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Research Article

Existence and Global Exponential Stability of Pseudo Almost Periodic Solutions for Neutral Type Quaternion-Valued Neural Networks with Delays in the Leakage Term on Time Scales

Yongkun Li and Xiaofang Meng

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China

Correspondence should be addressed to Yongkun Li; yklie@ynu.edu.cn

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We propose a class of neutral type quaternion-valued neural networks with delays in the leakage term on time scales that can unify the discrete-time and the continuous-time neural networks. In order to avoid the difficulty brought by the noncommutativity of quaternion multiplication, we first decompose the quaternion-valued system into four real-valued systems. Then, by applying the exponential dichotomic theory of linear dynamic equations on time scales, Banach's fixed point theorem, the theory of calculus on time scales, and inequality techniques, we obtain some sufficient conditions on the existence and global exponential stability of pseudo almost periodic solutions for this class of neural networks. Our results are completely new even for both the case of the neural networks governed by differential equations and the case of the neural networks governed by difference equations and show that, under a simple condition, the continuous-time quaternion-valued network and its corresponding discrete-time quaternion-valued network have the same dynamical behavior for the pseudo almost periodicity. Finally, a numerical example is given to illustrate the feasibility of our results.

1. Introduction

The quaternion, which was discovered by the Irish mathematician Hamilton [1] in order to generalize complex number properties to multidimensional space, is extensively used in several fields, such as modern mathematics, physics, and computer graphics [2–4]. One of the advantages by the use of quaternions is that it can treat and operate three- or four-dimensional vectors as one entity, which allows a significant decrease of computational complexity in three- or four-dimensional problems, so the effective information processing can be achieved by the operations for quaternionic variables. Therefore, the quaternion-valued neural network is able to cope with multidimensional issues more efficiently by employing quaternion directly.

In this respect, the quaternion-valued neural network is a fast growing field of research in both theoretical and application points of view (see [5–9]). Quaternion neural networks have been widely used in many fields and demonstrated better performances than the real number neural networks in chaotic time series prediction [10], approximate quaternion-valued functions [11], 3D wind forecasting [12, 13], image processing [14, 15], color-face recognition [16], vector sensor processing [17], and so on.

In reality, it is well known that the time delay is inevitable. In the circuit implementation of neural networks, time delays occur naturally due to the processing and transmission of signals in the network and the finite switching speed of amplifiers. And they may change the dynamical behaviors of considered neural networks. Therefore, the consideration of time delays is more and more significant in the study of the dynamics of neural networks.

Many scholars have devoted themselves into the dynamics analysis of neural networks with various types of time delays and many valuable results have been achieved in the existing literature see [18–26]. There are three typical types of time delays for incorporating time delays into neural networks: (i) introduce transmission delays into the neural networks, and consider discrete delays, distributed delays, mixed delays, even state depended delays, or complex delays; (ii) consider the delays in the leakage term; (iii) take into account neutral type delays. All of the above three types of time delays may alter the dynamics of the neural network under consideration.

On the one hand, the concept of pseudo-almost periodicity was introduced by Zhang [27, 28] in the early 1990s. It quickly aroused the interest of some mathematical researchers [29–31]. The pseudo almost periodicity is more general and complicated than the periodicity and the almost periodicity. In the last few years, the pseudo almost periodicity has become a hot research topic, especially for the pseudo almost periodic oscillation of neural networks [32–39].

On the other hand, as it is known, both continuous-time and discrete-time neural networks are important in theocratic studies and applications. Moreover, discrete-time neural networks are more convenient for computation and numerical simulation than continuous-time neural networks. Therefore, we must not only study continuous-time neural networks, but also study discrete-time neural networks. Fortunately, the theory of time scales, which was initiated by Hilger [40] in his Ph.D. thesis in 1988, can unify the continuous and discrete cases. Studying dynamic equations on time scales can unify the differential equation case and the difference equation case. In recent years, the time scale theory has been widely concerned and rapidly developed [41–45]. And, many authors have studied the dynamical behavior of neural networks on time scales [46–54].

However, to the best of our knowledge, there is no paper published on the existence and stability of pseudo almost periodic solutions of quaternion-valued neural networks on time scales. This is important in theory and application, and it is also a very challenging issue.

Motivated by the above statement, in this paper, we propose the following neutral type quaternion-valued neural network with delays in the leakage term on time scales:

$$\begin{aligned} x_p^{\Delta}(t) = {} & -c_p(t)\, x_p\left(t-\delta_p(t)\right) \\ & + \sum_{q=1}^{n} a_{pq}(t)\, f_q\left(x_q\left(t-\tau_{pq}(t)\right)\right) \\ & + \sum_{q=1}^{n} b_{pq}(t)\, g_q\left(x_q^{\Delta}\left(t-\eta_{pq}(t)\right)\right) + u_p(t), \\ & t \in \mathbb{T}, \end{aligned} \tag{1}$$

where $\mathbb{T}$ is an almost periodic time scale, $p \in \{1, 2, \ldots, n\} := \Lambda$, $x_p(t) \in \mathbb{H}\otimes\mathbb{T}$ is the state of the pth neuron at time t; $c_p(t) > 0$ is the self-feedback connection weight, $\mathbb{H}\otimes\mathbb{T}$ denotes the set of all quaternion-valued functions defined on time scale $\mathbb{T}$; $a_{pq}(t)$ and $b_{pq}(t) \in \mathbb{H}\otimes\mathbb{T}$ are the delay connection weight and the neutral delay connection weight from neuron q to neuron p at time t, respectively; $u_p(t)$ is an external input on the pth unit at time t; $\delta_p(t)$ denotes the leakage delay satisfying $t-\delta_p(t) \in \mathbb{T}$ for $t \in \mathbb{T}$; $\tau_{pq}(t)$ and $\eta_{pq}(t)$ are transmission delays satisfying $t-\tau_{pq}(t) \in \mathbb{T}$ and $t-\eta_{pq}(t) \in \mathbb{T}$ for $t \in \mathbb{T}$.

The initial condition of system (1) is of the form

$$x_p(s) = \varphi_p(s), \quad p \in \Lambda,\ s \in [-\theta, 0]_{\mathbb{T}}, \tag{2}$$

where $\theta = \max\{\delta, \tau, \eta\}$, $\delta = \max_{p\in\Lambda}\{\sup_{t\in\mathbb{T}}\delta_p(t)\}$, $\tau = \max_{p,q\in\Lambda}\{\sup_{t\in\mathbb{T}}\tau_{pq}(t)\}$, $\eta = \max_{p,q\in\Lambda}\{\sup_{t\in\mathbb{T}}\eta_{pq}(t)\}$, $\varphi_p(s) \in C([-\theta, 0]_{\mathbb{T}},\ \mathbb{H}^n\otimes\mathbb{T})$.

Throughout this paper, we denote $[a, b]_{\mathbb{T}} = \{t \mid t \in [a, b] \cap \mathbb{T}\}$. For convenience, for an rd-continuous pseudo almost periodic function $f : \mathbb{T} \to \mathbb{R}$, we denote $f^- = \inf_{t\in\mathbb{T}}|f(t)|$ and $f^+ = \sup_{t\in\mathbb{T}}|f(t)|$.

Our main purpose of this paper is to study the existence and global exponential stability of pseudo almost periodic solutions of (1). Our results are completely new even for both the case of the neural networks governed by quaternion-valued differential equations and the case of the neural networks governed by quaternion-valued difference equations.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and preliminary lemmas and transform the quaternion-valued system (1) into four real-valued systems. In Section 3, we establish some sufficient conditions for the existence and global exponential stability of pseudo almost periodic solutions of (1). In Section 4, we give an example to demonstrate the feasibility of our results. This paper ends with a brief conclusion in Section 5.

2. Preliminaries

In this section, we shall first recall some fundamental definitions, lemmas which are used in what follows.

The skew field of quaternions is denoted by

$$\mathbb{H} := \left\{q = q^R + q^I i + q^J j + q^K k\right\}, \tag{3}$$

where q^R, q^I, q^J, and q^K are real numbers and the elements i, j, and k obey Hamilton's multiplication rules:

$$\begin{aligned} ij &= -ji = k, \\ jk &= -kj = i, \\ ki &= -ik = j, \\ i^2 &= j^2 = k^2 = ijk = -1. \end{aligned} \tag{4}$$

The quaternion conjugate is defined by $\overline{q} = q^R - \underline{q} = q^R - q^I i - q^J j - q^K k$, and the norm $|q|$ of q is defined as $|q|^2 = q\overline{q} = \overline{q}q = (q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}$, $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T},\ s < t\}$, $t \in \mathbb{T}$, and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered maximum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (provided its exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \varepsilon |\sigma(t) - s|, \quad (5)$$

for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable on $\mathbb{T}^k$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

By writing $f \in \mathbb{H} \otimes \mathbb{T}$ in the form of $f = f^R + if^I + jf^J + kf^K$ with $f^l \in \mathbb{H} \otimes \mathbb{T}$, $l \in \{R, I, J, K\} := E$, it is easy to verify that f is delta differentiable if and only if f^R, f^I, f^J, f^K are delta differentiable. Moreover, if f is delta differentiable, then

$$\begin{aligned} f^\Delta(t) = \left(f^R\right)^\Delta(t) + i\left(f^I\right)^\Delta(t) + j\left(f^J\right)^\Delta(t) \\ + k\left(f^K\right)^\Delta(t). \end{aligned} \quad (6)$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \ne 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathscr{R} = \mathscr{R}(\mathbb{T})$. We define $\mathscr{R}^+ = \{p \in \mathscr{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. For more knowledge about calculus on time scales, we refer to [41, 42].

