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The logic of ground

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Abstract
I explore the logic of ground. I first develop a logic of weak ground. This logic strengthens the logic of weak ground presented by Fine in his 'Guide to Ground.' This logic, I argue, generates many plausible principles which Fine's system leaves out. I then derive from this a logic of strict ground. I argue that there is a strong abductive case for adopting this logic. It's elegant, parsimonious and explanatorily powerful. Yet, so I suggest, adopting it has important consequences. First, it means we should think of ground as a type of identity. Second, it means we should reject much of Fine's logic of strict ground. I also show how the logic I develop connects to other systems in the literature. It is definitionally equivalent both to Angell's logic of analytic containment and to Correia's system **G**.

Keywords Logic of ground · Grounding · Identification · Analytic containment

1 Introduction

Many philosophers think there is a distinctive type of non-causal explanation. The term '*in virtue of*' can express this type of explanation. But it is now commonly expressed with the term '*grounds*.' We can locate the intended notion by pointing to paradigmatic examples. Consider the relationship between sets and their members. The existence of these members is thought to explain the existence of sets. This explanatory connection is a connection of ground. The same is true of the relation between composite objects and their parts and between abstracta and concreta. The existence of parts is often thought to explain the existence of wholes. The existence

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of concreta is often thought to explain the existence of abstracta. These explanatory connections are connections of ground. We also express such connections when we say that the physical explains the mental, the natural explains the normative and the determinate explains the determinable. These are all connections of ground.

Many philosophers think ground is important. Several think it has general application throughout philosophy. Several think it plays a central role in metaphysics. Some think that metaphysics should aim to determine what grounds what. Some think that we should state key metaphysical theories in terms of ground. Some think that questions of ground play a crucial role in determining what is real.[1] If any of this is right, it's important to gain a theoretical understanding of ground. In this paper I aim to contribute to such an understanding. In particular, I aim to explore the logic of ground. The logic of ground comprises general claims about what grounding connections things stand in. I'll focus on what grounding connections truth-functional propositions stand in. For instance, perhaps disjunctions are grounded in their disjuncts. Perhaps conjunctions are grounded in their conjuncts.[2] To explore the logic of ground is to explore this kind of claim.

The most influential logic of ground is the one Kit Fine develops in 'Guide to Ground.'[3] Fine takes a specific notion of ground –weak ground– as fundamental. I will do the same. I aim to first get the logic of weak ground right. I then define other notions of ground in terms of weak ground. In Section 3 I'll outline Fine's logic of weak ground. In Sections 4 and 5 we'll look at various pressures to strengthen Fine's system. In Section 5 we'll also see the most interesting consequence of strengthening Fine's system: it pushes us to think of ground as a type of identity. I sum up the strengthened system –called LWG– in Section 6. In Section 8 we'll see the logic of strict ground this system generates. In Sections 7, 9 and 10 we'll see how it connects to others in the literature. In Section 7 we'll see it's definitionally equivalent to Angell's logic of analytic containment. In Section 9 we'll see it's incompatible with almost all the rules Fine thinks govern strict full ground. And in Section 10 we'll see that it's definitionally equivalent to the system Correia considers in [2]. In some sense, then, I will present a case for a reformulation of the system Correia considers. But before doing all that we better make clear a few distinctions between different notions of ground.

2 Distinctions of ground

Several distinctions between notions of ground will be important to this paper. The most important such distinction is between strict ground and weak ground. Strict ground never holds between a proposition and itself. Weak ground always holds between a proposition and itself. It is the latter which is most central to the system Fine presents in [13]. He proposes we understand strict ground as asymmetric weak ground: A strictly grounds B iff A weakly grounds B and B doesn't even help weakly

[1] See [19], [18, 110-14] and [10] for defences of each claim.

[2] Rosen [18, 117], Schnieder [20, 449], Fine [13] and Correia [3] all suggest this.

[3] See [13].

ground A. Weak ground will also be central to my system. I aim to first formulate a logic of weak ground (Sections 3–6) and then derive from this a logic of strict ground (Section 8).

We can characterize weak ground a couple ways. One nice characterization is in terms of strict ground and identification. On this characterization, A weakly grounds B iff A strictly grounds B *or* for A to be the case just is for B to be the case.[4] The idea is that weak ground is strict ground or identification. What is identification? It's an identity-like connection between sentences. It's the notion expressed by claims like 'for water to be wet just is for H_2O to be wet' and 'for there to be squares just is for there to be rectangular equilaterals.' In these claims 'water is wet' is identified with 'H_2O is wet' and 'there are squares' with 'there are rectangular equilaterals.' Like identity, identification is reflexive, symmetric and transitive. And, like identity, identification obeys a version of Leibniz's Law. We'll see what version in Section 5. If the reader doesn't get the notion, they can consult [9].

Let's give a second characterization of weak ground. This turns on explanatory subsumption. On this characterization, A weakly grounds B iff A explains everything B explains.[5] A weak ground of a proposition shares that proposition's explanatory role. We could take the relevant explanations to just be ground-theoretic explanations. Then, this allows us to define weak ground solely in terms of strict ground. We can say that A weakly grounds B iff A strictly grounds everything B strictly grounds. So we have two ways to think of weak ground.[6]

Another important distinction is between factive and non-factive ground. Factive ground connects only truths. Non-factive ground also connects falsehoods. One can define factive (full) ground in terms of non-factive ground. The definition is: A factively grounds B iff A non-factively grounds B and A obtains. It doesn't seem possible to define non-factive ground in terms of factive ground. This provides reason to take non-factive ground as more basic. The greater elegance of the rules for non-factive ground provides more reason. Quite generally, if we have the rule: A non-factively grounds B, then we have the rule: if A, then A factively grounds B. The rules for factive grounds are just restrictions of the rules for non-factive ground. As a result up until Section 10 I will be entirely concerned with non-factive ground. Only in Section 10 will we consider any rules for factive ground.

