefly as follows. (i) The nature of quantum SSB (markedly different from classical SSB) is itself of interest to metaphysics, just as the nature of quantum measurement is. (ii) Quantum SSB is closely related to the problem of quantum measurement (in a trivial sense, a quantum measurement is a kind of SSB, but the question is whether there is a non-trivial sense that sheds light on the nature of quantum measurement). Without getting clear first on what quantum SSB is, the questions about the relation cannot even be properly formulated. (iii) We address the question of what it means to explain quantum SSB as a natural phenomenon, which concerns the nature of explanation.

2. Two accounts of quantum SSB (the no-go results for finite systems)

In a recent paper broaching the concept of SSB for philosophers, John Earman (2003) brings to our attention two main features of quantum SSB, which we combine into the following.

[RA] A symmetry, α, of a system is *spontaneously broken* if and only if
(i) it is *not unitarily implementable*;[2]
or
(ii) some of the fundamental states of the system related by α generate (via the GNS construction) *unitarily inequivalent representations* of the algebra.

[RA] constitutes the first of two accounts of quantum SSB we examine in this paper, whose content will be explained step by step as we proceed and which we shall call the *representational account* (hence, [RA]). First let us see why there is no possible SSB in *finite* quantum systems (a 'no-go' theorem) according to [RA] (cf. Emch 1977). The theoretical framework for this account is the algebraic approach to quantum physics, which is suitable for both QFT and QSM.[3] For the algebraic account of a quantum system, we begin with an infinite n-dimensional Euclidean world, X. For the sake of simplicity, we shall study the algebras on a lattice world, $X = \mathbf{Z}^n$, unless explicitly stated otherwise. To each finite region, $\Lambda \subset X$, we associate a C*-algebra $\mathcal{A}_\Lambda$, the self-adjoint elements of which are interpreted as the observables relative to Λ. On these algebras, a corresponding state ω_Λ is given that associates to any observable, A_Λ, a real number, $\left\langle \omega_\Lambda ; A_\Lambda \right\rangle$, which is its expectation value. These algebras are most instructively studied in their representations as algebras of operators acting on Hilbert spaces. To be effective, representations *must* be tailored to fit the specific physical situations at hand which are given by the states (more on this point when we discuss the GNS construction of representations in Section 4).

The ('no-go') argument may now be given as follows (some technical terms are explained later). First, for any finite region Λ, all irreducible representations of $\mathcal{A}_\Lambda$ are

unitarily equivalent, i.e. for any two irreducible representations of $\mathcal{A}_{\Box}$, $\Box^{k}_{\Box}$:
$\mathcal{A}_{\Box}$ □ $\mathcal{B}(\mathcal{H}^{k}_{\Box})$, $k = 1,2$, (where $\mathcal{B}(\mathcal{H}_{\Box})$ is the set of all bounded operators on the Hilbert space associated with □), there is a unitary transformation, $V : \mathcal{H}^{1}_{\Box}$ □ $\mathcal{H}^{2}_{\Box}$, such that $\Box^{2}_{\Box}(A) = V\Box^{1}_{\Box}(A)V^{\Box 1}$ for all A □ $\mathcal{A}_{\Box}$. Then, for any two states, $\Box^{k}$ ($k = 1,2$), on $\mathcal{A}_{\Box}$ and their corresponding GNS representations, $(\mathcal{H}_{\Box^1}, \Box_{\Box^1})$ and $(\mathcal{H}_{\Box^2}, \Box_{\Box^2})$, the two states are said to be spatially equivalent whenever the two representations are unitarily equivalent. An automorphism, $| |$, of $\mathcal{A}_{\Box}$ is said to be unitarily implementable with respect to a state □ if, and only if, $\Box \circ \Box$ is spatially equivalent to □, and similarly it is unitarily implementable with respect to a representation □ whenever the representation $| | \circ | |$ is unitarily equivalent to □. Since $\mathcal{A}_{\Box} = \mathcal{B}(\mathcal{H}_{\Box})$ (with $\mathcal{H}_{\Box}$ finite or infinite dimensional), all automorphisms of $\mathcal{A}_{\Box}$ are inner and thus unitarily implementable in every representation. Now, given [RA], there cannot be SSB in finite quantum systems.

This argument remains opaque until one understands why and how the lack of unitary implementability of the relevant automorphism is a case of SSB. As we will see in Section 4, this first account of quantum SSB is not nearly as helpful in facilitating such an understanding as the second account we now announce (cf. Emch 1977).

[DA] A symmetry □ □ $Aut(\mathcal{A})$ of a system S is *spontaneously broken* in a $| |$-invariant KMS state □ on $\mathcal{A}$ if and only if

(i) □ is *not extremal KMS*,

and

(ii) some of its extremal KMS components -- in a unique convex decomposition of □ into extremal KMS states -- are *not □-invariant*.

The details of [DA] will be discussed in the next two sections; for now one only needs to remember that a KMS state is one of the fundamental states mentioned in [RA] (roughly, it is the canonical equilibrium state in QSM); and moreover, in order to *detect* the SSB in a system, one must find a *witness* among the macroscopic observables associated with □ (see [W] in Section 4). We shall call this account the *decompositional account* (hence, [DA]).

In this account, there is also a 'no-go' theorem for finite systems, i.e. systems in □; it runs as follows. For finite systems, the Hilbert space formalism of quantum mechanics suffices. That means that the algebra of observables can be written as $\mathcal{A}_{\Box} = \mathcal{B}(\mathcal{H}_{\Box})$. Let the Hamiltonian H be a self-adjoint operator acting on $\mathcal{H}_{\Box}$ with, for the sake of convenience, a discrete spectrum $Sp(H) = \{\Box\}$, such that the partition function,

$Z \equiv \sum_i exp(-\beta \varepsilon_i)$, converges for all $0 < \beta = 1/kT < \infty$. The most general states in $\mathcal{A}_\Lambda$ are then positive, normalized, linear functionals on $\mathcal{B}(\mathcal{H}_\Lambda)$, which are of the form, $\omega = \omega_1 + \omega_2$, where ω_1 vanishes on the compact operators, and ω_2 is countably additive. Therefore, we only need to consider density matrices, dear to von Neumann and apt for the Hilbert space formalism. Let ρ be a density matrix, i.e. a positive operator of trace-class on $\mathcal{H}_\Lambda$ with $Tr(\rho) = 1$. It is proven in Appendix A that a state $\omega : A \in \mathcal{A}_\Lambda \mapsto Tr(\rho A) \in \mathbf{C}$ is KMS if, and only if, its entropy, $S(\rho) = -Tr(\rho ln \rho)$ is maximal, given the constraint $Tr(\rho H) = E$ (the total energy). And then $\rho = exp(-\beta H)/Tr[exp(-\beta H)]$ is the Gibbsian distribution. From the RHS of the expression for ρ we see immediately that for any symmetry, $\alpha[A] = VAV^{-1}$, such that $[V, H] = 0$, we have $[V, \rho] = 0$ as well. In other words, there is no possible (equilibrium) states of the system that can fail to commute with, and hence break, any symmetry of the Hamiltonian.

3. Infinite systems and KMS states

In theory, SSB becomes possible only in infinite systems. This fact holds for classical thermo-systems as well as quantum thermo- or field-systems. Both conceptually and technically, taking the macroscopic limit is by no means a trivial matter.[4] If one finds it conceptually difficult to stomach the notion of using infinite (model) systems to study the behavior of real systems that one would think are obviously finite, let us begin with the following observation. Taking a finite system to the limit of infinite size is certainly an act of idealization, but regarding a system as finite, which assumes either perfectly isolating walls or an infinite distance between systems, is also a drastic idealization. Unless specific surface or boundary effects are of interest, taking the infinite limit is (from a philosophical point of view) more sensible than not taking it. However, it is one thing to regard taking the macroscopic limit a desirable or acceptable idealization, it is quite another to find out that it is necessary for the emergence of SSB. We assume here that the quantum thermo-systems to be considered are such that the macroscopic limit has been implemented, and we have a triple, $\{\mathcal{A}, \omega, \alpha(t)\}$, where $\mathcal{A}$ is a C*-algebra of observables, ω a state on $\mathcal{A}$, and α a one-parameter group of automorphisms of $\mathcal{A}$ representing the time evolution (for more on taking the macroscopic limit, see Liu 2001; Batterman 2002, 2004; Belot 2004).