Definition 1 (see [47]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T},\ \forall t \in \mathbb{T}\} \ne \{0\}. \quad (7)$$

Definition 2 (see [47]). Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic on $\mathbb{T}$ if for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau(\varepsilon) \in \Pi$ such that

$$\left| f(t + \tau) - f(t) \right| < \varepsilon, \quad \forall t \in \mathbb{T}. \quad (8)$$

Let $\mathrm{AP}(\mathbb{T}, \mathbb{R}^n) = \{f \in C(\mathbb{T}, \mathbb{R}^n) : f \text{ be almost periodic}\}$ and $\mathrm{BC}(\mathbb{T}, \mathbb{R}^n)$ denote the space of all bounded continuous functions from $\mathbb{T}$ to $\mathbb{R}^n$.

Similar to Definition 4.1 in [55], we introduce the following definition.

Definition 3. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called pseudo almost periodic if $f = g + h$, where $g \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^n)$ and $h \in \mathrm{PAP}_0(\mathbb{T}, \mathbb{R}^n) = \{f \in \mathrm{BC}(\mathbb{T}, \mathbb{R}^n) : f \text{ is } \Delta\text{-measurable such that } \lim_{r\to+\infty}(1/2r)\int_{t_0-r}^{t_0+r} |f(s)|\Delta s = 0, \text{ where } t_0 \in \mathbb{T},\ r \in \Pi\}$.

We denote by $\mathrm{PAP}(\mathbb{T}, \mathbb{R}^n)$ the set of all pseudo almost periodic functions defined on $\mathbb{T}$.

Lemma 4 (see [56]). *If $f, g \in PAP(\mathbb{T}, \mathbb{R}^n)$, then $f + g$, $fg \in PAP(\mathbb{T}, \mathbb{R}^n)$; if $f \in PAP(\mathbb{T}, \mathbb{R}^n)$, $g \in AP(\mathbb{T}, \mathbb{R}^n)$, then $fg \in PAP(\mathbb{T}, \mathbb{R}^n)$.*

Similar to the proof of Lemma 2.10 in [56], one can show the following.

Lemma 5. *If $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition, $\varphi \in PAP(\mathbb{T}, \mathbb{R})$, $\tau \in C^1(\mathbb{T}, \Pi)$, and $\inf_{t\in\mathbb{T}}(1 - \tau^\Delta(t)) > 0$, then $f(\varphi(t - \tau(t))) \in PAP(\mathbb{T}, \mathbb{R})$.*

Definition 6. Function $f = f^R + if^I + jf^J + kf^K \in C(\mathbb{T}, \mathbb{H}^n \otimes \mathbb{T})$ is called pseudo almost periodic if for each $l \in E$, $f^l \in C(\mathbb{T}, \mathbb{R}^n)$ is pseudo almost periodic.

Definition 7 (see [47]). Let $A(t)$ be an $n \times n$ matrix-valued function on $\mathbb{T}$. Then the linear system

$$x^\Delta(t) = A(t)x(t), \quad t \in \mathbb{T}, \quad (9)$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant K, α, projection P, and the fundamental solution matrix $X(t)$ of (9), satisfying

$$\begin{aligned} \left\| X(t)PX^{-1}(s) \right\|_0 &\le Ke_{\ominus\alpha}(t, s), \\ & \qquad s, t \in \mathbb{T},\ t \ge s, \\ \left\| X(t)(I - P)X^{-1}(s) \right\|_0 &\le Ke_{\ominus\alpha}(s, t), \\ & \qquad s, t \in \mathbb{T},\ t \le s. \end{aligned} \quad (10)$$

Consider the following pseudo almost periodic system:

$$x^\Delta(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \quad (11)$$

where $A(t)$ is an almost periodic matrix function and $f(t)$ is a pseudo almost periodic vector function.

Lemma 8 (see [47]). *If the linear system (9) admits an exponential dichotomy, then the pseudo almost periodic system (11) has a unique pseudo almost periodic solution $x(t)$ as follows:*

$$\begin{aligned} x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s \\ - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s, \end{aligned} \quad (12)$$

where $X(t)$ is the fundamental solution matrix of (9).

Lemma 9 (see [46]). *Let $c_p(t) : \mathbb{T} \to \mathbb{R}^+$ be an almost periodic function, $-c_p(t) \in \mathscr{R}^+$, $p \in \Lambda$, $t \in \mathbb{T}$, and $\min_{1\le p\le n}\{\inf_{t\in\mathbb{T}} c_p(t)\} > 0$; then the linear system*

$$x^\Delta(t) = \operatorname{diag}\left(-c_1(t), -c_2(t), \dots, -c_n(t)\right)x(t) \quad (13)$$

admits an exponential dichotomy on $\mathbb{T}$.

Throughout the rest of this paper, we assume the following:

(H_1) Let $x_p = x_p^R + i x_p^I + j x_p^J + k x_p^K$, where $x_p^R, x_p^I, x_p^J, x_p^K \in \mathbb{R}$, $p \in \Lambda$. Then $f_p(x_p)$ and $g_p(x_p)$ can be expressed as

$$\begin{aligned} f_p(x_p) &= f_p^R(x_p^R) + i f_p^I(x_p^I) + j f_p^J(x_p^J) \\ &\quad + k f_p^K(x_p^K), \quad p \in \Lambda, \\ g_p(x_p) &= g_p^R(x_p^R) + i g_p^I(x_p^I) + j g_p^J(x_p^J) \\ &\quad + k g_p^K(x_p^K), \quad p \in \Lambda. \end{aligned} \tag{14}$$

By (H_1), we can transform system (1) into the following four real-valued systems:

$$\begin{aligned}
(x_p^R(t))^\Delta &= -c_p(t) x_p^R(t - \delta_p(t)) \\
&\quad + \sum_{q=1}^{n} \left(a_{pq}^R(t) f_q^R(x_q^R(t - \tau_{pq}(t))) \right. \\
&\quad - a_{pq}^I(t) f_q^I(x_q^I(t - \tau_{pq}(t))) \\
&\quad - a_{pq}^J(t) f_q^J(x_q^J(t - \tau_{pq}(t))) \\
&\quad \left. - a_{pq}^K(t) f_q^K(x_q^K(t - \tau_{pq}(t))) \right) \\
&\quad + \sum_{q=1}^{n} \left(b_{pq}^R(t) g_q^R((x_q^R)^\Delta(t - \eta_{pq}(t))) \right. \\
&\quad - b_{pq}^I(t) g_q^I((x_q^I)^\Delta(t - \eta_{pq}(t))) \\
&\quad - b_{pq}^J(t) g_q^J((x_q^J)^\Delta(t - \eta_{pq}(t))) \\
&\quad \left. - b_{pq}^K(t) g_q^K((x_q^K)^\Delta(t - \eta_{pq}(t))) \right) + u_p^R(t), \\
(x_p^I(t))^\Delta &= -c_p(t) x_p^I(t - \delta_p(t)) \\
&\quad + \sum_{q=1}^{n} \left(a_{pq}^R(t) f_q^I(x_q^I(t - \tau_{pq}(t))) \right. \\
&\quad + a_{pq}^I(t) f_q^R(x_q^R(t - \tau_{pq}(t))) \\
&\quad + a_{pq}^J(t) f_q^K(x_q^K(t - \tau_{pq}(t))) \\
&\quad \left. - a_{pq}^K(t) f_q^J(x_q^J(t - \tau_{pq}(t))) \right) \\
&\quad + \sum_{q=1}^{n} \left(b_{pq}^R(t) g_q^I((x_q^I)^\Delta(t - \eta_{pq}(t))) \right. \\
&\quad + b_{pq}^I(t) g_q^R((x_q^R)^\Delta(t - \eta_{pq}(t))) \\
&\quad + b_{pq}^J(t) g_q^K((x_q^K)^\Delta(t - \eta_{pq}(t))) \\
&\quad \left. - b_{pq}^K(t) g_q^J((x_q^J)^\Delta(t - \eta_{pq}(t))) \right) + u_p^I(t), \\
(x_p^J(t))^\Delta &= -c_p(t) x_p^J(t - \delta_p(t)) \\
&\quad + \sum_{q=1}^{n} \left(a_{pq}^R(t) f_q^J(x_q^J(t - \tau_{pq}(t))) \right. \\
&\quad + a_{pq}^J(t) f_q^R(x_q^R(t - \tau_{pq}(t))) \\
&\quad - a_{pq}^I(t) f_q^K(x_q^K(t - \tau_{pq}(t))) \\
&\quad \left. + a_{pq}^K(t) f_q^I(x_q^I(t - \tau_{pq}(t))) \right) \\
&\quad + \sum_{q=1}^{n} \left(b_{pq}^R(t) g_q^J((x_q^J)^\Delta(t - \eta_{pq}(t))) \right. \\
&\quad + b_{pq}^J(t) g_q^R((x_q^R)^\Delta(t - \eta_{pq}(t))) \\
&\quad - b_{pq}^I(t) g_q^K((x_q^K)^\Delta(t - \eta_{pq}(t))) \\
&\quad \left. + b_{pq}^K(t) g_q^I((x_q^I)^\Delta(t - \eta_{pq}(t))) \right) + u_p^J(t), \\
(x_p^K(t))^\Delta &= -c_p(t) x_p^I(t - \delta_p(t)) \\
&\quad + \sum_{q=1}^{n} \left(a_{pq}^R(t) f_q^K(x_q^K(t - \tau_{pq}(t))) \right. \\
&\quad + a_{pq}^K(t) f_q^R(x_q^R(t - \tau_{pq}(t))) \\
&\quad + a_{pq}^I(t) f_q^J(x_q^J(t - \tau_{pq}(t))) \\
&\quad \left. - a_{pq}^J(t) f_q^I(x_q^I(t - \tau_{pq}(t))) \right) \\
&\quad + \sum_{q=1}^{n} \left(b_{pq}^R(t) g_q^K((x_q^K)^\Delta(t - \eta_{pq}(t))) \right. \\
&\quad + b_{pq}^K(t) g_q^R((x_q^R)^\Delta(t - \eta_{pq}(t))) \\
&\quad + b_{pq}^I(t) g_q^J((x_q^J)^\Delta(t - \eta_{pq}(t))) \\
&\quad \left. - b_{pq}^J(t) g_q^I((x_q^I)^\Delta(t - \eta_{pq}(t))) \right) + u_p^K(t),
\end{aligned} \tag{15}$$

$$\begin{aligned}
a_{pq}(t) &= a_{pq}^R(t) + i a_{pq}^I(t) + j a_{pq}^J(t) + k a_{pq}^K(t), \\
b_{pq}(t) &= b_{pq}^R(t) + i b_{pq}^I(t) + j b_{pq}^J(t) + k b_{pq}^K(t), \\
u_p(t) &= u_p^R(t) + i u_p^I(t) + j u_p^J(t) + k u_p^K(t), \\
&\qquad p \in \Lambda.
\end{aligned} \tag{16}$$