The third important distinction is between partial and full ground. I'll say A is a partial ground for B iff A, perhaps together with some other propositions, fully grounds B. In this case, A *helps* ground B. For example, A might partially ground

[4] Correia makes the same suggestion in [5, 516].

[5] Amongst theorists of ground, deRosset has expressed the most scepticism of weak ground in print [8]. He points out that this characterization may be problematic when the generalization is satisfied vacuously [8, 16-7]. For instance, suppose there are only three propositions, A, B and $A \wedge B$. Then plausibly both A and $A \wedge B$ strictly fully ground nothing, and so strictly fully ground the same things. Hence, by this characterization, they weakly ground one another. But this seems implausible. Although insightful, I think this is non-problematic. As deRosset also points out, we can escape this particular problem by stipulating that every proposition strictly grounds some further proposition. This makes the denuded worlds deRosset envisages impossible. Moreover, this is a straightforward implication of popular rules such as $\frac{}{A < \neg\neg A}$ which are in Fine's original system. With this rule, A would strictly ground $\neg\neg A$, and $A \wedge B$ would not.

[6] Fine [13, 51-2] gives some more characterizations of weak ground.

$A \wedge C$ without fully grounding $A \wedge C$. This is because A, C fully ground $A \wedge C$. Up until Section 7 we will only be concerned with full ground. Only when deriving the logic of strict ground in Section 8 will partial ground become important.

These three distinction will play an important role in this paper. Some theorists think there are further important distinctions between notions of ground. The most influential such distinction is probably that between worldly and representational ground.[7] Correia describes the distinction as follows:

> The two kinds of conceptions differ to an important extent in how fine-grained they take grounding to be. Assuming grounding to be a relation, on a worldly conception it is natural to take the items related to be worldly items, say states of affairs or situations, whereas on a representational conception it is natural to take them to be representations, say propositions of some kind [5, 508].

I take no stand on whether we should make this further distinction. But if we do the following questions arises: for which notion of ground do I aim to provide a logic for? My answer is: whatever notion the explanatory claims at the start of this paper express. This answer is insufficient only if those claims are ambiguous between worldly and representational ground. But I doubt that those claims are so ambiguous. This is because they fail the standard tests of ambiguity and, as Grice says, 'senses are not to be multiplied beyond necessity.'[8] We should not posit ambiguity unless we have compelling theoretical or intuitive reason to do so. So I will assume that the explanatory claims with which we began are univocal. They express *the* target notion of ground. It is for this notion which I will provide a logic. Perhaps there is some other important notion of ground for which a different logic would be appropriate. But if there is it would not seem to be the one at work in these central explanatory claims.

3 Fine's logic of weak ground

We can now start exploring the logic of ground. I'll begin by presenting a version of Fine's logic. This version will include those of Fine's rules which deal with weak full ground and some anodyne additions.[9] It is this version which I think we should strengthen. I will restrict myself to the truth-functional case. The vocabulary of this systems contains the truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction) and $\vee$ (disjunction), with $A \to B$ (material implication) defined as $\neg A \vee B$ and $A \leftrightarrow B$ (material biconditional) defined as $A \to B \wedge B \to A$. Finally, we take $\leq$ to express weak full ground.

We then define basic formulas as follows:[10]

[7] See [2, 255-57], [5, 508], [6, 58]. See also [3, 31-32] for a distinction between metaphysical, conceptual and logical ground and [13, 38-40] for a distinction between metaphysical, natural and normative ground.

[8] For the tests see [22]. The quote expresses Grice's 'Modified Occam's Razor' from [15, 47]. Kripke articulates similar sentiments in [16, 278].

[9] It also allows truth-functional compounds of grounding statements (e.g. $\neg(A \leq B)$). This is necessary for articulating adequate introduction rules for strict ground in Section 7.

[10] My definition is quite similar to Correia's in [2, 259].

- $a, b, c \ldots$ with or without numerical subscripts are basic formulas
- If a and b are basic formulas, then so are $(a \wedge b)$, $(a \vee b)$ and $\neg a$

Now let a *list* consist of a sequence of any finite number of basic formulas separated by ','. Such sequences may contain just a single formula, or even no formulas whatsoever. We also interpret the grammar of ',' such that lists are invariant under both permutation and repetition: *a, b, c*..., for instance, is treated as the same list as *c, b, a*... and *a* is treated as the same list as *a, a, a*....

We define *well-formed formulas* as follows:

- The basic formulas are wffs
- If Δ is a list and c is a basic formula, then $(\Delta \leq c)$ is a wff[11]
- If A and B are wffs, then $(A \wedge B)$, $(A \vee B)$ and $\neg A$ are wffs

I use $A, B, C \ldots$ to indicate arbitrary formulas, $\Delta, \Gamma \ldots$ for arbitrary lists (with or without numerical subscripts) and will often omit brackets for readability. I will also often write Δ_1, Δ_2 to denote the list consisting of all the sentences in Δ_1 and Δ_2.