We can now proceed to define equilibrium states and thermodynamic phases of a quantum thermo-system. In the following, $\langle \omega; A \rangle$ denotes as usual the expectation value of the observable A in the state ω, and $\alpha(t)[B]$ the time-evolute at time t of the observable B under the time-evolution α. Accordingly, $F_{AB}(t) = \langle \omega; A\alpha(t)[B] \rangle$ and

$F_{AB}(t+i\beta) = \langle \varphi; \alpha(t)[B]A \rangle$ denote time correlations. The single most important concept for the following discussion is that of a KMS state, whose formal definition is given below.

[KMS]: Given a C*-algebra $\mathcal{A}$ and a group of automorphisms $\alpha : t \in \mathbf{R} \mapsto \alpha(t) \in Aut(\mathcal{A})$, a state φ on $\mathcal{A}$ is said to be KMS with respect to $\alpha(\mathbf{R})$ if:

(i) $\langle \varphi; A\alpha(t)[B] \rangle$ and $\langle \varphi; \alpha(t)[B]A \rangle$ are continuous in t, $\forall A, B \in \mathcal{A}$;

(ii) $\forall A, B \in \mathcal{A}$, there exists a function, F_{AB}, defined on the closure of the strip $S_\beta = \{z \in C \mid Im(z) \in (0, \beta)\}$ and analytic inside it, such that

$F_{AB}(t) = \langle \varphi; A\alpha(t)[B] \rangle$ and $F_{AB}(t+i\beta) = \langle \varphi; \alpha(t)[B]A \rangle$, $\forall t \in \mathbf{R}$.

(Here, and through out this paper, the units are chosen such that $\hbar = 1$.)

Note that if φ is a KMS state on $\mathcal{A}$ with respect to $\alpha(\mathbf{R})$, then $\varphi \circ \alpha(t) = \varphi$ for all $t \in \mathbf{R}$, which is a sense of mininal stability of the system under time evolution. A large body of evidence is known to support taking KMS as the definition of canonical equilibrium states in QSM, such as that a KMS state satisfies both local and global thermodynamical stability conditions against perturbations of state (cf. Sewell 2002, 113-123). Further, since every KMS state is uniquely decomposable into *extremal* KMS states, (i.e. states that are not decomposable to other KMS states), thermodynamic phases are then naturally defined as extremal KMS states.

4. Two accounts of quantum SSB (continued)

We now argue that the decompositional account, [DA], is better than the representational account, [RA], as far as understanding the nature of quantum SSB is concerned, and we show in some detail how the former explains quantum SSB.

Even if the problems and solutions concerning SSB in QFT were more important than the ones in QSM, QFT would *not* be the best realm in which an understanding of the concept of quantum SSB should be sought. In this respect, we argue that the decompositional account in QSM has *at least six advantages*. *First*, unlike the classical model of SSB that cannot explain the essential features of SSB in quantum systems (in QFT or QSM), this account certainly can. *Second*, there is a great deal of similarity between an infinite quantum thermo-systems and a quantum field; so much so that arguments by analogy from the former to the latter are commonly used in the literature of QFT (cf. Anderson 1963; Coleman 1975). In a nutshell, to understand SSB in QFT one must deal with both (i) an infinite number of degrees of freedom, and (ii) (special, or even general) relativity. Our view is that (i) is essential to the understanding of quantum SSB whereas (ii) is not; and the complexity introduced by (ii) tends to distract attention and/or obscure discussions of the SSB concept. *Third*, unlike the representational account in which the physical interpretations of the two features of quantum SSB -- the lack of unitary implementability and unitarily inequivalent representations -- are either obscure or not forthcoming, this account [DA] almost wears their physical meanings on its sleeves.[5] *Fourth*, adopting [DA] restitutes the correct conceptual order of explanation (or understanding): the two features in [RA] may be sure signs of a quantum SSB, but they should not be taken to define the concept. It is rather because a quantum system has SSB that it exhibits the two features of [RA]; not the other way around (more on this in Section 6). In the decompositional account, [DA] looks like a proper defining feature of quantum SSB, and as we will see the account shows how a quantum system satisfying [DA] will also have the two features in [RA], thus setting the right conceptual order. *Fifth*, there are simple and experimentally implementable models in QSM for the decompositional account, while models in QFT for the representational account are either simple but unrealistic (i.e. not experimentally realizable) or realistic but complex (i.e. too complicated to be used to exhibit in controllable or simple terms what the account means).[6] And finally, *sixth*, [DA] has the advantage over [RA] in making it clear that the essence of SSB in general, for any systems, is *the coexistence of solutions, each of which has less symmetry than the symmetry of the problem, while the symmetry of the problem acts transitively on the set of all solutions.* The two features in [RA] are but alternative mathematical characterizations of the above, namely, the symmetry associated with such solutions are not unitarily implementable, and some of these solutions generate unitarily inequivalent representations.

Let us assume next that the existence conditions for an infinite quantum thermo-system are met and we have $\{\mathcal{A},\omega,\alpha(t)\}$ with ω being KMS (see Section 3). As we said earlier, algebras as abstract objects are best studied and understood in a representation that is tailored to the physical situation in question summarized here by the state ω. A representation in general, $(\mathcal{H},\pi)$, of $\mathcal{A}$ comprises a separable Hilbert space $\mathcal{H}$ and a morphism, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, of $\mathcal{A}$ into the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ of $\mathcal{H}$. The meaning of tailoring a representation to the physical situation at hand can be seen in the result (proven by Gelfand, Naimark, and Segal, hence GNS), which says that every state ω on $\mathcal{A}$ induces uniquely (up to unitary equivalence) a representation, π_ω, of $\mathcal{A}$ in a Hilbert space, $\mathcal{H}_\omega$, with a cyclic vector, Ω_ω, such that $\langle\omega;A\rangle = (\Omega_\omega,\pi_\omega(A)\Omega_\omega)$, $\forall A \in \mathcal{A}$.[7] (To relieve notational congestion, we drop the subindex $_\omega$.) Given a GNS representation $\pi(\mathcal{A})$ of $\mathcal{A}$, we consider the algebra, $\pi(\mathcal{A})'$, that comprises all the bounded operators in $\mathcal{H}$ that commute with all elements in $\pi(\mathcal{A})$, and we enlarge $\pi(\mathcal{A})$ to $\pi(\mathcal{A})''$, the bicommutant of $\pi(\mathcal{A})$, that comprises all the bounded operators that commute with all elements in $\pi(\mathcal{A})'$. The bicommutant set, $\pi(\mathcal{A})''$, which is known as a von Neumann algebra, is of special importance because it is the *weak-operator closure* of $\pi(\mathcal{A})$, which means that unlike the abstract $\mathcal{A}$, it is anchored to a specific representation -- hence to a specific physical situation at hand -- and contains macroscopic observables while $\mathcal{A}$ only contains quasi-local ones, i.e. observables that can be approximated by local observables in the norm topology (whereas, the macroscopic observables require the much more encompassing weak-operator topology).

Because of its central importance, we let $\mathcal{N} = \pi(\mathcal{A})''$. Among various subsets of $\mathcal{N}$ is its center, $\mathcal{Z} = \mathcal{N} \cap \mathcal{N}'$, which is important because it is the intersection of all complete sets of commuting operators -- these sets contain *all the observables that can be measured simultaneously without dispersion.* In other words, it contains the *essential observables* of the system in question (cf. Wick & Wightman & Wigner 1952).

We are now ready to state the *central decomposition* Scholium which is the core result for the decompositional account.

[CD] Let $\mathcal{A}$, ω, α such that $\omega \circ \alpha = \omega$, $\mathcal{H}, \pi, \mathcal{N},$ and $\mathcal{Z}$ be the objects as defined above. And suppose that for simplicity that the spectrum, $Sp(\mathcal{Z})$, is discrete, i.e. there is a partition $\{P_k\}$ of the identity in $\mathcal{H}$ by projection operators, P_k, such that $\mathcal{Z} = \{Z = \sum_k c_k P_k \mid c_k \in \mathbf{C}\}$.