According to (15), we can get

$$X_p^{\Delta}(t) = -c_p(t) X_p(t-\delta_p(t)) + \sum_{q=1}^{n} A_{pq}(t) F_q(X_q(t-\tau_{pq}(t))) + \sum_{q=1}^{n} B_{pq}(t) G_q(X_q^{\Delta}(t-\eta_{pq}(t))) + U_p(t), \quad p \in \Lambda, \tag{17}$$

where

$$A_{pq}(t) = \begin{pmatrix} a_{pq}^R(t) & -a_{pq}^I(t) & -a_{pq}^J(t) & -a_{pq}^K(t) \\ a_{pq}^I(t) & a_{pq}^R(t) & -a_{pq}^K(t) & a_{pq}^J(t) \\ a_{pq}^J(t) & a_{pq}^K(t) & a_{pq}^R(t) & -a_{pq}^I(t) \\ a_{pq}^K(t) & -a_{pq}^J(t) & a_{pq}^I(t) & a_{pq}^R(t) \end{pmatrix},$$

$$B_{pq}(t) = \begin{pmatrix} b_{pq}^R(t) & -b_{pq}^I(t) & -b_{pq}^J(t) & -b_{pq}^K(t) \\ b_{pq}^I(t) & b_{pq}^R(t) & -b_{pq}^K(t) & b_{pq}^J(t) \\ b_{pq}^J(t) & b_{pq}^K(t) & b_{pq}^R(t) & -b_{pq}^I(t) \\ b_{pq}^K(t) & -b_{pq}^J(t) & b_{pq}^I(t) & b_{pq}^R(t) \end{pmatrix},$$

$$X_p(t) = (x_p^R(t), x_p^I(t), x_p^J(t), x_p^K(t))^T,$$

$$U_p(t) = (u_p^R(t), u_p^I(t), u_p^J(t), u_p^K(t))^T,$$

$$F_q = (f_q^R, f_q^I, f_q^J, f_q^K)^T,$$

$$G_q = (g_q^R, g_q^I, g_q^J, g_q^K)^T. \tag{18}$$

The initial condition associated with (17) is of the form

$$X_p(s) = \Phi_p(s), \quad p \in \Lambda, \ s \in [-\theta, 0]_{\mathbb{T}}, \tag{19}$$

where $\Phi_p(s) = (\phi_p^R(s), \phi_p^I(s), \phi_p^J(s), \phi_p^K(s))$, $\phi_p^l(s) \in C([-\theta, 0]_{\mathbb{T}}, \mathbb{R})$, $l \in E$.

Remark 10. It is obvious that if $x(t) = (x_1^R(t), x_1^I(t), x_1^J(t), x_1^K(t), x_2^R(t), x_2^I(t), x_2^J(t), x_2^K(t), \ldots, x_n^R(t), x_n^I(t), x_n^J(t), x_n^K(t))^T$ is a solution to system (17), then

$$y(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T, \tag{20}$$

where $x_p(t) = x_p^R(t) + i x_p^I(t) + j x_p^J(t) + k x_p^K(t)$, $p = 1, 2, \ldots, n$ must be a solution to (1). Thus, the problem of finding a pseudo almost periodic solution for (1) reduces to finding one for system (17). For considering the stability of solutions of (1), we just need to consider the stability of solutions of system (17).

3. Main Results

In this section, we will study the existence and global exponential stability of pseudo almost periodic solutions of system (17).

Let $\mathbb{X} = \{f \mid f, f^{\Delta} \in PAP(\mathbb{T}, \mathbb{R}^{4n})\}$ with the norm $\|f\|_{\mathbb{X}} = \max\{\|f\|, \|f^{\Delta}\|\}$, where $\|f\| = \max_{1\le h\le 4n}\{f_h^+\}$, $\|f^{\Delta}\| = \max_{1\le h\le 4n}\{(f_h^{\Delta})^+\}$; then $\mathbb{X}$ is a Banach space.

Throughout this paper, we assume that the following conditions hold:

(H_2) $c_p \in C(\mathbb{T}, \mathbb{R}^+)$ with $-c_p \in \mathscr{R}^+$ is an almost periodic function, $A_{pq}, B_{pq} \in AP(\mathbb{T}, \mathbb{R}^{4\times 4})$, $U_p \in PAP(\mathbb{T}, \mathbb{R}^{4\times 1})$, $\delta_p \in C(\mathbb{T}, \Pi)$, $\tau_{pq}, \eta_{pq} \in C^1(\mathbb{T}, \Pi)$, $\inf_{t\in\mathbb{T}}(1 - \tau_{pq}^{\Delta}(t)) > 0$, $\inf_{t\in\mathbb{T}}(1 - \eta_{pq}^{\Delta}(t)) > 0$, $p, q \in \Lambda$.

(H_3) Functions $f_q^l, g_q^l \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants α_q^l, β_q^l such that for all $x^l, y^l \in \mathbb{R}$

$$\left|f_q^l(x^l) - f_q^l(y^l)\right| \le \alpha_q^l \left|x^l - y^l\right|,$$
$$\left|g_q^l(x^l) - g_q^l(y^l)\right| \le \beta_q^l \left|x^l - y^l\right|, \tag{21}$$

and $f_q^l(0) = g_q^l(0) = 0$, $q \in \Lambda$, $l \in E$.

(H_4) There exists a positive constant κ such that

$$\max_{p\in\Lambda}\left\{\max_{l\in E}\left\{\frac{\Gamma_p^l \kappa + u_p^{l^+}}{c_p^-}, \left(1 + \frac{c_p^+}{c_p^-}\right)\left(\Gamma_p^l \kappa + u_p^{l^+}\right)\right\}\right\} \le \kappa,$$
$$\max_{p\in\Lambda}\left\{\max_{l\in E}\left\{\frac{\Gamma_p^l}{c_p^-}, \left(1 + \frac{c_p^+}{c_p^-}\right)\Gamma_p^l\right\}\right\} := \rho < 1, \tag{22}$$

where

$$\Gamma_p^l = c_p^+ \delta_p^+ + A_p^l + B_p^l, \quad p \in \Lambda, \ l \in E,$$

$$A_p^R = \sum_{q=1}^{n}\left(a_{pq}^{R^+}\alpha_q^R + a_{pq}^{I^+}\alpha_q^I + a_{pq}^{J^+}\alpha_q^J + a_{pq}^{K^+}\alpha_q^K\right), \quad p \in \Lambda,$$

$$B_p^R = \sum_{q=1}^{n}\left(b_{pq}^{R^+}\beta_q^R + b_{pq}^{I^+}\beta_q^I + b_{pq}^{J^+}\beta_q^J + b_{pq}^{K^+}\beta_q^K\right), \quad p \in \Lambda,$$

$$A_p^I = \sum_{q=1}^{n}\left(a_{pq}^{R^+}\alpha_q^I + a_{pq}^{I^+}\alpha_q^R + a_{pq}^{J^+}\alpha_q^K + a_{pq}^{K^+}\alpha_q^J\right), \quad p \in \Lambda,$$

$$B_p^I = \sum_{q=1}^{n}\left(b_{pq}^{R^+}\beta_q^I + b_{pq}^{I^+}\beta_q^R + b_{pq}^{J^+}\beta_q^K + b_{pq}^{K^+}\beta_q^J\right), \quad p \in \Lambda,$$

$$A_p^J = \sum_{q=1}^{n} \left(a_{pq}^{R^+} \alpha_q^J + a_{pq}^{J^+} \alpha_q^R + a_{pq}^{I^+} \alpha_q^K + a_{pq}^{K^+} \alpha_q^I \right), \quad p \in \Lambda,$$
$$B_p^J = \sum_{q=1}^{n} \left(b_{pq}^{R^+} \beta_q^J + b_{pq}^{J^+} \beta_q^R + b_{pq}^{I^+} \beta_q^K + b_{pq}^{K^+} \beta_q^I \right), \quad p \in \Lambda,$$
$$A_p^K = \sum_{q=1}^{n} \left(a_{pq}^{R^+} \alpha_q^K + a_{pq}^{K^+} \alpha_q^R + a_{pq}^{I^+} \alpha_q^J + a_{pq}^{J^+} \alpha_q^I \right), \quad p \in \Lambda,$$
$$B_p^K = \sum_{q=1}^{n} \left(b_{pq}^{R^+} \beta_q^K + b_{pq}^{K^+} \beta_q^R + b_{pq}^{I^+} \beta_q^J + b_{pq}^{J^+} \beta_q^I \right), \quad p \in \Lambda. \tag{23}$$

Theorem 11. *Assume that (H_1)–(H_4) hold; then system (17) has a unique pseudo almost periodic solution in the region $\mathbb{X}^* = \{\varphi \in \mathbb{X} \mid \|\varphi\|_{\mathbb{X}} \le \kappa\}$.*

Proof. System (17) can be written as

$$\begin{aligned} X_p^\Delta(t) = {} & -c_p(t) X_p(t) + c_p(t) \int_{t-\delta_p(t)}^{t} X_p^\Delta(s)\,\Delta s \\ & + \sum_{q=1}^{n} A_{pq}(t) F_q\left(X_q\left(t - \tau_{pq}(t)\right)\right) \\ & + \sum_{q=1}^{n} B_{pq}(t) G_q\left(X_q^\Delta\left(t - \eta_{pq}(t)\right)\right) + U_p(t), \\ & p \in \Lambda,\ t \in \mathbb{T}. \end{aligned} \tag{24}$$