With grammar established, we move on to proof theory. Fine presents a system of natural deduction. Derivations take the form of a tree:

$$\frac{\Phi_1; \Phi_2 \ldots}{\Psi}$$

Nodes in the tree are expressions of the grounding language. Ψ is the root of the tree. The leaves of the tree are either inferred by rules of the form $\overline{\Psi}$ or from hypotheses $\Phi_1; \Phi_2 \ldots$ via the rules below. The intended interpretation of these trees is ordinary validity: if the outermost leaves of a tree are true, then so must be the root. So, they express general principles about how certain facts are grounded. I will sometimes compress such trees by writing:

$$\frac{\overline{}\\ \vdots}{\Psi}$$

Where the vertical dots represent an unwritten set of the below steps. I'll state the steps on which an inference depends with a label on the left-hand side of each line.

We can now present the version of Fine's system which interests us. Again, this is a system which deals with weak full ground and the truth-functors. We can split this up into a *pure* and *impure* logic. The pure logic abstracts away from the logical structure of the grounding or grounded sentence. The impure logic takes account of this structure. The former consists of just the following rules:

THE PURE LOGIC

$$\text{CUT}(\leq/\leq)\ \frac{\Delta_1 \leq A_1 \qquad \Delta_2 \leq A_2 \ldots \qquad A_1, A_2 \ldots \leq C}{\Delta_1, \Delta_2 \ldots \leq C}$$

$$\text{Identity}\ \frac{}{A \leq A} \qquad\qquad \text{IMP}\ \frac{A \qquad A \leq B}{B}$$

[11] This clause means I'm formulating ground as a sentential operator. This is common in the literature.

CUT allows the chaining together of statements of weak ground. Identity ensures every proposition weakly grounds itself. IMP says that if A weakly grounds B, then A implies B. Fine omits IMP from his system because his interest is in the inference relations between ground-theoretic forumulas.[12] But it clearly comports with his notion of ground. I add it because it will be essential in Section 5.

We now turn to the impure logic. The main idea here is that the strict grounds of logically complex truths conform to the classical truth-conditional semantic clauses (see [12, 105-6]). For instance, the classical truth-conditional semantic clause for $A \wedge B$ is: A ∧ B is true iff A is true and B is true. So, the impure logic comprises the following rules:[13]

THE IMPURE LOGIC

$$\vee\text{-I}_1 \frac{}{A \leq A \vee B} \qquad \vee\text{-I}_2 \frac{}{B \leq A \vee B}$$

$$\wedge\text{-I} \frac{}{A, B \leq A \wedge B}$$

$$\neg\wedge\text{-I}_1 \frac{}{\neg A \leq \neg(A \wedge B)} \qquad \neg\wedge\text{-I}_2 \frac{}{\neg B \leq \neg(A \wedge B)}$$

$$\neg\vee\text{-I} \frac{}{\neg A, \neg B \leq \neg(A \vee B)}$$

$$\neg\neg\text{-I} \frac{}{A \leq \neg\neg A}$$

With these rules we can introduce logical complexity on the right-hand side of the grounding operator. So they capture an attractive theory of the grounds of logically complex propositions. We also add the classical rules for truth-functors. I trust these are familiar. So I won't outline them.

This completes our review of the relevant version of Fine's system. Note that amalgamation and transitivity are valid in this system:

$$\text{Amalg} \frac{\Delta_1 \leq C \qquad \Delta_2 \leq C \ldots}{\Delta_1, \Delta_2 \ldots \leq C} \qquad \text{Trans} \frac{A \leq B \qquad B \leq C}{A \leq C}$$

These will be useful in some of the succeeding proofs. Apart from this note that Fine's system never allows us to introduce logical complexity on the left-hand side of the grounding operator (except for a single ¬). So, although his system tells us rather a lot about what grounds logically complex propositions, it tells us little about what logical complex propositions ground.[14] It does not, for instance, tell us whether $A \vee B$ weakly grounds $B \vee A$. I think this creates quite general pressure to strengthen the logic. That is because it seems to me that a logic of ground should give us some general guidance about what logically complex propositions ground. In particular, certain relevant principles seem to me valid. I will explore this in the next section.

[12] He confirmed this to me in conversation.

[13] In [13], Fine derives these from rules for strict ground. I don't want to deal with strict ground yet. So, I take them to be underived in this fragment.

[14] Fine briefly suggests some relevant principles in [13, 67]. But he does not go into detail.

4 What do logically complex propositions ground?

4.1 Conjunction

In this section, I'll explore some general pressure to strengthen this version of Fine's logic. We'll begin by focusing on conjunction. Fine's system contains no left-hand-side introduction rules for weak ground. I think this renders it overly weak and intuitively incomplete. It renders the system overly weak in the sense that it means the system provides no general account of what conjunctions ground. It renders it intuitively incomplete in that it means the system doesn't generate intuitively valid principles. This is where the pressure to strengthen Fine's system comes from.

Let's illustrate these problems by looking at some principles Fine's system fails to generate. Consider the following four principles:

$$\text{Commutativity-}\wedge\ \frac{}{A \wedge B \leq B \wedge A}$$

$$\text{Associativity-}\wedge\ \frac{}{(A \wedge B) \wedge C \leq A \wedge (B \wedge C)}$$

$$\text{DeMorgan(1)-}\wedge\vee\ \frac{}{(\neg A \wedge \neg B) \leq \neg(A \vee B)}$$

$$\text{Supplementation-}\wedge\ \frac{A \leq B \qquad C \leq D}{A \wedge C \leq B \wedge D}$$

These principles say certain grounding relations always hold. For instance, the commutativity principle says that $A \wedge B$ always grounds $B \wedge A$. As I've said, Fine's system cannot generate these principles. That's because it contains no left-hand side introduction rules for conjunction. It doesn't tell us what conjunctions ground. This seems to me a problem in itself. We should prefer a system which gives us a general characterization of the grounding relations conjunctions stand in. This connects to the theoretical virtue of strength. I think we should quite generally prefer theories which tell us more about the world.[15] When it comes to conjunction Fine's system suffers on this metric.