Then, for $\mathcal{N}_k = P_k \mathcal{N} (= P_k \mathcal{N} P_k)$ and ω_k defined, $\forall N \in \mathcal{N}$, by $\langle \omega_k ; N \rangle = \langle \omega; P_k N P_k \rangle / \langle \omega; P_k \rangle$ ($\langle \omega; P_k \rangle \neq 0$), we have

1. $\mathcal{N} = \oplus \mathcal{N}_k$;
2. $\omega = \sum_k \lambda_k \omega_k$ with $\lambda_k = \langle \omega; P_k \rangle > 0$, and $\sum_k \lambda_k = 1$;
3. either $\omega_k \circ \alpha = \omega_k$ or $\omega_k \circ \alpha = \omega_j$, $k \neq j$.

(For a more complete version of [CD] and the proof, see Appendix B; in particular, [CD]can be straightforwardly tailored to accommodate the case where $Sp(\mathcal{Z})$ is not necessarily discrete.) [CD] is a scholium for a degenerate state that is uniquely decomposable into non-degenerate ones.[8]

From [CD], if we add that the system in question is not extremal KMS *and* it is the case that $□_k \circ □ = □_j, k \neq j$, we have the decompositional account of quantum SSB, i.e. [DA]. The first condition satisfies (i) in [DA], and the second satisfies (ii) in [DA] because condition (2) in [CD] above gives a unique convex decomposition of □ into extremal KMS states.

The notion of witness concerns the essential observables in the center of von Neumann algebra. For QFT, they are usually various 'charges,' such as the electric charge or the baryon number. In a translation-invariant QSM, we have *macroscopic* observables, which are defined as space-averages of local observables. As they obtain only as weak-operator limits, they belong to $\mathcal{N} = □(\mathcal{A})□$, where □ is the GNS representation associated with the translation-invariant state □ in question. They also belong to $\mathcal{N}□ = □(\mathcal{A})□$ as a consequence of causality, namely the fact that local observables that belong to spatially disjoint regions commute among themselves. These observables provide us with witnesses of SSB, which in general are defined as follows.

[W] An observable $W □ \mathcal{N}$ is a *witness* for SSB if it satisfies the following:

1. $W □ Z$;
2. $\langle □_k ; W \rangle \neq \langle □_j ; W \rangle$, for some $k \neq j$.
3. W has a natural physical interpretation (e.g. being a measurable property).

In physical terms, if the state is invariant under a translation group, say $\mathbf{Z}^n$ (for a lattice system) or $\mathbf{R}^n$ (for a continuous system), the locality condition requires that macroscopic observables (as space-averages of local observables) belong to the center of the GNS representation of □, which means that they satisfy points (1) and (3) above. And an example of (2) is the spontaneous magnetization of a lattice spin-1/2 system, namely, below the critical temperature, there are two values of magnetization, $\langle □_{\pm} ; m \rangle \neq 0$, (corresponding to spin up or spin down) that show us -- as witnesses do -- the presence of SSB.

Two questions arise from the above characterization of quantum SSB. (i) Does this also satisfy the two features given in [RA] so that it is also sufficient (though perhaps not necessary) for the alleged SSBs in QFT? (ii) In what exact sense is this a case of SSB for infinite quantum systems?

Because, as we mentioned earlier (see also Earman 2003), it is true for C*-algebra in general that an automorphism □ of $\mathcal{A}$, mapping one state to another, is unitarily implementable if the GNS representations of the two states are unitarily equivalent, we only need to find out whether if [DA] then one or the other condition in [RA] holds. It turns out that the answer to the latter is affirmative. Assume [DA] and take the simplest case (where

$k = +, \square$): $\square_{+}$ and $\square_{\square}$, where $\square_{\pm} \circ \square = \square_{\mp}$. Since the two states are extremal KMS states, the corresponding GNS states are primary, which means their centers are trivial (multiples of identity). Suppose (for reductio) that the two representations were unitarily equivalent. We then would have a unitarily operator, U, such that $U W_{+} U^{\square 1} = W_{\square}$, where the $W_{\pm} = \langle \square_{\pm}; W \rangle P_{\pm}$ are the corresponding witnesses in the two centers; but that means $\langle \square_{+}; W \rangle = \langle \square_{\square}; W \rangle$, which contradicts point (2) in [W]. This shows that *all quantum SSB systems in sense [DA] are also SSB systems in sense [RA].*

The demand for a witness in the condition for SSB is a general feature pertinent to quantum systems. In a classical system, it is sufficient if the symmetry in question maps one lowest-energy state to a different such state, since states of classical systems are determined by the values (precise or average) of the observables. For a quantum system, it is possible that the symmetry in question maps one extremal KMS state, e.g. $\square_{k}$, to another state, e.g. $\square_{j}$ (where $k \neq j$), and yet there is no observable in the center that satisfy both (2) and (3) of [W].

Further, the result of the decomposition is a mixture, not a coherent superposition; therefore, when a system crosses the critical value of the relevant parameter towards a SSB, the symmetry in question is *broken*, e.g. the system is in one of $\square_{k}$ not in a coherent superposition of these states. The probabilities are of course given by e.g. $\square_{k} = \langle \square; P_{k} \rangle > 0$.

For SSB in QSM, in addition to the possibility of a central decomposition, it is necessary for the systems to be in a KMS state; for otherwise the decomposition, though unique, is not necessarily one of extremal KMS states, which represent pure thermodynamical phases. One should also note that although the central decomposition of a KMS state is unique, a decomposition of such into primary states is not. A different condition has yet to be found for quantum field-systems that play a similar role. Whatever that condition is, it cannot be something such as the lack of unitary implementability or the inequivalent of representations, because none of them has been conceived as a condition for guaranteeing that the central decomposition of a state of quantum field results in states that spontaneously break the symmetry in question. Unlike the KMS condition, which can be proven to be equivalent to a condition of stable equilibrium states, the conditions of unitary implementability (or the lack of it) and unitary inequivalent of representations lack such direct physical meaning.

5. Models for the decompositional account

Among the various models of quantum SSB -- which may be used to show the *consistency* of our account -- the BCS model for superconductivity is probably the most famous; recently, it even caught the attention of philosophers of science (see, e.g. Cartwright

1999). It occupies an important place in the history of physics, and it is a case of the breaking of a U(1) gauge group (NB: the popular SSBs in QFT are also of gauge groups); however, since the symmetry group is continuous, which requires technical precautions, we first discuss a simple discrete model: the Weiss-Heisenberg model for (anti)ferromagnetism (cf. Emch & Knops 1970; Streater 1967; Bianchi et al 2004).

To make it simple and yet still sufficiently general, we take the model to consist of a one-dimensional chain of quantum spins with a long-range, Ising-type interaction.[9] At each site k in the chain, $\mathbf{Z}$, sits a particle with a quantum half-spin, σ_k, whose components are the three Pauli matrices. These components as observables generate an algebra $\mathcal{A}_k$ at each site, which is a copy of the algebra $M(2,\mathbf{C})$ of 2×2 matrices with complex entries: $M(2,\mathbf{C}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbf{C}\}$. For any finite string Λ of sites along the chain, we have the algebra, $\mathcal{A}_\Lambda = \bigotimes_{k\in\Lambda} \mathcal{A}_k$, which is a copy of $M(2^{|\Lambda|},\mathbf{C})$; and an evolution described in the Heisenberg picture as,

$$\alpha_{\Lambda,B}(t) : A \in \mathcal{A}_\Lambda \mapsto \alpha_\Lambda(t)[A] = e^{iH_{\Lambda,B}t} A e^{-iH_{\Lambda,B}t} \in \mathcal{A}_\Lambda, \qquad (1)$$

where we choose the ferromagnetic Weiss-Ising Hamiltonian:

$$H_{\Lambda,B} = -\sum_{k} [B + B_{\Lambda,k}]\sigma_k^z, \text{ with } B_{\Lambda,k} = \frac{1}{2}\sum_{j} J_{\Lambda,jk}\sigma_j^z, \qquad (2)$$

$$\text{where } J_{\Lambda,jk} = \begin{cases} 0, & \text{if } j = k \\ 2J/|\Lambda| > 0, & \text{if } j \neq k \end{cases}.$$