For any $\varphi \in \mathbb{X}$, consider the linear dynamic system

$$\begin{aligned} X_p^\Delta(t) = {} & -c_p(t) X_p(t) + c_p(t) \int_{t-\delta_p(t)}^{t} \varphi_p^\Delta(s)\,\Delta s \\ & + \sum_{q=1}^{n} A_{pq}(t) F_q\left(\varphi_q\left(t - \tau_{pq}(t)\right)\right) \\ & + \sum_{q=1}^{n} B_{pq}(t) G_q\left(\varphi_q^\Delta\left(t - \eta_{pq}(t)\right)\right) + U_p(t), \\ & p \in \Lambda,\ t \in \mathbb{T}. \end{aligned} \tag{25}$$

Since $\min_{1\le p\le n}\{\inf_{t\in\mathbb{T}} c_p(t)\} > 0$ and $-c_p \in \mathscr{R}^+$, it follows from Lemma 9 that the linear system

$$X_p^\Delta(t) = -c_p(t) X_p(t) \tag{26}$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, by Lemma 8, we see that system (25) has exactly one pseudo almost periodic solution which can be expressed as follows:

$$\begin{aligned} X_p^\varphi(t) = {} & \int_{-\infty}^{t} e_{-c_p}(t, \sigma(s)) \left(c_p(s) \int_{s-\delta_p(s)}^{s} \varphi_p^\Delta(u)\,\Delta u \right. \\ & + \sum_{q=1}^{n} A_{pq}(s) F_q\left(\varphi_q\left(s - \tau_{pq}(s)\right)\right) \\ & \left. + \sum_{q=1}^{n} B_{pq}(s) G_q\left(\varphi_q^\Delta\left(s - \eta_{pq}(s)\right)\right) + U_p(s) \right) \Delta s, \\ & p \in \Lambda. \end{aligned} \tag{27}$$

Now, we define the operator $\Phi : \mathbb{X}^* \to \mathbb{X}^*$ as

$$\left(\varphi_1, \varphi_2, \ldots, \varphi_n\right)^T \longrightarrow \left(X_1^\varphi, X_2^\varphi, \ldots, X_n^\varphi\right)^T, \tag{28}$$

where $\varphi_p = (\varphi_p^R, \varphi_p^I, \varphi_p^J, \varphi_p^K)$, X_p^φ is defined by (27), $p \in \Lambda$.

First, we show that, for any $\varphi \in \mathbb{X}^*$, we have $\Phi\varphi \in \mathbb{X}^*$. From (27), we have

$$\begin{aligned} \left|(\Phi\varphi)_p^R(t)\right| = {} & \left| \int_{-\infty}^{t} e_{-c_p}(t, \sigma(s)) \left(c_p(s) \int_{s-\delta_p(s)}^{s} \left(\varphi_p^R\right)^\Delta(u)\,\Delta u \right. \right. \\ & + \sum_{q=1}^{n} \left(a_{pq}^R(s) f_q^R\left(\varphi_q^R\left(s - \tau_{pq}(s)\right)\right) - a_{pq}^I(s) f_q^I\left(\varphi_q^I\left(s - \tau_{pq}(s)\right)\right) - a_{pq}^J(s) f_q^J\left(\varphi_q^J\left(s - \tau_{pq}(s)\right)\right) - a_{pq}^K(s) f_q^K\left(\varphi_q^K\left(s - \tau_{pq}(s)\right)\right) \right) \\ & + \sum_{q=1}^{n} \left(b_{pq}^R(s) g_q^R\left(\left(\varphi_q^R\right)^\Delta\left(s - \eta_{pq}(s)\right)\right) - b_{pq}^I(s) g_q^I\left(\left(\varphi_q^I\right)^\Delta\left(s - \eta_{pq}(s)\right)\right) - b_{pq}^J(s) g_q^J\left(\left(\varphi_q^J\right)^\Delta\left(s - \eta_{pq}(s)\right)\right) - b_{pq}^K(s) g_q^K\left(\left(\varphi_q^K\right)^\Delta\left(s - \eta_{pq}(s)\right)\right) \right) \\ & \left. \left. + u_p^R(s) \right) \Delta s \right| \le \int_{-\infty}^{t} e_{-c_p}(t, \sigma(s)) \left(c_p^+ \int_{s-\delta_p(s)}^{s} \left| \left(\varphi_p^R\right)^\Delta(u) \right| \Delta u \right. \end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{n}\left(a_{pq}^{R^+}\left|f_q^R\left(\varphi_q^R\left(s-\tau_{pq}(s)\right)\right)\right|+a_{pq}^{I^+}\left|f_q^I\left(\varphi_q^I\left(s-\tau_{pq}(s)\right)\right)\right|+a_{pq}^{J^+}\left|f_q^J\left(\varphi_q^J\left(s-\tau_{pq}(s)\right)\right)\right|+a_{pq}^{K^+}\left|f_q^K\left(\varphi_q^K\left(s-\tau_{pq}(s)\right)\right)\right|\right)\\
&+\sum_{q=1}^{n}\left(b_{pq}^{R^+}\left|g_q^R\left(\left(\varphi_q^R\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right|+b_{pq}^{I^+}\left|g_q^I\left(\left(\varphi_q^I\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right|+b_{pq}^{J^+}\left|g_q^J\left(\left(\varphi_q^J\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right|+b_{pq}^{K^+}\left|g_q^K\left(\left(\varphi_q^K\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right|\right)\\
&+u_p^{R^+}\Bigg)\Delta s\le\int_{-\infty}^{t}e_{-c_p}(t,\sigma(s))\left(c_p^+\delta_p^+\kappa+\sum_{q=1}^{n}\left(a_{pq}^{R^+}\alpha_q^R\kappa+a_{pq}^{I^+}\alpha_q^I\kappa+a_{pq}^{J^+}\alpha_q^J\kappa+a_{pq}^{K^+}\alpha_q^K\kappa\right)+\sum_{q=1}^{n}\left(b_{pq}^{R^+}\beta_q^R\kappa+b_{pq}^{I^+}\beta_q^I\kappa+b_{pq}^{J^+}\beta_q^J\kappa+b_{pq}^{K^+}\beta_q^K\kappa\right)\right.\\
&\left.+u_p^{R^+}\right)\Delta s\le\frac{1}{c_p^-}\left(c_p^+\delta_p^+\kappa+A_p^R\kappa+B_p^R\kappa+u_p^{R^+}\right)=\frac{\Gamma_p^R\kappa+u_p^{R^+}}{c_p^-},\quad p\in\Lambda.
\end{aligned}
\tag{29}
$$

In a similar way, we have

$$
\left|(\Phi\varphi)_p^l(t)\right|\le\frac{\Gamma_p^l\kappa+u_p^{l^+}}{c_p^-},\quad p\in\Lambda,\ l=I,J,K. \tag{30}
$$

On the other hand, we have

$$
\begin{aligned}
&\left|\left((\Phi\varphi)_p^R\right)^{\Delta}(t)\right|=\Bigg|c_p(t)\int_{t-\delta_p(t)}^{t}\left(\varphi_p^R\right)^{\Delta}(u)\Delta u+\sum_{q=1}^{n}\left(a_{pq}^R(t)f_q^R\left(\varphi_q^R\left(t-\tau_{pq}(t)\right)\right)-a_{pq}^I(t)f_q^I\left(\varphi_q^I\left(t-\tau_{pq}(t)\right)\right)-a_{pq}^J(t)f_q^J\left(\varphi_q^J\left(t-\tau_{pq}(t)\right)\right)\right.\\
&\left.-a_{pq}^K(t)f_q^K\left(\varphi_q^K\left(t-\tau_{pq}(t)\right)\right)\right)+\sum_{q=1}^{n}\left(b_{pq}^R(t)g_q^R\left(\left(\varphi_q^R\right)^{\Delta}\left(t-\eta_{pq}(t)\right)\right)-b_{pq}^I(t)g_q^I\left(\left(\varphi_q^I\right)^{\Delta}\left(t-\eta_{pq}(t)\right)\right)-b_{pq}^J(t)g_q^J\left(\left(\varphi_q^J\right)^{\Delta}\left(t-\eta_{pq}(t)\right)\right)\right.\\
&\left.-b_{pq}^K(t)g_q^K\left(\left(\varphi_q^K\right)^{\Delta}\left(t-\eta_{pq}(t)\right)\right)\right)+u_p^R(t)-c_p(t)\int_{-\infty}^{t}e_{-c_p}(t,\sigma(s))\Bigg(c_p(s)\int_{s-\delta_p(s)}^{s}\left(\varphi_p^R\right)^{\Delta}(u)\Delta u\\
&+\sum_{q=1}^{n}\left(a_{pq}^R(s)f_q^R\left(\varphi_q^R\left(s-\tau_{pq}(s)\right)\right)-a_{pq}^I(s)f_q^I\left(\varphi_q^I\left(s-\tau_{pq}(s)\right)\right)-a_{pq}^J(s)f_q^J\left(\varphi_q^J\left(s-\tau_{pq}(s)\right)\right)-a_{pq}^K(s)f_q^K\left(\varphi_q^K\left(s-\tau_{pq}(s)\right)\right)\right)\\
&+\sum_{q=1}^{n}\left(b_{pq}^R(s)g_q^R\left(\left(\varphi_q^R\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-b_{pq}^I(s)g_q^I\left(\left(\varphi_q^I\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-b_{pq}^J(s)g_q^J\left(\left(\varphi_q^J\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-b_{pq}^K(s)g_q^K\left(\left(\varphi_q^K\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right)\\
&+u_p^R(s)\Bigg)\Delta s\Bigg|\le\left(1+\frac{c_p^+}{c_p^-}\right)\left(c_p^+\delta_p^+\kappa+A_p^R\kappa+B_p^R\kappa+u_p^{R^+}\right)=\left(1+\frac{c_p^+}{c_p^-}\right)\left(\Gamma_p^R\kappa+u_p^{R^+}\right),\quad p\in\Lambda.
\end{aligned}
\tag{31}
$$