There is also a second –more serious– problem raised by these principles: they seem to me pretty clearly valid. Consider first commutativity. Not only should one's logic of ground weigh in on when conjunction is commutative over grounding. It should say conjunction *is* commutative over grounding. We can argue for this from our characterizations of weak ground. The first –more important– argument rests on our characterization of weak ground in terms of strict ground and identification. On this characterization, A weakly grounds B iff A strictly grounds B or for A to be the case just is for B to be the case. It seems to me that for $A \wedge B$ to be the case just is for $B \wedge A$ to be the case. This is an intuition about identifications. From this it follows that Commutativity-$\wedge$ must hold. The second –less important– argument

[15] Williamson is probably the most influential recent advocate of strength as a virtue of metaphysical theories. See e.g., [21, 276-77]. He takes it to be one of the 'normal criteria of scientific theory choice' (ibid). Of course, some philosophers don't think strength is a virtue at all. They differ with me (and Williamson) on methodological grounds. Unfortunately, this sort of dispute is often intractable.

rests on our characterization of weak ground as explanatory subsumption. It seems to me that $A \wedge B$ explains everything that $B \wedge A$ explains. This is an intuition about explanatory subsumption. From this it follows that $A \wedge B$ weakly grounds $B \wedge A$. So both characterizations of weak ground support the commutativity principle.[16]

We can make similar arguments for associativity and DeMorgan. Consider the former first. It seems to me that $((A \wedge B) \wedge C)$ just is $(A \wedge (B \wedge C))$. The differing location of the brackets creates no real distinction. So it must be that the former weakly grounds the latter. Now consider the DeMorgan principle. It seems to me that for $\neg A$ and $\neg B$ to be the case just is for neither A nor B to be the case. If this is true, then $(\neg A \wedge \neg B)$ must weakly ground $\neg(A \vee B)$. We can also give arguments from the explanatory subsumption characterization for these rules. But I leave this to the reader. I think this makes these rules very plausible.

The argument for Supplementation-$\wedge$ is not quite so straightforward. But it still seems to me a very intuitive principle. In arguing for it, I'll appeal to just our explanatory subsumption characterization of weak ground. It seems plausible to me that if A explains everything B explains, and C explains everything D explains, then $A \wedge C$ explains everything $B \wedge D$ explains. This is because otherwise there would have to be something which A, C explained and $B \wedge D$ explained, but $A \wedge C$ did not explain. I think it is implausible that there is such a thing. So, by the explanatory subsumption characterization of weak ground, Supplementation-$\wedge$ must follow.

So there are arguments from the characterization of weak ground for all these principles. If these arguments are sound, then Fine's system leaves something out. Fortunately, there is a simple addition to Fine's system which generates all these principles. This addition allows us to introduce a conjunction on the left-hand side of the grounding operator. In other words, it is a rule which tells us what conjunctions ground. The rule is the following:

$$\wedge\text{-Agglomeration}\ \frac{A, B, \Delta \leq C}{A \wedge B, \Delta \leq C}$$

This says that everything (weakly) grounded by A, B, Δ is (weakly) grounded by $A \wedge B$, Δ. This addition generates the above four rules (proofs below). So it deals with the incompleteness from which Fine's system suffers. And it enables a simple, elegant answer to the general question about what conjunctions ground: $A \wedge B$ grounds C if and only if A, B ground C.[17] So it also deals with the lack of strength we saw in Fine's system. I think this creates a strong abductive case for strengthening this system by adding $\wedge$-Agglomeration.

The proofs that $\wedge$-Agglomeration generates the above rules are below. Note that when I label a step 'permutation' I'm taking advantage of the permutation invariance of lists (so $A, B \ldots$ is treated as the same list as $B, A \ldots$) to reiterate the same premise.

[16] Why is the first argument more important? Because the intuition in the second seems to rely, at least to some extent, on the intuition in the first.

[17] $\wedge$-I and transitivity entail that, if $A \wedge B$ grounds C, then A, B ground C.

Proof of Commutativity-∧

$$\wedge\text{-A}\ \dfrac{\text{Permutation}\ \dfrac{\wedge\text{-I}\ \dfrac{}{B, A \leq B \wedge A}}{A, B \leq B \wedge A}}{A \wedge B \leq B \wedge A}$$

Proof of Associativity-∧

$$\wedge\text{-A}\ \dfrac{\wedge\text{-A}\ \dfrac{\text{CUT}\ \dfrac{\text{Identity}\ \dfrac{}{A \leq A} \qquad \wedge\text{-I}\ \dfrac{}{B, C \leq B \wedge C} \qquad \wedge\text{-I}\ \dfrac{}{A, B \wedge C \leq A \wedge (B \wedge C)}}{A, B, C \leq A \wedge (B \wedge C)}}{A \wedge B, C \leq A \wedge (B \wedge C)}}{(A \wedge B) \wedge C \leq A \wedge (B \wedge C)}$$

Proof of Supplementation-∧

$$\wedge\text{-A}\ \dfrac{\text{CUT}\ \dfrac{A \leq B \qquad C \leq D \qquad \wedge\text{-I}\ \dfrac{}{B, D \leq B \wedge D}}{A, C \leq B \wedge D}}{A \wedge C \leq B \wedge D}$$

Proof of DeMorgan(1)-∧∨

$$\wedge\text{-A}\ \dfrac{\neg\vee\text{-I}\ \dfrac{}{\neg A, \neg B \leq \neg(A \vee B)}}{\neg A \wedge \neg B \leq \neg(A \vee B)}$$

As I've said, this generates an abductive case for ∧-Agglomeration. The agglomeration rule settles questions Fine's system leaves open. And it settles these questions in the intuitively correct way. But once we've made this addition, it becomes very tempting to make further additions to Fine's system. In the next section we will see how this plays out with disjunction.