Here, B is an external magnetic field in the z-direction and $J_{\Lambda,jk} \geq 0$ the strength of interaction between any two spins which favorizes their parallel, rather than anti-parallel, alignment. And we use $B_{\Lambda,k}$ to indicate the magnetic field (surrounding the *k*-site) that derives from the averaged effect of spins on other sites. The Hamiltonian not only determines the evolution of the system, it also determines the canonical equilibrium state, $\varphi_{\Lambda,B,\beta}$, at natural (i.e. inverse) temperature $\beta = 1/kT > 0$:

$$\varphi_{\Lambda,B,\beta} : A \in \mathcal{A}_\Lambda \mapsto \langle \varphi_{\Lambda,B,\beta}; A \rangle = Tr\,(\rho_{\Lambda,B,\beta}A) \qquad (3)$$

$$\text{where } \rho_{\Lambda,B,\beta} = Z^{-1} e^{-\beta H_{\Lambda,B}} \text{ and } Z = Tr(e^{-\beta H_{\Lambda,B}}).$$

So far, the model only comprises finitely many spins; and hence (3) is the unique state on the local algebra, $\mathcal{A}_\Lambda = \bigotimes_{k\in\Lambda} \mathcal{A}_k$, that satisfies the KMS condition at natural temperature β with respect to the given time evolution $\alpha_{\Lambda,B}$. In addition, when $B = 0$, the system is invariant under the symmetry, [illegible], that is geometrically interpreted as a 180° rotation around the y-axis. One can check that this rotational symmetry is unitarily implementable (with the matrix $V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acting on each site); and when $B = 0$ the model is also invariant under rotations around the other two axes; and even when $B \neq 0$ the model is invariant under the rotation around the z-axis. This means that the components of the magnetization per site of the string Λ, $\mathbf{m}_\Lambda = |\Lambda|^{-1} \sum_k \sigma_k$, have the following properties: $\langle \omega_{\Lambda,B,\beta}; m^x_\Lambda \rangle = 0 = \langle \omega_{\Lambda,B,\beta}; m^y_\Lambda \rangle$ and $\lim_{B\to 0} \langle \omega_{\Lambda,B,\beta}; m^z_\Lambda \rangle = 0$, which in turns means that there is no spontaneous magnetization (or no SSB), agreeing with the general 'no-go' theorem.

It is proven that taking this model to the macroscopic limit -- $|\Lambda| \to \infty$ -- produces from (3) a well behaved state $\omega_{B\beta}$ on $\mathcal{A}$, where $\mathcal{A} = \overline{\bigcup_{\Lambda\subset\mathbf{Z}} \mathcal{A}_\Lambda}$ is a C*-algebra of quasi-local observables obtained as the C*-inductive limit of $\mathcal{A}_\Lambda$. The evolution at the limit, $\alpha(t)$, is defined on the von Neumann algebra $\mathcal{N}_{B,\beta} = \pi_{B,\beta}(\mathcal{A})''$ (not on $\mathcal{A}$ itself). It is then proven that the natural extension of $\omega_{B\beta}$ to $\mathcal{N}_{B,\beta}$ satisfies the KMS condition with respect to $\alpha(t)$. (For the sake of notational simplicity, we use $\omega_{B\beta}$ to denote both the state on $\mathcal{A}$ and its extension to $\mathcal{N}_{B,\beta}$.) And finally, the magnetization per site at the limit whose components are $m^{x,y,z}_{B\beta}$ is also well defined as follows. Because taking the limit $\Lambda \uparrow \mathbf{Z}$ for these components depends on the state of the chain, we have the following weak-limit:

$$w-\lim_{\Lambda\uparrow\mathbf{Z}} \pi_{B\beta}(m_\Lambda)^{x,y,z} = m^{x,y,z}_{B\beta},$$

which means that $\forall\varepsilon > 0$, and every pair (ϕ, ψ) of vectors in the Hilbert space of the representation, $\exists\Lambda_0$ with $|\Lambda_0| < \infty$, such that $\forall\Lambda \supseteq \Lambda_0$, $(\phi, [m^{x,y,z}_{\Lambda,B\beta} - m^{x,y,z}_{B,\beta}]\psi) < \varepsilon$. Hence, $m^{x,y,z}_{B\beta}$ is representation-dependent, and that is so because different states of such a system, which produce different GNS representations, give different expectation values for the magnetization. Further, because $m^{x,y,z}_{B\beta}$ commute with all observables in the algebra, they are in its center: $\mathcal{Z}_{B.\beta} \equiv N_{B,\beta} \cap N'_{B,\beta}$.

We can now see how the general concept of SSB according to the decompositional account (i.e. [CD], [DA], and [W] above) applies to this model. For [CD], the automorphism under consideration in the model is □ and the GNS-associated objects are all defined in terms of B and $\square$ because $\square_{B,\square}$ is. It is proven that (i) $\square_{B,\square}$ is extremal KMS with respect to the evolution $\square(t)$ if, and only if, $\square \square J^{\square 1}$, and (ii) if $\square_{B,\square}$ is extremal KMS, the three components $m_{B,\square}^{x,y,z}$ are as follows:

$$m_{B,\square}^{x} = 0 = m_{B,\square}^{y} \qquad (4a)$$
$$m_{B,\square}^{z} = tanh\{\square(B + Jm_{B,\square}^{z})\}, \qquad (4b)$$

where (4b) is a self-consistent equation -- i.e. the same variable $m_{B,\square}^{z}$ appearing on both sides of the equation. At fixed B, exactly one value of $m_{B,\square}^{z}$ satisfies (4b); it is positive (negative) when $B > 0$ ($B < 0$), and $\lim_{B \square 0} m_{B,\square}^{z} = 0$. In other words, there is no SSB when $\square \square J^{\square 1}$. However, when $\square > J^{\square 1}$ and $B = 0$, we have,

$$\square_{0,\square} = \frac{1}{2}(\square_{0,\square}{}^{+} + \square_{0,\square}{}^{\square}) \qquad (5)$$

(which is clearly a case of $\square^{k}$, with $k = +, \square$), such that

1. $\square_{B,\square}{}^{\pm} \circ \square = \square_{B,\square}{}^{\mp}$,
2. $\square_{0,\square}{}^{\pm}$ are extremal KMS and the corresponding GNS representations, $\square_{0,\square}{}^{\pm}$, are primary sub-representations of $\square_{0,\square}$.

And since $m_{\square}^{\pm} = \left\langle \square_{0,\square}{}^{\pm}; m_{0,\square}^{z} \right\rangle = P^{\pm}\mathbf{m}_{0,\square}P^{\pm}$, and (from (4b)) $m_{\square}^{\pm} = tanh(\square J m_{\square}^{\pm})$, we have

3. $m_{\square}^{\pm}$ are in the center, $Z_{0.\square} \equiv N_{0,\square} \square N_{0,\square}^{\square}$;
4. $m_{\square}^{\pm}$ has two non-zero solutions: $m_{\square}^{\pm} = \square m_{\square}^{\mp}$; they serve as our witnesses for the SSB ('witness' as defined in [W]).

And hence,

5. the two representations, $\square_{0,\square}{}^{\pm}$, for the states, $\square_{0,\square}{}^{\pm}$, are unitarily inequivalent.

The last result, 5, follows from our earlier general discussion of the decompositional account of SSB. This model of the 'flip-flop' symmetry, $\square$, *therefore* satisfies [DA]; and moreover, it *also* satisfies [RA] because the corresponding GNS representations, $\square_{0,\square}{}^{\pm}$, are unitarily inequivalent (which implies that the symmetry is not unitarily implementable). And the conceptual advantage of [DA] is 4 above, which involves explicitly a macroscopic observable: the magnetization.