In a similar way, we have

$$
\left|\left((\Phi\varphi)_p^l\right)^{\Delta}(t)\right|\le\left(1+\frac{c_p^+}{c_p^-}\right)\left(\Gamma_p^l\kappa+u_p^{l^+}\right),\quad p\in\Lambda,\ l=I,J,K. \tag{32}
$$

It follows from (29) to (32) and (H_4) that

$$
\|\Phi\varphi\|_{\mathbb{X}}\le\kappa, \tag{33}
$$

which implies that $\Phi\varphi\in\mathbb{X}^*$, so the mapping Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Next, we shall prove that Φ is a contraction mapping. In fact, for any $\varphi,\psi\in\mathbb{X}^*$, we have

$$
\begin{aligned}
&\left|(\Phi\varphi-\Phi\psi)_p^R(t)\right|\\
&=\Bigg|\int_{-\infty}^{t}e_{-c_p}(t,\sigma(s))\\
&\cdot\Bigg(c_p(s)\int_{s-\delta_p(s)}^{s}\left(\left(\varphi_p^R\right)^{\Delta}(u)-\left(\psi_p^R\right)^{\Delta}(u)\right)\Delta u\\
&+\sum_{q=1}^{n}\left(a_{pq}^R(s)\left(f_q^R\left(\varphi_q^R\left(s-\tau_{pq}(s)\right)\right)-f_q^R\left(\psi_q^R\left(s-\tau_{pq}(s)\right)\right)\right)\right.\\
&-a_{pq}^I(s)\left(f_q^I\left(\varphi_q^I\left(s-\tau_{pq}(s)\right)\right)-f_q^I\left(\psi_q^I\left(s-\tau_{pq}(s)\right)\right)\right)\\
&-a_{pq}^J(s)\left(f_q^J\left(\varphi_q^J\left(s-\tau_{pq}(s)\right)\right)-f_q^J\left(\psi_q^J\left(s-\tau_{pq}(s)\right)\right)\right)\\
&\left.-a_{pq}^K(s)\left(f_q^K\left(\varphi_q^K\left(s-\tau_{pq}(s)\right)\right)-f_q^K\left(\psi_q^K\left(s-\tau_{pq}(s)\right)\right)\right)\right)\\
&+\sum_{q=1}^{n}\left(b_{pq}^R(s)\left(g_q^R\left(\left(\varphi_q^R\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-g_q^R\left(\left(\psi_q^R\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right)\right.\\
&-b_{pq}^I(s)\left(g_q^I\left(\left(\varphi_q^I\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-g_q^I\left(\left(\psi_q^I\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right)\\
&-b_{pq}^J(s)\left(g_q^J\left(\left(\varphi_q^J\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-g_q^J\left(\left(\psi_q^J\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right)\\
&\left.-b_{pq}^K(s)\left(g_q^K\left(\left(\varphi_q^K\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)-g_q^K\left(\left(\psi_q^K\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right)\right)\Bigg)\Delta s\Bigg|
\end{aligned}
$$

$$\leq \frac{1}{c_p^-}\left(c_p^+\delta_p^+ + \sum_{q=1}^{n}\left(a_{pq}^{R^+}\alpha_q^R + a_{pq}^{I^+}\alpha_q^I + a_{pq}^{J^+}\alpha_q^J + a_{pq}^{K^+}\alpha_q^K + b_{pq}^{R^+}\beta_q^R + b_{pq}^{I^+}\beta_q^I + b_{pq}^{J^+}\beta_q^J + b_{pq}^{K^+}\beta_q^K\right)\right) \cdot \|\varphi-\psi\|_{\mathbb{X}} = \frac{\Gamma_p^R}{c_p^-}\|\varphi-\psi\|_{\mathbb{X}}, \quad p \in \Lambda. \tag{34}$$

In a similar way, we have

$$\left|(\Phi\varphi-\Phi\psi)_p^l(t)\right| \leq \frac{\Gamma_p^l}{c_p^+}\|\varphi-\psi\|_{\mathbb{X}}, \quad p \in \Lambda,\ l = I, J, K. \tag{35}$$

On the other hand, we have

$$\left|\left((\Phi\varphi-\Phi\psi)_h^R\right)^\Delta(t)\right| \leq \left(c_p^+\delta_p^+ + \sum_{q=1}^{n}\left(a_{pq}^{R^+}\alpha_q^R + a_{pq}^{I^+}\alpha_q^I + a_{pq}^{J^+}\alpha_q^J + a_{pq}^{K^+}\alpha_q^K + b_{pq}^{R^+}\beta_q^R + b_{pq}^{I^+}\beta_q^I + b_{pq}^{J^+}\beta_q^J + b_{pq}^{K^+}\beta_q^K\right)\right)\|\varphi-\psi\|_{\mathbb{X}} + \frac{c_p^+}{c_p^-}\left(c_p^+\delta_p^+ + \sum_{q=1}^{n}\left(a_{pq}^{R^+}\alpha_q^R + a_{pq}^{I^+}\alpha_q^I + a_{pq}^{J^+}\alpha_q^J + a_{pq}^{K^+}\alpha_q^K + b_{pq}^{R^+}\beta_q^R + b_{pq}^{I^+}\beta_q^I + b_{pq}^{J^+}\beta_q^J + b_{pq}^{K^+}\beta_q^K\right)\right)\|\varphi-\psi\|_{\mathbb{X}} = \left(1+\frac{c_p^+}{c_p^-}\right)\Gamma_p^R\|\varphi-\psi\|_{\mathbb{X}}, \quad p \in \Lambda. \tag{36}$$

In a similar way, we have

$$\left|\left((\Phi\varphi-\Phi\psi)_p^l\right)^\Delta(t)\right| \leq \left(1+\frac{c_p^+}{c_p^-}\right)\Gamma_p^l\|\varphi-\psi\|_{\mathbb{X}}, \quad p \in \Lambda,\ l = I, J, K. \tag{37}$$

From (34) to (37) and (H_4) it follows that

$$\|\Phi\varphi-\Phi\psi\|_{\mathbb{X}} \leq \rho\|\varphi-\psi\|_{\mathbb{X}}. \tag{38}$$

Hence, we obtain that Φ is a contraction mapping. Then, system (17) has a unique pseudo almost periodic solution in the region $\mathbb{X}^* = \{\varphi \in \mathbb{X} : \|\varphi\|_{\mathbb{X}} \leq \kappa\}$. The proof is complete. □

Theorem 12. *Assume that (H_1)–(H_4) hold; then system (17) has a unique pseudo almost periodic solution that is globally exponentially stable.*

Proof. From Theorem 11, we see that system (17) has a pseudo almost periodic solution $X^*(t) = (X_1^*(t), X_2^*(t), \ldots, X_n^*(t))^T$ with initial value $\Phi^*(s) = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t))^T$. Suppose that $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T$ is an arbitrary solution of system (17) with initial value $\Phi(s) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$ and let $Z(t) = X(t) - X^*(t)$; then we have

$$Z_p^\Delta(t) = \Bigg\{-a_p(t)\left(X_p(t-\delta_p(t)) - X_p^*(t-\delta_p(t))\right) + \sum_{q=1}^{n}A_{pq}(t)\left(F_q\left(X_q(t-\tau_{pq}(t))\right) - F_q\left(X_q^*(t-\tau_{pq}(t))\right)\right) + \sum_{q=1}^{n}B_{pq}(t) \cdot \left(F_q\left(X_q^\Delta(t-\eta_{pq}(t))\right) - F_q\left(\left(X_q^*\right)^\Delta(t-\eta_{pq}(t))\right)\right)\Bigg\}, \quad p \in \Lambda. \tag{39}$$

For $p \in \Lambda$ and $l \in E$, we define Θ_p^l and Π_p^l as follows:

$$\begin{aligned}\Theta_p^l(\zeta) &= c_p^- - \zeta - \exp\left(\zeta\sup_{t\in\mathbb{T}}\mu(s)\right)\left(c_p^+\delta_p^+\exp(\zeta\delta_p^+) + A_p^l\exp(\zeta\tau_{pq}^+) + B_p^l\exp(\zeta\eta_{pq}^+)\right),\\ \Pi_p^l(\zeta) &= c_p^- - \zeta - \left(c_p^+\exp\left(\zeta\sup_{t\in\mathbb{T}}\mu(s)\right) + c_p^- - \zeta\right) \cdot \left(c_p^+\delta_p^+\exp(\zeta\delta_p^+) + A_p^l\exp(\zeta\tau_{pq}^+) + B_p^l\exp(\zeta\eta_{pq}^+)\right).\end{aligned} \tag{40}$$

By (H_4), we have

$$\begin{aligned}\Theta_p^l(0) &= c_p^- - \Gamma_p^l > 0,\\ \Pi_p^l(0) &= c_p^- - \left(c_p^+ + c_p^-\right)\Gamma_p^l > 0,\end{aligned} \quad p \in \Lambda,\ l \in E. \tag{41}$$