4.2 Disjunction

In this section we'll look at the pressure to supplement Fine's system with a left-hand side introduction rule for disjunction. As before the pressure has two sources. We want a general account of what disjunctions ground and we want to validate several specific principles. The specific principles are counterparts to the principles we discussed in the previous section:

$$\text{Commutativity-}\vee\ \dfrac{}{A \vee B \leq B \vee A}$$

$$\text{Associativity-}\vee\ \dfrac{}{(A \vee B) \vee C \leq A \vee (B \vee C)}$$

$$\text{DeMorgan(1)-}\vee\wedge\ \dfrac{}{(\neg A \vee \neg B) \leq \neg(A \wedge B)}$$

$$\text{Supplementation-}\vee\ \dfrac{A \leq B \qquad C \leq D}{A \vee C \leq B \vee D}$$

Fine's system fails to generate these principles. This is because it tells us so little about what logically complex propositions ground. But it seems to me that, like their counterparts for conjunction, these principles are intuitively plausible. Again, this is supported by our characterizations of weak ground. It seems to me that for $A \vee B$ to be the case just is for $B \vee A$ to be the case. So, by the identification characterization of weak ground, the former must weakly ground the latter. And it seems to me that $A \vee B$ explains everything $B \vee A$ explains. So, by the explanatory subsumption characterization of weak ground, the former must weakly ground the latter. If so, Fine's system is incomplete. Similar arguments can be made for the other principles.

The case for these principles seems even stronger in the presence of $\wedge$-Agglomeration. The issue is one of (dis)unity. If one doesn't accept these rules, then one treats disjunction and conjunction quite differently. This makes the resultant system disunified. This is a theoretical vice, at least in the sense that it incurs an obligation to explain why disjunction and conjunction behave so differently. It's not clear to me how such an explanation might go. So, this creates more pressure to treat disjunction and conjunction symmetrically.

We can generate these principles by strengthening Fine's system. Here we face a choice: there are two different ways we might strengthen Fine's system. The first way is by adding the following rule:

$$\vee\text{-Agglomeration}\ \frac{A, \Delta \leq C \qquad B, \Gamma \leq C}{A \vee B, \Delta, \Gamma \leq C}$$

Informally, this tells us that, given A and B are each individually part of some ground for C, then $A \vee B$ is always part of a ground of C. This give us a general account of what disjunctions ground: a disjunction grounds C if and only if both of its disjuncts ground C. And it generates the principles with which we began the section. This, it seems to me, makes up a good abductive case for supplementing Fine's system with this rule.

But I said we have a choice. We could instead add the following, slightly weaker, rule:

$$\text{Weak } \vee\text{-Agglomeration}\ \frac{A, \Delta \leq C \qquad B, \Delta \leq C}{A \vee B, \Delta \leq C}$$

The difference between this and $\vee$-Agglomeration is that, when applying this rule, the things which help A and B ground C must be the same. This gives us the same general account of when a disjunction grounds something. It also generates the principles with which we began the section. So the abductive case for each rule initially seems much the same. We should adopt at least one of these rules.

Which of these rules should we adopt? This hinges on how conjunction and disjunction interact. Consider the following *distributivity* rules:

$$\wedge\vee\ \text{D1}\ \frac{}{A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)}$$

$$\wedge\vee\ \text{D2}\ \frac{}{(A \wedge B) \vee (A \wedge C) \leq A \wedge (B \vee C)}$$

$$\vee\wedge\ \text{D1}\ \frac{}{A \vee (B \wedge C) \leq (A \vee B) \wedge (A \vee C)}$$

$$\vee\wedge\ \text{D2}\ \frac{}{(A \vee B) \wedge (A \vee C) \leq A \vee (B \wedge C)}$$

Both our disjunction agglomeration rules generate the first three of these rules. But they differ on $\vee\wedge$D2. Only $\vee$-Agglomeration generates this rule. So if $\vee\wedge$D2 is invalid we should prefer Weak $\vee$-Agglomeration. More than that, with this fourth rule we can derive $\vee$-Agglomeration from Weak $\vee$-Agglomeration. So if $\vee\wedge$D2 is valid we should prefer $\vee$-Agglomeration. I prove this in the Appendix A.1. So which agglomeration rule to adopt hinges on the validity of $\vee\wedge$D2.

Is $\vee\wedge$D2 valid? Here are two points in its favour. First, it seems plausible that for $A \vee (B \wedge C)$ to be the case just is for $(A \vee B) \wedge (A \vee C)$ to be the case.[18] Hence, by our characterization of weak full ground, $\vee\wedge$D2 must be valid. Second, it seems inelegant for three of these distributivity rules to be valid but one to be invalid. There is some unity in having all four distributivity rules. So, insofar as aesthetic considerations move us, we should endorse $\vee\wedge$D2.

But we can also raise a point against it. $\vee\wedge$D2 implies the following rule:[19]

$$\frac{}{A, B \leq (A \vee (B \wedge C))}$$

This causes problems for factive ground. We said A, B factively ground C iff A, B non-factively ground C and A, B are both true. So, if A and B are true, it follows that A, B factively grounds $(A \vee (B \wedge C))$. Yet, as Krämer and Roski point out in [17], there are at least *apparent* counterexamples to this result. For example, suppose C is $\neg B$. They suggest it is implausible that A, B factively ground $(A \vee (B \wedge \neg B))$. This is because this implies that B helps factively ground $(A \vee (B \wedge \neg B))$. But it seems like A is the only factive ground for $(A \vee (B \wedge \neg B))$. So perhaps we should reject $\vee\wedge$D2.