We now turn to a brief sketch of the BCS model, where the interaction between electrons and phonons in a metallic superconductor is mimicked by the interacting Cooper pairs, i.e. pairs of electrons, close to the Fermi surface, of opposite momenta and spin orientations (cf. Schrieffer 1964). Its finite-volume Hamiltonian in a box Λ with volume V is

$$H_{\Lambda} = \sum_{p,s} \varepsilon(p) a_s(p)^* a_s(p) + \sum_{p,q} b(p)^* V(p,q) b(q) \qquad (6)$$

where $a_s(p)^*$ is the creation operator of electrons with momentum p and spin s, and $a_s(p)$ the correponding annihilation operator; and therefore the first term of H_{Λ} describes the free energy of the electrons. Similarly, $b(p)^* = a_{\uparrow}(p)^* a_{\downarrow}(-p)^*$ is the creation operator of Cooper pairs; and $b(p)^* V(p,q) b(q)$ describes the interactional energy between two Cooper pairs.

In the *mean-field approximation*, the Hamiltonian (6) of the BCS model is diagonalized by a Bogoliubov-Valatin transformation,

$$\xi_{\uparrow}(p)^* = u_{\beta}(p)^* a_{\uparrow}(p)^* - v_{\beta}(p)^* a_{\downarrow}(-p)$$
$$\xi_{\downarrow}(-p) = v_{\beta}(p) a_{\uparrow}(p)^* + u_{\beta}(p) a_{\downarrow}(-p),$$

to give,

$$H_{\Lambda,\beta} = \sum_{p,s} E_{\beta}(p) \xi_{s,\beta}(p)^* \xi_{s,\beta}(p), \qquad (7)$$

which means that the system can now be viewed as having free elementary excitations created and annihilated by $\xi_{s,\beta}(p)^*$ and $\xi_{s,\beta}(p)$, respectively, with the energy

$$E_{\beta}(p) = [\varepsilon(p)^2 + \Delta_{\beta}(p)^* \Delta_{\beta}(p)]^{1/2}; \qquad (8)$$

originally, u, v, and Δ were considered to be complex-valued functions. In particular, the observable 'energy-gap' is obtainable from the self-consistent equation

$$\Delta_{\beta}(p) = -\sum_{q} V(p,q) \frac{\Delta_{\beta}(p)}{2E_{\beta}(p)} tanh(\frac{1}{2}\beta E_{\beta}(p)). \qquad (9)$$

There is a critical tempreture, $T_c = (k\beta_c)^{-1}$, such that

(a) for $T \geq T_c$, equation (9) has no non-zero solution; and

(b) for $T < T_c$, a non-zero solution appears, which can be computed numerically and which agreed with the available data in 1957 well enough to be regarded as giving a satisfactory explanation to the then prominent and yet puzzling phenomenological features of superconductivity.

Moreover, one notices that the Bogoliubov-Valatin transformation, with the u_κ and v_κ complex-valued, leads to a *violation* of the gauge-invariance of the theory. Specifically, the action of the gauge group, $S^1 = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$, is defined by

$$a_s(p) \to \alpha(\theta)[a_s(p)] = e^{+i\theta} a_s(p); \quad a_s(p)^* \to \alpha(\theta)[a_s(p)^*] = e^{-i\theta} a_s(p)^* . \qquad (10)$$

Clearly, the Hamiltonian (6) *is* invariant under this symmetry group, whereas the Hamiltonian (7) is *not*. This is indication that superconductivity is a SSB of gauge symmetry; it is however not properly accounted for in this mean-field approach for finite systems.

A flurry of researches in the 1960s reformulated the BCS model as an *infinite* system. The claims are that the mean-field approximation becomes *exact* in this limit, and that the system exhibits a controllable spontaneous breaking of a gauge symmetry (cf. Haag 1962; Ezawa 1964; Emch & Guenin 1966; Thirring 1969; Dubin 1974, ch. 4). In this approach, the algebra appropriate for the BCS model is the algebra, $\mathcal{A}_\Lambda$, associated with a representation, π_Λ, of the Fermi anti-commutation relations that the elementary excitations satisfy, i.e.

$$\{a_s(f), a_{s'}(g)\} = 0 \qquad \text{and} \qquad \{a_s(f), a_{s'}(g)^*\} = (f,g)\delta_{ss'} I \qquad (11)$$

where f and g run over $L^2(\mathbf{R}^3, dx)$, and (f,g) is the scalar product in this space; in particular, $p = n(2\pi / V^{1/2})$; $n \in \mathbf{Z}^3$ and $a_s(p) \equiv a_s(f_p)$ with

$$f_p(x) = \begin{cases} V^{-1/2} e^{ixp} & x \in \Lambda \\ 0 & otherwise \end{cases} .$$

The key to the understanding of how the mean-field Hamiltonian (7) becomes an exact diagnization of the BCS Hamiltonian (6) in the limit $|\Lambda| \to \infty$ is to rewrite (6) as

$$H_\Lambda = \sum_{p,s} \varepsilon(p) a_s(p)^* a_s(p) + \frac{1}{2} \sum_p \left[b(p)^* \beta_\Lambda(p)^* b(p) \right]$$

where $\beta_\Lambda(p) = \sum_q \lambda_\Lambda(p,q) b(q)$ so that the canonical commutation relations (11) entail

$$[\beta_\Lambda(p), a(q)] = 0 \qquad \text{and} \qquad [\beta_\Lambda(p), a(q)^*] = \lambda(p,q).$$

One needs now to recognize two mathematical facts. First, the physical constraints on the interaction between the Cooper pairs entail that there exists a constant γ such that

$$|\lambda(p,q)| < \frac{\gamma}{|\Lambda|} \qquad \text{so that} \qquad \lim_{|L| \to \infty} |\lambda(p,q)| = 0.$$

And second,

$$\beta_\omega(p) = w - \lim_{|\Lambda| \to \infty} \beta_\Lambda(p)$$

exists in the representation space associated with the canonical equilibrium state ω of the infinite BCS model. As a consequence, $\beta_\omega(p)$ belongs to the center of the von Neumann algebra obtained from this representation. Below the critical temperature T_c this representation turns out to be a direct integral over primary representations that correspond to the pure thermodynamical phases of the model. In each of these phases, $\beta_\omega(p)$ is a pure number (more precisely a scalar multiple of the identity operator) and satisfies the self-consistency equation (9).

Note further that for each $T \geq T_c$ the solution (9) is unique, *up to a complex number of modulus 1*. This ambiguity is essential; indeed while the gauge symmetry is broken in each phase, the gauge group acts non-trivially on the center of the integral representation π_ω; in particular

$$\beta_\omega(p) \to \alpha(\theta)[\beta_\omega(p)] = e^{2i\theta} \beta_\omega(p),$$

so that the gauge group acts transitively on the space of pure thermodynaimcal phases, as should happen in accordance with [DA]: one gets from any one of the component to any other by the action of an element of the gauge group S^1. Similarly, in connection with the Bogoliubov-Valatin elementary excitations, we have

$$\square(\square)[u(p)] = u(p) \qquad \text{and} \qquad \square(\square)[v(p)] = e^{2i\square}v(p)$$

so that

$$\square(\square)[\square_s(p)] = e^{i\square}\square_s(p) \qquad \text{and} \qquad \square(\square)[\square_s(p)^*] = e^{\square i\square}\square_s(p)^*$$

and the gauge symmetry of the evolution in the representation space of $\square_{\square}$ is preserved. It is only the decomposition into pure thermodynamical phases that breaks the gauge symmetry.

6. Explaining quantum SSB

To further argue for the decompositional account [DA] we turn to the question of what constitutes an explanation of quantum SSB. We first briefly state the two senses of explanation that are involved in explaining classical SSB, and we then argue that the representational account [RA] can only be said to 'explain' quantum SSB in one of these two senses while the decompositional account explains it in both.