Since Θ_p^l and Π_p^l are continuous on $[0, +\infty)$ and $\Theta_p^l(\zeta), \Pi_p^l(\zeta) \to -\infty$, as $\zeta \to +\infty$, there exist $\xi_p{}^l, \xi_p^{*l} > 0$ such that $\Theta_p^l(\xi_p{}^l) = \Pi_p^l(\xi_p^{*l}) = 0$ and $\Theta_p^l(\zeta) > 0$ for $\zeta \in (0, \xi_p{}^l)$, $\Pi_p^l(\zeta) > 0$ for $\zeta \in (0, \xi_p^{*l})$, $p \in \Lambda$, $l \in E$. Take $\gamma = \min_{p\in\Lambda, l\in E}\{\xi_p{}^l, \xi_p^{*l}\}$; we have $\Theta_p^l(\gamma) \geq 0$, $\Pi_p^l(\gamma) \geq 0$. So, we can choose a positive constant $0 < \lambda < \min\{\gamma, \min_{p\in\Lambda}\{c_p^-\}\}$ such that

$$\begin{aligned}\Theta_p^l(\lambda) &> 0,\\ \Pi_p^l(\lambda) &> 0,\end{aligned} \quad p \in \Lambda,\ l \in E, \tag{42}$$

which imply that, for $p \in \Lambda, l \in E$,

$$\begin{aligned}&\frac{\exp\left(\lambda \sup_{t\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda}\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right)\right.\\&\quad\left.+A_p^l\exp\left(\lambda\tau_{pq}^+\right)+B_p^l\exp\left(\lambda\eta_{pq}^+\right)\right)<1,\\&\left(1+\frac{c_p^+\exp\left(\lambda\sup_{t\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda}\right)\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right)\right.\\&\quad\left.+A_p^l\exp\left(\lambda\tau_{pq}^+\right)+B_p^l\exp\left(\lambda\eta_{pq}^+\right)\right)<1.\end{aligned} \tag{43}$$

Let $M = \max_{p\in\Lambda}\max_{l\in E}\{c_p^-/\Gamma_p^l\}$; then by (H_4) we have $M > 1$. Thus,

$$\begin{aligned}&\frac{1}{M}-\min_{p\in\Lambda,l\in E}\left\{\frac{\exp\left(\lambda\sup_{t\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda}\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right)\right.\right.\\&\quad\left.\left.+A_p^l\exp\left(\lambda\tau_{pq}^+\right)+B_p^l\exp\left(\lambda\eta_{pq}^+\right)\right)\right\}<0.\end{aligned} \tag{44}$$

Let

$$\begin{aligned}\|Z(t)\| &= \max_{p\in\Lambda}\left\{\max_{l\in E}\left\{\left|x_p^l(t)-x^{*l}_p(t)\right|,\left|\left(x_p^l\right)^\Delta(t)-\left(x^{*l}_p\right)^\Delta(t)\right|\right\}\right\},\\ \|\phi\|_0 &= \max_{1\le p\le n}\left\{\max_{l\in E}\left\{\sup_{s\in[-\theta,0]_{\mathbb{T}}}\left|\varphi_p^l(s)-\varphi^{*l}_p(s)\right|,\sup_{s\in[-\theta,0]_{\mathbb{T}}}\left|\left(\varphi_p^l\right)^\Delta(s)-\left(\varphi^{*l}_p\right)^\Delta(s)\right|\right\}\right\}.\end{aligned} \tag{45}$$

Then, for any $\epsilon > 0$, it is obvious that

$$\|Z(0)\| < \left(\|\phi\|_0+\epsilon\right), \tag{46}$$

$$\|Z(t)\| < M\left(\|\phi\|_0+\epsilon\right)e_{\ominus\lambda}(t,0), \quad \forall t\in[-\theta,0]_{\mathbb{T}}. \tag{47}$$

We claim that

$$\|Z(t)\| < M\left(\|\phi\|_0+\epsilon\right)e_{\ominus\lambda}(t,0), \quad \forall t\in(0,+\infty)_{\mathbb{T}}. \tag{48}$$

If (48) is not true, then there must be some $t_1 \in (0,+\infty)_{\mathbb{T}}$ such that

$$\begin{aligned}&\|Z(t_1)\| \ge M\left(\|\phi\|_0+\epsilon\right)e_{\ominus\lambda}(t_1,0),\\&\|Z(t)\| < M\left(\|\phi\|_0+\epsilon\right)e_{\ominus\lambda}(t,0), \quad t\in[-\theta,t_1)_{\mathbb{T}}.\end{aligned} \tag{49}$$

Therefore, there must exist a constant $c \ge 1$ such that

$$\begin{aligned}&\|Z(t_1)\| = cM\left(\|\phi\|_0+\epsilon\right)e_{\ominus\lambda}(t_1,0),\\&\|Z(t)\| < cM\left(\|\phi\|_0+\epsilon\right)e_{\ominus\lambda}(t,0),\\&\qquad t\in[-\theta,t_1)_{\mathbb{T}}.\end{aligned} \tag{50}$$

Multiplying the both sides of (39) by $e_{-c_p}(0,\sigma(t))$ and integrating over $[0,t]_{\mathbb{T}}$, we get

$$\begin{aligned}Z_p(t) = &\left\{Z_p(0)e_{-c_p}(t,0)+\int_0^t e_{-c_p}(t,\sigma(s))\left(a_p(s)\int_{s-\delta_p(s)}^s Z_p^\Delta(u)\Delta u\right.\right.\\&+\sum_{q=1}^n A_{pq}(s)\left(F_q\left(X_q\left(s-\tau_{pq}(s)\right)\right)-F_q\left(X_q^*\left(s-\tau_{pq}(s)\right)\right)\right)\\&\left.\left.+\sum_{q=1}^n B_{pq}(s)\left(F_q\left(X_q^\Delta\left(s-\eta_{pq}(s)\right)\right)-F_q\left(\left(X_q^*\right)^\Delta\left(s-\eta_{pq}(s)\right)\right)\right)\right)\Delta s\right\}, \quad p\in\Lambda.\end{aligned} \tag{51}$$

In view of (46), (47), and (50) and $M > 1$, we have

$$\begin{aligned}&\left|(x-x^*)_p^R(t_1)\right|\\&\quad=\left|\left(x_p^R(0)-x_p^{*R}(0)\right)e_{-c_p}(t_1,0)\right.\\&\quad+\int_0^{t_1}e_{-c_p}(t_1,\sigma(s))\\&\quad\cdot\left(c_p(s)\int_{s-\delta_p(s)}^s\left(\left(x_p^R\right)^\Delta(u)-\left(x_p^{*R}\right)^\Delta(u)\right)\Delta u\right.\\&\quad+\sum_{q=1}^n\left(a_{pq}^R(s)\left(f_q^R\left(x_q^R\left(s-\tau_{pq}(s)\right)\right)\right.\right.\\&\quad-f_q^R\left(x_q^{*R}\left(s-\tau_{pq}(s)\right)\right)\right)\end{aligned}$$

$$
\begin{aligned}
& - a_{pq}^{I}(s)\left(f_q^I\left(x_q^I\left(s-\tau_{pq}(s)\right)\right) - f_q^I\left(x_q^{*I}\left(s-\tau_{pq}(s)\right)\right)\right) \\
& - a_{pq}^{J}(s)\left(f_q^J\left(x_q^J\left(s-\tau_{pq}(s)\right)\right) - f_q^J\left(x_q^{*J}\left(s-\tau_{pq}(s)\right)\right)\right) \\
& - a_{pq}^{K}(s)\left(f_q^K\left(x_q^K\left(s-\tau_{pq}(s)\right)\right)\right. \\
& \left.\left.\left. - f_q^K\left(x_q^{*K}\left(s-\tau_{pq}(s)\right)\right)\right)\right)\right) \\
& + \sum_{q=1}^{n}\left(b_{pq}^R(s)\left(g_q^R\left(\left(x_q^R\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right.\right. \\
& \left. - g_q^R\left(\left(x_q^{*R}\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right) \\
& - b_{pq}^I(s)\left(g_q^I\left(\left(x_q^I\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right. \\
& \left. - g_q^I\left(\left(x_q^{*I}\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right) \\
& - b_{pq}^J(s)\left(g_q^J\left(\left(x_q^J\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right. \\
& \left. - g_q^J\left(\left(x_q^{*J}\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right) \\
& - b_{pq}^K(s)\left(g_q^K\left(\left(x_q^K\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right. \\
& \left.\left.\left. - g_q^K\left(\left(x_q^{*K}\right)^{\Delta}\left(s-\eta_{pq}(s)\right)\right)\right)\right)\right)\Delta s\Bigg| \\
& \le e_{-c_p}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& + cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& \cdot \int_0^{t_1} e_{-c_p\oplus\lambda}(t_1,\sigma(s))\left(c_p^+\delta_p^+ \times e_\lambda\left(\sigma(s), s-\delta_p(s)\right)\right. \\
& + A_p^R e_\lambda\left(\sigma(s), s-\tau_{pq}(s)\right) \\
& \left. + B_p^R e_\lambda\left(\sigma(s), s-\eta_{pq}(s)\right)\right)\Delta s \\
& \le cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& \cdot \left\{\frac{e_{-c_p\oplus\lambda}(t_1,0)}{cM} + \int_0^{t_1} e_{-c_p\oplus\lambda}(t_1,\sigma(s))\right. \\
& \cdot \exp\left(\lambda \sup_{s\in\mathbb{T}}\mu(s)\right) \\
& \times\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right) + A_p^R\exp\left(\lambda\tau_{pq}^+\right)\right. \\
& \left.\left. + B_p^R\exp\left(\lambda\eta_{pq}^+\right)\right)\Delta s\right\} \\
& < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& \cdot \left\{\frac{e_{-(c_p-\lambda)}(t_1,0)}{M} + \frac{1-e_{-(c_p-\lambda)}(t_1,0)}{c_p^- - \lambda}\right. \\
& \cdot \exp\left(\lambda \sup_{s\in\mathbb{T}}\mu(s)\right) \\
& \times\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right) + A_p^R\exp\left(\lambda\tau_{pq}^+\right)\right. \\
& \left.\left. + B_p^R\exp\left(\lambda\eta_{pq}^+\right)\right)\right\} \\
& = cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& \cdot \left\{\left[\frac{1}{M} - \frac{\exp\left(\lambda\sup_{s\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda}\right.\right. \\
& \cdot\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right) + A_p^R\exp\left(\lambda\tau_{pq}^+\right)\right. \\
& \left.\left. + B_p^R\exp\left(\lambda\eta_{pq}^+\right)\right)\right] e_{-(c_p-\lambda)}(t_1,0) \\
& + \frac{\exp\left(\lambda\sup_{s\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda} \\
& \cdot\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right) + A_p^R\exp\left(\lambda\tau_{pq}^+\right)\right. \\
& \left.\left. + B_p^R\exp\left(\lambda\eta_{pq}^+\right)\right)\right\} \\
& < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right), \quad p\in\Lambda.
\end{aligned} \tag{52}
$$