I am not certain whether $\vee\wedge$D2 valid. But I find the points in its favour more convincing. This is because if one endorses $\vee\wedge$D2 one can give a good explanation of why B helps grounds $(A \vee (B \wedge \neg B))$. One can say it is because $(A \vee (B \wedge \neg B))$ just is $(A \vee B) \wedge (A \vee \neg B))$ and B clearly helps ground this latter proposition. This seems to reduce the force of Krämer and Roski's apparent counter-example.[20] So I think the argument for $\vee\wedge$D2 wins out. It seems we should adopt $\vee$-Agglomeration. But that isn't completely decisive: we'll return to this at the end of Section 5. In the next section we turn to agglomeration rules for negated conjunction and disjunction.

4.3 Negated disjunction and conjunction

Agglomeration rules for disjunction and conjunction are silent on what negations ground. In this section, I deal with what negated conjunctions and disjunctions

[18] Correia [4, 111-12] rejects this, because the semantics he discusses for a logic of identification invalidates it. But I see this as a *prima facie* problem for his semantics rather than a problem for this claim. Generally, I think semantics should be fashioned to fit our judgements about intuitive validities. Our judgements about intuitive validities should not be refashioned so as to fit with a semantics.

[19] The derivation goes via $\wedge$-I, $\vee$-I_1, $\vee$-I_2 and CUT.

[20] An alternative approach is to modify the connection between factive and non-factive ground. I see no problem with B helping to non-factively ground $(A \vee (B \wedge \neg B))$. The problem arises only if B helps to factively ground $(A \vee (B \wedge \neg B))$. But exploring how to modify this connection would take us too far afield.

ground. Again, Fine's system does not provide any general account of this. To remedy that I endorse the following rules:[21]

$$\neg\vee\text{-Agglomeration}\ \frac{\neg A, \neg B, \Delta \leq C}{\neg(A \vee B), \Delta \leq C}$$

$$\neg\wedge\text{-Agglomeration}\ \frac{\neg A, \Delta \leq C \qquad \neg B, \Gamma \leq C}{\neg(A \wedge B), \Delta, \Gamma \leq C}$$

My reasons for endorsing these rules parallel my reasons for endorsing the other agglomeration rules: they make the system stronger and generate a host of intuitive principles. These principles are as follows:

$$\text{Commutativity-}\neg\vee\ \frac{}{\neg(A \vee B) \leq \neg(B \vee A)}$$

$$\text{Associativity-}\neg\vee\ \frac{}{\neg((A \vee B) \vee C) \leq \neg(A \vee (B \vee C))}$$

$$\text{DeMorgan(2)-}\vee\wedge\ \frac{}{\neg(A \vee B) \leq \neg A \wedge \neg B}$$

$$\text{Supplementation-}\neg\vee\ \frac{\neg A \leq \neg B \qquad \neg C \leq \neg D}{\neg(A \vee C) \leq \neg(B \vee D)}$$

$$\text{Commutativity-}\neg\wedge\ \frac{}{\neg(A \wedge B) \leq \neg(B \wedge A)}$$

$$\text{Associativity-}\neg\wedge\ \frac{}{\neg((A \wedge B) \wedge C) \leq \neg(A \wedge (B \wedge C))}$$

$$\text{DeMorgan(2)-}\wedge\vee\ \frac{}{\neg(A \wedge B) \leq \neg A \vee \neg B}$$

$$\text{Supplementation-}\neg\wedge\ \frac{\neg A \leq \neg B \qquad \neg C \leq \neg D}{\neg(A \wedge C) \leq \neg(B \wedge D)}$$

Again we can buttress the case for these principles by arguing from our characterization(s) of weak ground. I think this makes up a strong case for the negated disjunction and conjunction agglomeration principles. More generally, I think the rules we have discussed so far form a nice unified package. It would seem to me odd to have any of these rules and lack the others. This means that insofar as one accepts *any* of these agglomeration rules, there is some pressure to accept them all. In the next section we turn to a somewhat more peripheral area: double negations.

4.4 Double negation

What do double negations ground? Here I'm less confident. But I think the best answer to this question is: double negations ground everything which the proposition they double negate grounds. This means I endorse the following:

$$\neg\neg\text{-Idempotence}\ \frac{}{\neg\neg A \leq A}$$

[21] Although here we might again endorse a slightly weaker ¬∧-Agglomeration rule.

This is also a left-hand side introduction rule, in that it allows us to introduce double negations on the left of the grounding operator. There are three reasons to endorse this rule. First, it increases the strength of the system. Second, if we accept the agglomeration rules, accepting ¬¬-Idempotence creates unity. This is because the agglomeration rules imply that conjunction, disjunction and their negated counterparts are idempotent. In other words:

∨-Idempotence $\dfrac{}{A \vee A \leq A}$ ∧-Idempotence $\dfrac{}{A \wedge A \leq A}$

¬∨-Idempotence $\dfrac{}{\neg(A \vee A) \leq \neg A}$ ¬∧-Idempotence $\dfrac{}{\neg(A \wedge A) \leq \neg A}$

If these truth-functors are idempotent, it seems nicely unified if double negation is idempotent. Another way of putting this is: if neither self-conjunction, self-negated-conjunction, self-disjunction nor self-negated-disjunction generate a ground-theoretic difference, why should double negation do so? The point is not that it couldn't possibly do so. But if it does that requires some explanation. No explanation is obvious.