The essential features of classical SSB are culled from some simple mechanical systems in which some symmetry, say a rotational symmetry around the z-direction, is spontaneously broken because when the value of certain parameter passes a critical point, the original single fundamental state become unstable; and without any apparent asymmetrical causal antecedents, the multiple possible stable fundamental states individually break the symmetry (cf. Liu 2003). The set of possible stable states is closed under the symmetry in that $\forall s \in S$ *and* $\forall g \in G, g[s] \in S$, but $g[s] \neq s$ for some g and s, where S is the set of stable fundamental states and G the symmetry group. Here we have the '*structural/kinematic*' aspect (or sense) of classical SSB. It refers to the conditions under which a SSB is possible. The '*causal/dynamic*' aspect (or sense) contains those features that characterize how (typically) actual SSBs take place. SSBs are mostly, though not necessarily, caused by arbitrarily small random fluctuations, and the conditions (or mechanisms) responsible for making SSBs possible necessarily differ from those actually producing them (otherwise, the breakings can no longer be regarded as spontaneous). To separate the two aspects, we will label as *structural* an explanation of SSB that tells us under what conditions a SSB is possible (and the obtaining of the conditions certainly has its own causal explanation, e.g. by lowering the temperature across the critical temperature); and we label as *causal* an explanation that tells us the mechanisms for actual breakings: how systems harboring SSB end up in those symmetry-breaking fundamental states.

The causal aspect appears to be absent in both accounts (i.e. [RA] and [DA]) of quantum SSB; both seem to only tell us when a SSB is possible in an infinite quantum systems (a magnet or a quantum field). However, we shall argue that the decompositional account is more conducive to providing a causal (or dynamical) explanation for quantum SSB than is the representational account. The causal explanation of classical SSBs already essentially involves classical SM because what actually causes mechanical SSB systems to go from an unstable fundamental state to a set of stable ones -- some of which breaks the symmetry -- cannot be explicitly stated as initial conditions in a mechanical (non-statistical) theory (detailed arguments for this point can be found in Liu 2003). And what happens in the Weiss-Heisenberg model (and many other similar models[10]) can be roughly described as follows. Two opposing tendencies exist in a Weiss-Heisenberg system, namely, the one caused by the interactions to align the spins in the same direction and the other caused by thermal agitation to randomize the directions of individual spins. The strength of the latter obviously depends on the system's temperature and the strength of the former depends on how many spins are already aligned in parts of the system. When the temperature is above the critical value, the balance of the strengths is in favor of thermal agitation. Any large fluctuation of aligned segments of spins will quickly disappear rather than growing larger. But when the temperature drops below the critical value, the balance tilts in favor of interactional alignment of spins that will tend to grow larger and eventually result in spontaneous magnetization, i.e. having an average net magnetization even when no external field is present.

Since quantum SSB is a natural phenomenon (not merely a theoretical concept[11]), we deem it necessary for a theory of quantum SSB to provide a conception of how it is causally explained. In fact, authors who contemplate causal explanations for the SSBs in QFT models heavily depend on analogies from what we sketched above for the thermo-systems (cf. Anderson 1963; Streater 1965; Guralnik et al 1968; Coleman 1975). For instance, attributing the causal mechanism of SSB in a gauge field to a correlation or coherence of phases within a vacuum state is helpful when one recalls what actually happens in a phase transition in a thermo-system, but not so helpful without such an analogy (cf. Moriyasu 1983).[12] Therefore, although [DA] does not, and cannot, include the specific causal mechanism that spontaneously breaks a symmetry -- for the mechanisms may differ in different systems -- it better accommodates the inclusion of causal accounts by stating in explicit terms the physical consequence of all such mechanisms. It provides clues for, or constraints on, what such mechanisms may be like, whereas one does not see this in [RA]: neither not being unitarily implementable nor having fundamental states that generate unitarily inequivalent representations seems to provide any clue as to what may have caused a system to break its symmetry spontaneously.

Even if we were to disregard the demand for a causal explanation, the representational account would still be weaker because it is not clear that it even provides a clear conception of a structural explanation. Knowing that an automorphism is not unitarily implementable does not by itself tell, even structurally or formally, why the associated symmetry is spontaneously broken. It itself needs explanation more than it can explain. Granted, we do gain some understanding of quantum SSB (at least in the sense it is possible) in the notion of having, say, vacuum states generate unitarily inequivalent representations. We shall explain how this is so in the next section; but it seems obvious that from the decompositional account we gain a better understanding. What we learn from [DA], together with [CD], is how under the KMS condition, the appearance of a non-trivial center, i.e. $\mathcal{Z} = \{Z = \sum_k c_k P_k \mid c_k \in \mathbf{C}\}$, (together with having witnesses in it) makes a SSB possible. For since such KMS states decompose uniquely into sets of extremal KMS states and the representations associated with these extremal states are primary (i.e. having trivial centers), it is easy to see why these representations are unitarily inequivalent. If unitarily equivalent representations can be regarded as different mathematical descriptions of the same physical situation, it follows that two different extremal KMS states and their associated representations cannot be unitarily equivalent. Take the case of the Weiss-Heisenberg model. The two extremal states, spin-up, or spin-down, of the entire chain $\mathbf{Z}$, cannot be regarded as about the same physical situation. In other words, from the decompositional account one gets a structural explanation of how a SSB is possible, and *in addition*, one gets an account of why the states that spontaneously break the symmetry belong to unitarily inequivalent representations. As we said earlier, this is the right conceptual order in the explanation (even if only on the formal level) of SSB, which one does not see in [RA].

One may argue that we have misconstrued the representational account, especially when it concerns SSB in QFT. Is not the $P(\phi)_2$ field (quantum fields in a 2-d space-time) a concrete model in QFT, from which cases of SSB, strikingly analogous to the classical cases, can be derived (cf. Glimm & Jaffe 1981, Simon 1974)? To this we have two brief responses. First, this model provides a vivid picture of what a SSB looks like in a quantum-field model precisely because it bears a striking resemblance, as far as the geometry for the set of symmetry-breaking vacuum states are concerned, to the Weiss-Heisenberg model of ferromagnetism. Hence, one would think that it serves as a better illustration of [DA] than of [RA]. It is not even clear how the lack of unitary implementability and having vacuum states generating inequivalent representations play any crucial role in this QFT model of SSB. Second, unlike in the cases in which $\beta = 1/kT$ (or temperature T) is the control-parameter, the physical meaning of σ is not entirely clear, especially when $\sqrt{\sigma}$ is to be constantly interpreted as some kind of mass. Hence, even if we have in the $P(\phi)_2$ field model a control parameter, σ, that serves an analogous role as does $\beta = 1/kT$ in the Weiss-Heisenberg model and a double-dip curve when $\sigma < \sigma_c (\sigma_c = 0)$ that is characteristic of SSB, it is not clear what is physically responsible for the possibility of SSB in this case.

7. Conclusion

Even though a study of classical SSB systems is not sufficient for an understanding of SSB in general (for quantum SSB has its own unique features), it is neither necessary nor advisable to tackle the recondite cases in QFT; or so we have argued. Theories and models of SSB in QSM provide just the right material for the purpose. The discussion of two 'no-go' theorems for finite systems and Section 3 provides an answer to why quantum SSBs only occur in infinite systems (question 1 in Section 1). We discuss an account of what SSB is for quantum thermo-systems. We then give two concrete models to show that the account is indeed adequate. A connection is established between the decompositional account of SSB, which originates from QSM, and the representational account, which has been given for cases in QFT; and from this connection one can understand what is otherwise puzzling, namely, why SSB should be characterized as a symmetry that is not unitarily implementable or as having vacuum states (in QFT) that belong to unitarily inequivalent representations (an answer to questions 2 and 3). In section 6 we give an account of what it means to physically explain a quantum SSB (question 4).

Appendix A: KMS for the neophyte

Let H be a self-adjoint operator denoting the Hamiltonian, and assume that $\forall \beta > 0$ the partition function $Z = Tr\exp(-\beta H)$ is finite; let $\varphi : A \in \mathcal{B}(\mathcal{H}) \mapsto \langle \varphi; A \rangle \in \mathbf{R}$ be the canonical equilibrium state defined by $\langle \varphi; A \rangle = Tr\rho A$ with $\rho = Z^{-1}\exp(-\beta H)$. The evolution is defined $\forall t \in \mathbf{R}$ and $\forall A \in \mathcal{B}(\mathcal{H})$ by $\alpha_t[A] = \exp(iHt)A\exp(-iHt)$. Note that the state φ is time invariant and faithful.