Similarly, we can get

$$
\left|\left(x-x^*\right)_p^l(t_1)\right| < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right), \quad p\in\Lambda,\ l=I,J,K. \tag{53}
$$

On the other hand, we have

$$
\begin{aligned}
& \left|\left(\left(x-x^*\right)_p^R\right)^{\Delta}(t_1)\right| \le -c_p^+ e_{-c_p}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& + cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right)\left(c_p^+\delta_p^+ e_\lambda\left(t_1, t_1\right.\right. \\
& \left. - \delta_p(t_1)\right) + A_p^R e_\lambda\left(t_1, t_1-\tau_{pq}(t_1)\right) + B_p^R e_\lambda\left(t_1, t_1\right. \\
& \left.\left. - \eta_{p_1q}(t_1)\right)\right) + c_p^+ cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right) \\
& \times\left(c_p^+\delta_p^+ e_\lambda\left(\sigma(s), s-\delta_p(s)\right) + A_p^R e_\lambda\left(\sigma(s), s\right.\right. \\
& \left.\left. - \tau_{pq}(s)\right) + B_p^R e_\lambda\left(\sigma(s), s-\eta_{pq}(s)\right)\right)\Delta s \\
& < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right)\left\{\left[\frac{1}{M}\right.\right. \\
& - \frac{\exp\left(\lambda\sup_{s\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda}\left(c_p^+\delta_p^+\exp\left(\lambda\delta_p^+\right)\right. \\
& \left.\left. + A_p^R\exp\left(\lambda\tau_{pq}^+\right) + B_p^R\exp\left(\lambda\eta_{p_1q}^+\right)\right)\right] c_p^+ e_{-(c_p-\lambda)}(t_1, \\
& 0) + \left(1 + \frac{c_p^+\exp\left(\lambda\sup_{s\in\mathbb{T}}\mu(s)\right)}{c_p^- - \lambda}\right)\times\left(c_p^+\delta_p^+\right. \\
& \left.\left. \cdot\exp\left(\lambda\delta_p^+\right) + A_p^R\exp\left(\lambda\tau_{pq}^+\right) + B_p^R\exp\left(\lambda\eta_{pq}^+\right)\right)\right\} \\
& < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right), \quad p\in\Lambda.
\end{aligned} \tag{54}
$$

Similarly, we have

$$\left|\left(\left(x-x^*\right)_p^l\right)^\Delta(t_1)\right| < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right),$$
$$p\in\Lambda,\ l=I,J,K. \tag{55}$$

It follows from (52) to (55) that

$$\|Z(t_1)\| < cMe_{\ominus\lambda}(t_1,0)\left(\|\phi\|_0+\epsilon\right), \tag{56}$$

which contradicts the first equation of (49). Therefore, (48) holds. Let $\epsilon\to 0^+$ leads to

$$\|Z(t)\|\le Me_{\ominus\lambda}(t,0)\|\phi\|_0,\quad \forall t\in(0,+\infty)_{\mathbb{T}}. \tag{57}$$

Hence, the pseudo almost periodic solution of system (17) is globally exponentially stable. The proof is complete. □

Remark 13. From Theorems 11 and 12, we can find that the time delays in the leakage term are harmful for the existence and stability of almost periodic solutions of quaternion-valued system (1). Therefore, the time delays in the leakage term cannot be ignored.

Remark 14. In view of Theorems 11 and 12, we can also find that the neutral terms have an influence on the existence and stability of the almost periodic solution. Therefore, they cannot be ignored too.

Remark 15. According to Theorems 11 and 12, it is clear that if the coefficients of leakage terms of (1) are positive regressive, then the continuous-time network and its corresponding discrete-time network have the same dynamics for the pseudo almost periodicity.

4. An Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Section 3.

Example 1. Let $n=2$. Consider the following quaternion-valued neural networks with time delays on almost periodic time scale $\mathbb{T}$:

$$\begin{aligned}x_p^\Delta(t) &= -c_p(t)\,x_p\left(t-\delta_p(t)\right)\\ &\quad+\sum_{q=1}^{n}a_{pq}(t)\,f_q\left(x_q\left(t-\tau_{pq}(t)\right)\right)\\ &\quad+\sum_{q=1}^{n}b_{pq}(t)\,g_q\left(x_q^\Delta\left(t-\eta_{pq}(t)\right)\right)+u_p(t),\end{aligned} \tag{58}$$

where $p=1,2$, $t\in\mathbb{T}$ and the coefficients are as follows:

$$\begin{aligned}
c_1(t) &= 0.4+0.1\left|\sin\sqrt{2}t\right|,\\
c_2(t) &= 0.5-0.1\left|\sin t\right|,\\
f_1(x_1) &= f_2(x_1)\\
&= \frac{1}{15}\sin x_1^R + i\frac{1}{15}\left|\cos x_1^I-1\right| + j\frac{1}{30}\sin^2 x_1^J\\
&\quad + k\frac{1}{15}\tanh x_1^K,\\
g_1(x_1) &= g_2(x_1)\\
&= \frac{1}{32}\left(\left|x_1^R+1\right|+\left|x_1^R\right|-1\right)+i\frac{1}{32}\sin^2 x_1^I\\
&\quad + j\frac{1}{16}\left|x_2^J\right| + k\frac{1}{32}\sin^2 x_2^K,\\
a_{11}(t) &= a_{12}(t)\\
&= 0.2\left|\cos\left(\sqrt{2}t\right)\right| + i0.1\left|\sin\left(\sqrt{3}t\right)\right|\\
&\quad + j0.24\left|\cos t\right| + k0.26\left|\sin t\right|,\\
a_{21}(t) &= a_{22}(t)\\
&= 0.28\left|\sin\left(\sqrt{2}t\right)\right| + i0.32\left|\cos t\right|\\
&\quad + j0.25\left|\cos\left(\sqrt{2}t\right)\right| + k0.22\left|\cos t\right|,\\
b_{11}(t) &= b_{12}(t)\\
&= 0.16\left|\cos t\right| + i0.2\left|\cos\left(\sqrt{3}t\right)\right|\\
&\quad + j0.15\left|\sin\left(\sqrt{3}t\right)\right| + k0.25\left|\cos\left(\sqrt{2}t\right)\right|,\\
b_{21}(t) &= b_{22}(t)\\
&= 0.2\left|\sin\left(\sqrt{3}t\right)\right| + i0.18\left|\sin t\right| + j0.3\left|\cos t\right|\\
&\quad + k0.26\left|\sin\left(\sqrt{2}t\right)\right|,\\
u_1(t) &= u_2(t)\\
&= 0.1\cos\left(\sqrt{2}t\right) + i0.15\sin t\\
&\quad + j0.09\sin\left(\sqrt{2}t\right) + k0.12\cos\left(\sqrt{2}t\right),\\
\delta_1(t) &= 0.01\left|\sin(2\pi t)\right|,\\
\delta_2(t) &= 0.02\sin^2(\pi t),\\
\tau_{pq}(t) &= 0.2\left|\cos\left(\pi t+\frac{\pi}{2}\right)\right|,\\
\eta_{pq}(t) &= 0.3\left|\cos\left(\pi t+\frac{3\pi}{2}\right)\right|,\\
&\qquad p,q=1,2.
\end{aligned} \tag{59}$$