A third reason to endorse ¬¬-Idempotence is that it generates some plausible principles. I think the most important such principles are those it generates in concert with our agglomeration rules. We rely on ¬¬-Idempotence (together with the agglomeration rules) to generate the bottom two of the following principles:[22]

∧∨ EQUIVALENCIES

$\dfrac{}{A \vee B \leq \neg(\neg A \wedge \neg B)}$ $\dfrac{}{A \wedge B \leq \neg(\neg A \vee \neg B)}$

$\dfrac{}{\neg(\neg A \wedge \neg B) \leq A \vee B}$ $\dfrac{}{\neg(\neg A \vee \neg B) \leq A \wedge B}$

These principles are plausible because $\vee$ and $\wedge$ are usually thought to be interdefinable in a way they clearly parallel. It's often though that for $(A \vee B)$ to be the case just is for $\neg(\neg A \wedge \neg B)$ to be the case and for $(A \wedge B)$ to be the case just is for $\neg(\neg A \vee \neg A)$ to be the case. By our characterization of weak ground, such claims entail the above equivalencies. Hence, that ¬¬-Idempotence allows us to generate them seems to me a major benefit.

This completes the case for ¬¬-Idempotence. The case for ¬¬-Idempotence seems to me less strong than that for the agglomeration rules. And its addition follows less directly from our addition of the agglomeration rules. But it seems to me strong enough to warrant endorsing this rule. In the next section we'll explore another more peripheral rule. This will allow us to generate a version of Leibniz's law for ground-theoretic equivalence.

[22] We generate the top two with the agglomeration rules.

5 Ground-theoretic equivalence

The final rule(s) I will consider involves ground-theoretic equivalence. Ground-theoretic equivalence is just mutual weak grounding. A and B are ground-theoretically equivalent when A weakly grounds B and B weakly grounds A. We could express this by introducing '$\approx$' (ground-theoretic equivalence) as a meta-linguistic abbreviation for $A \leq B \wedge B \leq A$. But I prefer to treat '$\approx$' as a term in the object language.[23] To do this, we add the following clause to the definition of a wff:

– If a is a basic formula and b is a basic formula, then $a \approx b$ is a wff

We then introduce the following definition:

$$\text{Def}(\approx)\ \frac{}{A \approx B \leftrightarrow A \leq B \wedge B \leq A}$$

This captures the idea that ground-theoretic equivalence just is mutual weak ground.

What rules should ground-theoretic equivalence obey? Ground-theoretic equivalence is clearly reflexive, transitive and symmetric. It also seems plausible that it obeys a version of Leibniz's law. This version of Leibniz's law says that if A and B are ground-theoretically equivalent, then replacing some As for B in any formula of the defined language does not change whether that formula is true. In other words, we want the following rule to be valid:

$$\text{LL}\ \frac{\text{C} \qquad A \approx B}{\text{D}}$$

Where D is the result of substituting some As for Bs in C

I'll now discuss what we need to add to the system to generate LL. In fact, we need to add *very little* to establish LL. IMP straightforwardly establishes that, if C is a truth-functional formula, the relevant instances of LL hold. So we just need to see whether it also holds when C is a ground-theoretic formula. The other rules establish this for almost all ground-theoretic formula. They just miss out those whose only logical operators are a single negation. To get this final case, we need the following addition:

$$\neg\text{-Introduction}\ \frac{A \approx B}{\neg A \approx \neg B}$$

This says that if two sentences are ground-theoretically equivalent, then so are their negations. I think this is intuitively plausible.[24] In the Appendix A.2 I show how to prove LL from this and the other rules. This strengthens the abductive case for the rules we've discussed so far. Together, they generate a plausible rule governing ground-theoretic equivalence.

In the rest of the section we'll explore ground-theoretic equivalence a bit more. First, let's look at the connection between weak full ground and ground-theoretic

[23]This is only because it makes it slightly easier to establish the definitional equivalence between this system and the version of Angell's system discussed in Section 7.

[24]It's worth noting that ¬-Introduction is already an admissible rule: if A and B are theorems, then the relevant instance of it holds. The proof in [14, 202-5] essentially establishes this.

equivalence. The connection is tight: we can actually just think of weak full ground as a type of ground-theoretic equivalence. To show this, let $\hat{\Delta}$ be the result of conjoining all the members of Δ, starting from the left. Then we can prove the following is a theorem of LWG:

$$\text{Reduction Theorem } (\leq/\approx)\ \frac{}{\Delta \leq B \leftrightarrow B \approx B \vee \hat{\Delta}}$$

Here is the proof for when Δ has two members, A_1 and A_2:

Proof of Left-Right.

$$\text{Def}(\approx)\ \cfrac{\vee\text{-A}\ \cfrac{\cfrac{}{B \leq B} \qquad \wedge\text{-A}\ \cfrac{[A_1, A_2 \leq B]}{A_1 \wedge A_2 \leq B}}{B \vee (A_1 \wedge A_2) \leq B} \qquad \vee\text{-I}_1\ \cfrac{}{B \leq B \vee (A_1 \wedge A_2)}}{\cfrac{\vdots}{\rightarrow\text{-Intro}\ \cfrac{B \approx B \vee (A_1 \wedge A_2)}{A_1, A_2 \leq B \rightarrow (B \approx B \vee (A_1 \wedge A_2))}}}$$

Proof of Right-Left.

$$\text{Trans}, \wedge\text{-I}\ \cfrac{\vee\text{-I}_2\ \cfrac{}{(A_1 \wedge A_2) \leq B \vee (A_1 \wedge A_2)} \qquad \text{Def}(\approx)\ \cfrac{[B \approx B \vee (A_1 \wedge A_2)]}{\cfrac{\vdots}{B \vee (A_1 \wedge A_2) \leq B}}}{\cfrac{\vdots}{\rightarrow\text{-Intro}\ \cfrac{A_1, A_2 \leq B}{(B \approx B \vee (A_1 \wedge A_2)) \rightarrow A_1, A_2 \leq B}}}$$

We can then combine these to deduce $A_1, A_2 \leq B \leftrightarrow B \approx B \vee (A_1 \wedge A_2)$. When Δ contains n members, the proof just contains n applications of $\wedge$-agglomeration in the first tree.[25] This theorem makes it plausible that for A to weakly ground B just is for A to be ground-theoretically equivalent to $B \vee A$. It means we can think of weak full ground as a type of ground-theoretic equivalence.