Alternatively, consider the following computation, which uses only the invariance of the trace under cyclic permutation, and that $\exp(X)\exp(Y) = \exp(X + Y)$, for any pair of $X, Y \in \mathcal{B}(\mathcal{H})$ that commute with each other:

$$
\begin{aligned}
&Z^{-1}Tr\exp(-\beta H)\exp(iHt)X\exp(-iHt)Y = \\
&\quad Z^{-1}Tr\exp(-\beta H)Y\exp\{i(t+i\beta)H\}X\exp\{-i(t+i\beta)H\}
\end{aligned}
$$

i.e.

$$
Tr\rho\alpha_t[X]Y = Tr\rho Y\alpha_{t+i\beta}[X]. \qquad (1)
$$

This is the KMS condition in its most naive form. It is essential to note that it characterizes φ completely. Indeed, let σ be any density matrix and suppose that the normal state $\psi : A \in \mathcal{B}(\mathcal{H}) \mapsto Tr\sigma A$ satisfies (1). In the case that $Y = I$, the condition reads: $Tr\sigma\alpha_t[X] = Tr\sigma\alpha_{t+i\beta}[X]$. As this relation holds for $\forall X \in \mathcal{B}(\mathcal{H})$, it entails

$$
\exp(-iHt)\sigma\exp(iHt) = \exp\{-iH(t+i\beta)\}\sigma\exp\{iH(t+i\beta)\}
$$

i.e. $\exp(-\beta H)\sigma = \sigma\exp(-\beta H)$ and thus σ commutes with H. Consequently ψ is a stationary state, i.e. $\forall t \in \mathbf{R} : \psi \circ \alpha_t = \psi$; more generally, this can be seen as a consequence of the Liouville theorem which says that any analytic function bounded over $\mathbf{C}$ must be constant.

Now let $t = 0$ and ψ be substituted by φ in (1), we have,

$$
Tr\sigma XY = Tr\,\sigma\{Y\exp(-\beta H)X\exp(\beta H)\}.
$$

Since $Y \in \mathcal{B}(\mathcal{H})$ is arbitrarily chosen, the above entails: $\{\exp(\beta H)\rho\}X = X\{\exp(\beta H)\rho\}$, i.e. every $X \in \mathcal{B}(\mathcal{H})$ commutes with $\{\exp(\beta H)\rho\}$, which in turn entails that this operator is a multiple of the identity, or equivalently that $\rho = a\exp(-\beta H)$, where a is a constant that, when normalized ($Tr\rho = 1$) is Z^{-1}. Hence, the density matrix ρ (w.r.t. ω) is none other than the canonical equilibrium density matrix, just as ω is (w.r.t. β). Hence, for finite systems, to say that a state satisfies the KMS condition for $\beta > 0$ is equivalent to saying that it is the canonical equilibrium state for β.

Further remarks:

1. The KMS condition can be naturally generalized to infinitely extended systems (see Section 3).
2. For infinite systems, the KMS condition can be viewed unambiguously as a stability condition (cf. e.g. Sewell 2002 or Emch & Liu 2002).
3. For infinite systems, there may be more than one state that satisfy the KMS condition.
4. In any infinite systems the convex set of all states satisfying the KMS condition for the same temperature is a simplex, i.e. the KMS states decompose uniquely into a set of extremal KMS components. Physically, that KMS states are taken as equilibrium states supports the pragmatic expectation that equilibrium states decompose uniquely into a set of pure thermodynamical phases (see Appendix B below).

Appendix B: Central decomposition

To illustrate the idea at the core of the general theorem on central decomposition, we consider the following simpler result, which in itself is all that is needed for much of the arguments in the main body of the paper.

Scholium 1. *Suppose $\alpha \in Aut(\mathcal{A})$ and $\varphi \circ \alpha = \varphi$, which implies that α is unitarily implementable in the GNS representation $\pi(\mathcal{A})$ associated with φ. Then,*
1. $\forall N \in \mathcal{N} \equiv \pi(\mathcal{A})''$, $\alpha[N] \equiv U^{*}\pi(N)U$ extends α to an automorphism on $\mathcal{N}$, and $\langle \varphi; N \rangle = (\Phi, N\Phi)$ extends φ to a normal state on $\mathcal{N}$.
2. $\mathcal{Z} = \mathcal{N} \cap \mathcal{N}'$ is stable under α (i.e. $\forall Z \in \mathcal{Z}, \alpha[Z] \in \mathcal{Z}$), and the restriction of α on $\mathcal{Z}$, is also an automorphism.
3. If φ is KMS w.r.t. evolution α, then the extention to $\mathcal{N}$ is again KMS w.r.t. the extention of α.

Scholium 2. *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra admitting a cyclic and separating vector Φ such that its norm $\|\Phi\| = 1$. Assume further that the spectrum $Sp\,\mathcal{Z}$ of the center $\mathcal{Z} = \mathcal{N} \cap \mathcal{N}'$ is discrete. Then the normal state, $\varphi : N \in \mathcal{N} \mapsto (\Phi, N\Phi)$, can be written as a convex combination, $\varphi = \sum_k \lambda_k \varphi_k$, of primary states on $\mathcal{N}$.*

Proof: The discreteness of the spectrum of the center of the algebra means that there exists a partition of the identity, $\{P_k\}$, into mutually orthogonal projectors such that $\mathcal{Z} = \{Z = \sum_k z_k P_k \mid z_k \in \mathbf{C}\}$; in particular $\sum_k P_k = I$. Correspondingly $\mathcal{H} = \oplus \mathcal{H}_k$, where $\mathcal{H}_k = P_k \mathcal{H}$ is the closed subspace $\overline{\mathcal{N} P_k \Phi}$. Let $\mathcal{N}_k \equiv \pi_k(N)$ be the von Neumann algebra obtained as the restriction of $\mathcal{N}$ to the $\mathcal{N}$-stable subspace $\mathcal{H}_k$. Let now $\lambda_k \equiv \langle \varphi; P_k \rangle$; and note that since Φ is separating for $\mathcal{N}$ and therefore φ is faithful, $\lambda_k \neq 0$. Then

$$\varphi = \sum_k \lambda_k \varphi_k \quad with \quad \lambda_k > 0, \quad \sum_k \lambda_k = 1$$

and (1)

$$\varphi_k : N \in \mathcal{N} \mapsto \langle \varphi_k ; N \rangle = \frac{\langle \varphi; P_k N P_k \rangle}{\langle \varphi; P_k \rangle} \in \mathbf{C}.$$

$\square_k$ are positive, linear complex valued function on $\mathcal{N}$ and are normalized by $\langle \square_k ; I \rangle = I$; i.e. they are states on $\mathcal{N}$. Moreover, the above equality is a particular case of the following equality

$$\square N \,\square\, \mathcal{N} \; and \; \square Z = \square_j z_j P_j \,\square\, \mathcal{Z} : \langle \square ; ZN \rangle = \square_k z_k \square_k \langle \square_k ; N \rangle . \qquad (2)$$

Upon noticing that $\{\square_k, \mathcal{H}_k, \square_k \equiv P_k \square\}$ is the GNS triple associated with $\square_k$, the fact that $\square Z = \square_j z_j P_j \,\square\, \mathcal{Z} : \square_k(Z) = z_k P_k$ entails that the states $\square_k$ are primary. q.e.d.

Scholium 3. *If $\square$ is KMS, the central decomposition in Scholium 2 provides a unique decomposition into extremal KMS states.*

For the extension of these results to not necessarily discrete spectrum of the center, see Emch 1972, 215.

Appendix C: Equivalence of representations

Three notions of equivalence between representations of a C*-algebra are pertinent to this paper: (1) unitary equivalence, (2) quasi-equivalence, and (3) weak equivalence, which we briefly explain below; for details see Emch (1972) or Kadison & Ringrose (1986). These notions satisfy the following:

(1) |= (2) |= (3).

The notion of quasi-equivalence was discussed already in Mackey (1953); and Haag & Kastler (1964) argue, on the basis of a theorem by Fell (1960), that weak-equivalence ought to be identified as 'physical' equivalence. In a broader philosophical context, this interpretation was re-examined and endorsed by Clifton & Halvorson (2001); see also Ruetsche (2002).