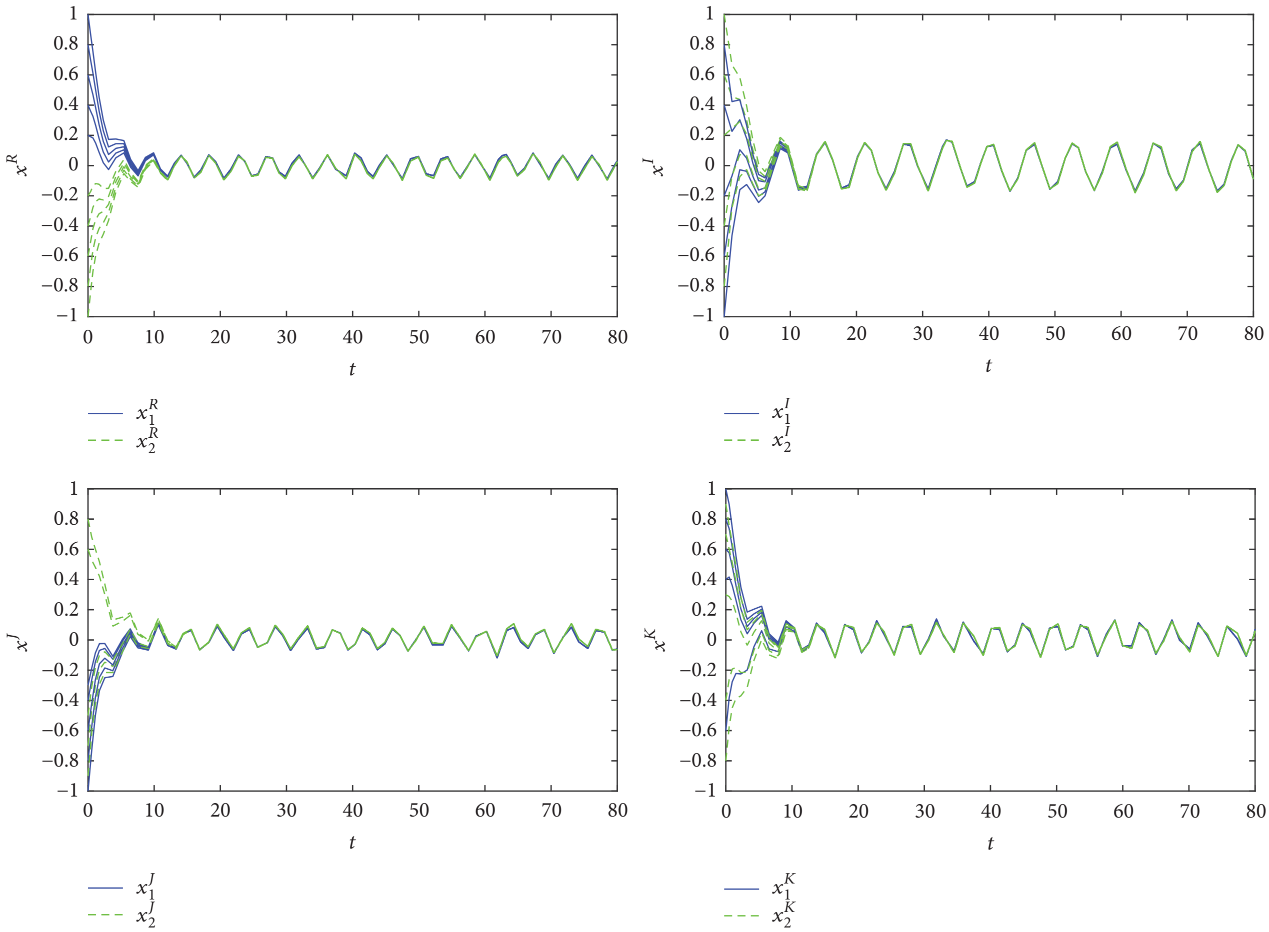

FIGURE 1: Transient states of four parts with continuous time t of (58) in Example 1.

Obviously, (H_1) holds. By calculating, we have

$$
\begin{aligned}
c_1^- &= 0.3, \\
c_2^- &= 0.4, \\
c_1^+ &= 0.4, \\
c_2^+ &= 0.5, \\
\alpha_1^l &= \alpha_2^l = \frac{1}{15}, \\
\beta_1^l &= \beta_2^l = \frac{1}{16}, \\
& l \in E, \\
a_{11}^{R^+} &= a_{12}^{R^+} = 0.2, \\
a_{11}^{I^+} &= a_{12}^{I^+} = 0.1, \\
a_{11}^{J^+} &= a_{12}^{J^+} = 0.24, \\
a_{11}^{K^+} &= a_{12}^{K^+} = 0.26, \\
a_{21}^{R^+} &= a_{22}^{R^+} = 0.28, \\
a_{21}^{I^+} &= a_{22}^{I^+} = 0.32, \\
a_{21}^{J^+} &= a_{22}^{J^+} = 0.25, \\
a_{21}^{K^+} &= a_{22}^{K^+} = 0.22, \\
b_{11}^{R^+} &= b_{12}^{R^+} = 0.16, \\
b_{11}^{I^+} &= b_{12}^{I^+} = 0.2, \\
b_{11}^{J^+} &= b_{12}^{J^+} = 0.15, \\
b_{11}^{K^+} &= b_{12}^{K^+} = 0.25, \\
b_{21}^{R^+} &= b_{22}^{R^+} = 0.2, \\
b_{21}^{I^+} &= b_{22}^{I^+} = 0.18, \\
b_{21}^{J^+} &= b_{22}^{J^+} = 0.3, \\
b_{21}^{K^+} &= b_{22}^{K^+} = 0.26, \\
u_1^{R^+} &= u_2^{R^+} = 0.1,
\end{aligned}
$$

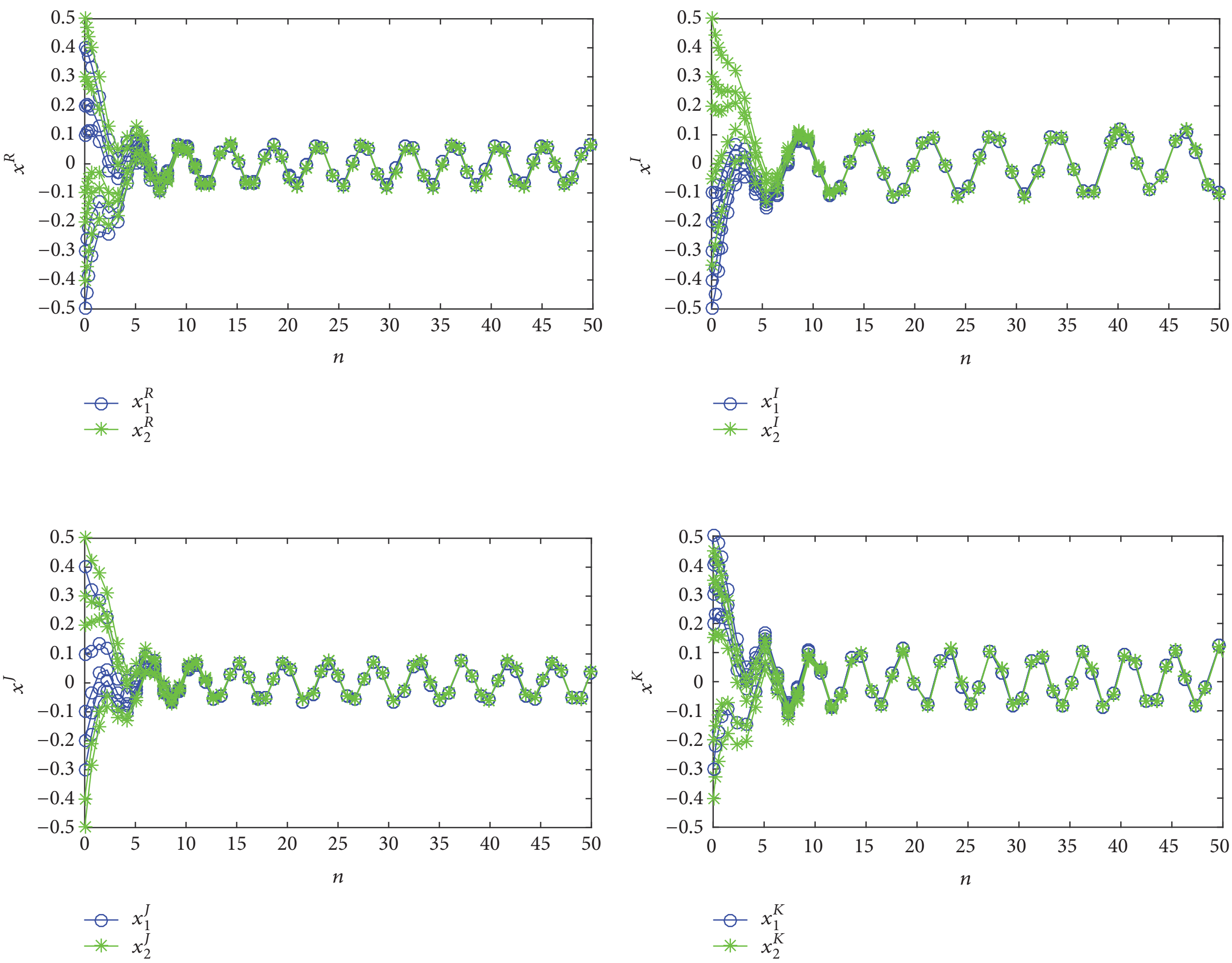

FIGURE 2: Transient states of four parts with discrete time t of (58) in Example 1.

$$\begin{aligned} &u_1^{I^+} = u_2^{I^+} = 0.15, \\ &u_1^{J^+} = u_2^{J^+} = 0.09, \\ &u_1^{K^+} = u_2^{K^+} = 0.12, \\ &\delta_1^+ = 0.01, \\ &\delta_2^+ = 0.02, \\ &\tau_{pq}^+ = 0.2, \\ &\eta_{pq}^+ = 0.3, \\ &\qquad p, q = 1, 2. \end{aligned} \tag{60}$$

It is easy to see that the following conditions hold. Take $\kappa = 2$; then, we have

$$\Gamma_1^R = \Gamma_1^I = \Gamma_1^J = \Gamma_1^K = 0.2057,$$

$$\Gamma_2^R = \Gamma_2^I = \Gamma_2^J = \Gamma_2^K = 0.2702,$$

$$\begin{aligned} &\max_{1\le p\le 2}\left\{\max_{l\in E}\left\{\frac{\Gamma_p^l\kappa + u_p^{l^+}}{c_p^-}, \left(1+\frac{c_p^+}{c_p^-}\right)\left(\Gamma_p^l\kappa + u_p^{l^+}\right)\right\}\right\} \\ &\quad = 1.8713 < \kappa = 2, \\ &\max_{1\le p\le 2}\left\{\max_{l\in E}\left\{\frac{\Gamma_p^l}{c_p^-}, \left(1+\frac{c_p^+}{c_p^-}\right)\Gamma_p^l\right\}\right\} = 0.6857 = \rho < 1. \end{aligned} \tag{61}$$

Therefore, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, all the conditions of Theorems 11 and 12 are satisfied; hence, we know that system (58) has a unique pseudo almost periodic solution, which is globally exponentially stable. This is, the continuous-time neural network and its discrete-time analogue have the same dynamical behaviors for the pseudo almost periodicity (see Figures 1 and 2).

5. Conclusion

In this paper, we have proposed a class of quaternion-valued neural networks of neutral type with delays in the leakage term on time scales. Based on the exponential dichotomy of linear dynamic equations on time scales, Banach's fixed point theorem and the theory of calculus on time scales, we obtain some sufficient conditions on the existence and global exponential stability of pseudo almost periodic solutions for the quaternion-valued neural networks. An example has been given to demonstrate the effectiveness of our results. To the best of our knowledge, this is the first time to study the pseudo almost periodic solutions for quaternion-valued neural networks on time scales. Our methods used in this paper can be applied to study other types of quaternion-valued systems on times scales.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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