Let $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})_i \; (i = 1,2)$ be any two non-zero representations of a C*-algebra $\mathcal{A}$; π_i is said to be primary whenever $\pi_i(\mathcal{A})' \cap \pi_i(\mathcal{A})'' = CI$, and it is said to be irreducible whenever $\pi_i(\mathcal{A})' = CI$. $\mathcal{N}_i \equiv \pi_i(\mathcal{A})''$ is the von Neumann algebra obtained as the weak-operator closure of $\pi_i(\mathcal{A})$ in $\mathcal{B}(\mathcal{H})_i$. $Ker\,\pi_i \equiv \{A \in \mathcal{A} \mid \pi_i(A) = 0\}$ denotes the kernel of the representation π_i. A C*-algebra is said to be simple whenever it admits no closed two-sided proper ideals; and then, in particular, $Ker\,\pi_i = \{0\}$.

Definition 1. *π_1 and π_2 are said to be* weakly equivalent *whenever* $Ker\,\pi_1 = Ker\,\pi_2$.

This condition can be variously reformulated in terms of the states associated with π_i, see e.g. Emch (1972) [Theorem II.1.7]; for the purpose of QSM, the most directly relevant reformulation may be the following result.

Scholium 1. *π_1 and π_2 are weakly equivalent whenever every density matrix associated with one representation can be approximated, point-wise on $\mathcal{A}$, but arbitrarily closely, by a net of density matrices associated with the other representation.*

Definition 2. *π_1 and π_2 are said to be* quasi-equivalent *whenever there exists a *-isomorphism $\alpha : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\alpha[\pi_1(A)] = \pi_2(A), \forall A \in \mathcal{A}$.*

Scholium 2. *π_1 and π_2 are quasi-equivalent whenever both of the following two conditions are satisfied: (i) π_1 and π_2 are weakly equivalent [and thus there exist a natural *-isomorphism $\alpha_0 : \pi_1(\mathcal{A}) \to \pi_2(\mathcal{A})$]; and (ii) α_0 extends to a *-isomorphism $\alpha : \mathcal{N}_1 \to \mathcal{N}_2$.*

Definition 3. *π_1 and π_2 are said to be* unitarily equivalent *whenever there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_2(A)U^{-1} = \pi_1(A), \forall A \in \mathcal{A}$.*

At the opposite extreme sits the following notion.

Definition 4. *π_1 and π_2 are said to be* disjoint *whenever every subrepresentation of π_1 is not unitarily equivalent to any subrepresentation of π_2, and vice versa.*

Scholium 3. *π_1 and π_2 are disjoint whenever they have no quasi-equivalent subrepresentations; conversely π_1 and π_2 are quasi-equivalent iff π_1 has no subrepresentation disjoint from π_2 and vice versa. Furthermore, two primary representations π_1 and π_2 are either quasi-equivalent or disjoint. Moreover, two irreducible representations are weakly equivalent iff they are unitarily equivalent, and thus iff they are quasi-equivalent. Finally, all representations of a simple C*-algebra are weakly equivalent.*

Scholium 4 (Takesaki 1970). *Two representations π_1 and π_2 that correspond to different temperatures are disjoint.*

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** Corresponding author. Email address: cliu@phil.ufl.edu.

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[1] The phrase, 'fundamental states,' denote the 'lowest energy states' in classical mechanics, the 'equilibrium states' in statistical physics, and the 'vacuum states' of quantum field theory. In the literature, 'ground states' are often used for such a purpose, but the term can be misleading. In this paper we use this phrase to mean the lowest energy states in all contexts except in QSM, where a system's total energy is infinite, the role of such states are played by the KMS states.

[2] In the algebraic language (on which more will be said later), we have, given $\{\mathcal{A}, G\}$, an automorphism $\alpha \in Aut(\mathcal{A})$ is *unitarily implementable* in a representation $\pi(\mathcal{A})$ of $\mathcal{A}$ on $\mathcal{H}$ if there exists a unitary operator, $U : \mathcal{H} \to \mathcal{H}$ such that $\pi(\alpha[A]) = U\pi(A)U^{-1}$, $\forall A \in \mathcal{A}$.

[3] For systematic treatments of the algebraic approach, see Emch 1972, Haag 1996, or Sewell 2002. For briefer and more philosophical discussions, see Clifton & Halvorson 2001; Arageorgis, et al 2002; Ruestche 2002, 2003; and Emch & Liu 2002.

[4] The corresponding technical term in the physics literature is 'thermodynamic limit.' We adopt the more general term to denote the method by which the size of Λ is increased without bounds while the density is kept finite and *the ratio of the volume to the surface of Λ is normal*. The result of this procedure in QSM is the emergence of thermodynamic properties, but in this paper we want to emphasize the emergence of 'macroscopic' properties; hence our term, 'macroscopic limit.'

[5] As Earman (2003) rightly pointed out, neither of the two features in [RA] explicitly say whether the symmetry in question is broken or in what manner it is broken. One wonder, from the outset, whether the two features, which seem to independently identify quantum SSB, are equivalent to the commonsense notion of it in classical SSB, and what explanation, causal or formal, these characterizations can offer to our understanding of quantum SSB (or SSB in general).

[6] As Earman (2003) again rightly pointed out in the conclusion of his paper, the quantum SSB as characterized by [RA] requires a great deal of interpretation before we can even make sense of it. The toy model of a simple quantum field Earman used to illulstrate [RA] does not show how the lack of unitary implementability of some symmetry amounts to a case of SSB. The discussion of the Weyl algebra (a species of C*-algebra) gives us an illustration of how the lack of unitary implementability leads to having vacuum states in inequivalent representations; but again it does not shed any light on why the latter implies a SSB. Invoking the notion of a folium of states on a C*-algebra and of disjoint representations does not help either, since there are many cases of disjoint representations that are not connected to SSB. Earman is right in asserting that 'for pure algebraic states, spatial inequivalence equals disjointness,' but the connection between this fact and quantum SSB is still obscure. States of a system at different temperatures (above the critical temperature) are disjoint, but this has nothing to do with SSB. Hence, inequivalence or

disjointness per se does not necessarily tell us anything on whether or not systems having such states harbor quantum SSB.

[7] A vector Ω is cyclic for a representation π if $\pi(\mathcal{A})\Omega \equiv \{\pi(A)\Omega \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

[8] The decomposition in (2) of [CD] is reminiscent of Lüders' version of the von Neumann description of the quantum measurement process $\omega \to \sum_k \lambda_k \omega_k$. Here however, the process that starts from the LHS, namely ω, and ends with the QM-conditional expectation in the RHS, namely $\sum_k \lambda_k \omega_k$, reduces to the statement that LHS = RHS since we refer to a situation where the P_k's belong to the center; in contrast, recall that the center consists only of the scalar multiples of the identity operator in QSM of finite systems that predate the introduction of superselection rules (cf. Wick et al 1952). Here the possibility that the center be non-trivial comes from the consideration of systems with an infinite number of degrees of freedom, which appears in QSM involving the macroscopic limit. Furthermore, [CD] is akin to the decomposition in superselection sectors in QFT.

[9] The long-rangeness of the interaction makes this model differ from the usual Ising models where nearest-neighbor interaction is assumed: the long-range interaction makes it possible for a 1-d model to harbor a phrase transition.

[10] There is a difference between the Onsager spontaneous magnetization (of a two-dimensional Ising model) (cf. Onsager 1944; Schultz et al 1964) and the one in the Weiss-Heisenberg model we discuss here. In the former, the exact solution of the magnetization at $B \neq 0$ is not obtained explicitly, while such a solution for the Weiss-Heisenberg model is, as in (4b) in the previous section. Therefore, we can 'see' (in the solution) what happens with the magnetization when B approaches zero in the WH model, but we cannot in the Onsager solution.

[11] For example, renormalization, first- or second- quantization, etc.

[12] One may argue that there are plenty of instances of causal explanations in the discussion of the Goldstone theorem (or zero-mass gauge particles) and Higgs mechanism in QFT. This is true; however, such causal stories are not about how the relevant gauge symmetries are spontaneously broken but rather about what happens as consequences of such breakings. The breakings are assumed from the outset of that discussion.