Now let's look at the connection between ground-theoretic equivalence and identifications. As discussed in Section 2, identifications are identity-like connections between two sentences. Consider the claims 'for there to be bachelors just is for there to be unmarried men' and 'for Cicero to be a good speaker just is for Tully to be a good speaker.' These are identifications. Identification has a few general features. It too is transitive, reflexive and symmetric. And it too obeys a version of Leibniz's Law. In other words, if A just is B, one can substitute As for Bs in any formula (at least of our defined language) without disturbing the truth of that formula.

What is the connection between ground-theoretic equivalence and identification? I think A is ground-theoretically equivalent to B if and only if A just is B. Ground-theoretic equivalence holds just in case identification holds. I think this because it

[25] It is thus important that lists are finite. We stipulated this in Section 3.

provably follows from some plausible principles. Let '$<$' stand for strict full ground and '$\equiv$' stand for identification. The principles are:

(i)	$A \approx B \leftrightarrow ((A \leq B) \wedge (B \leq A))$	Definition of ($\approx$)
(ii)	$A \leq B \leftrightarrow ((A \equiv B) \vee (A < B))$	Characterization of ($\leq$)
(iii)	$((A < B) \wedge (B \leq C)) \rightarrow A < C$	Trans ($</\leq$)
(iv)	$\neg(A < A)$	Irreflexivity of ($<$)
(v)	$A \leq A$	Identity
(vi)	$A \equiv B \rightarrow (C \rightarrow C^{[A/B]})$	Leibniz's Law ($\equiv$)

Here $C^{[A/B]}$ is the result of substituting any A in C for B. Here's the proof:

Left-to-right. Suppose $A \approx B$. Then by (i) it follows that $A \leq B$. So by (ii) it follows that $(A \equiv B) \vee (A < B)$. So suppose that $A < B$. Since by assumption $A \approx B$, by (i) it follows that $B \leq A$. So, by (iii), it follows that A $<$ A. But by (iv) $\neg(A < A)$. So, by reductio, $\neg(A < B)$. So, by disjunctive syllogism, $A \equiv B$.

Right-to-left. Suppose $A \equiv B$. By (v), $A \leq A$. So by (vi), we infer $A \leq B$ and by another application of (vi) we infer $B \leq A$. So by (i), $A \approx B$.

The proof is obviously valid. And the principles on which it relies are compelling. (i) is just a version of our definition of ground-theoretic equivalence. (ii) is our initial characterization of weak full ground. This isn't indisputable, but it seems hard to do without it. (iii) is a type of transitivity principle. This is valid in both the logic of strict ground I'll present in Section 8 and in the logic Fine presents in [13]. (iv) says strict full ground is irreflexive. (v) is core to both Fine's and my logic of weak full ground. And (vi) is a version of Leibniz's Law that identification obeys. So it seems that A is ground-theoretically equivalent to B if and only if A just is B.

This result makes it tempting to think that ground-theoretic equivalence just is identification. And the combination of these two points sheds light on the nature of weak full ground. For suppose that weak full ground really just is a type of ground-theoretic equivalence. And suppose that ground-theoretic equivalence really just is identification. Then weak full ground just is a type of identification: for A to weakly fully grounds B just is for B to be the same as $B \vee A$.[26] That our logic generates this smooth connection between these notions is another point in its favour. Indeed, this seems to me the most interesting consequence of adopting this logic. We'll later see how this illuminates both the logic of identification and the logic of ground (Section 7).

Before doing this, let's extract one final pay-off from our discussion of ground-theoretic equivalence. At the end of Section 4.2 I left it somewhat open whether $\vee\wedge$D2 was valid. So I left it open whether we should prefer the stronger or the weaker disjunction agglomeration rule. But since then we've added some more rules. With these extra rules, Weak $\vee$-Agglomeration entails $\vee\wedge$D2. The proof is in the Appendix A.1. This proof relies on the DeMorgan rules, Leibniz's Law and $\neg\neg$-Idempotence. This puts the defender of just the weaker agglomeration rule in a sticky situation. They must deny some of these rules. But the case for all these rules seems

[26] Correia and Skiles [7, 18-21] defend a similar view.

strong. This seems to me to clinch the case for the stronger disjunction agglomeration rule.[27]

6 The system LWG

Let's summarize the logic for weak ground I've proposed. We'll call this the system LWG. This logic takes its grammar from Sections 3 and 5. Its basic rules are all the rules from Section 3 as well as:

$$\wedge\text{-Agglomeration}\ \frac{A, B, \Delta \leq C}{A \wedge B, \Delta \leq C}$$

$$\vee\text{-Agglomeration}\ \frac{A, \Delta \leq C \qquad B, \Gamma \leq C}{A \vee B, \Delta, \Gamma \leq C}$$

$$\neg\vee\text{-Agglomeration}\ \frac{\neg A, \neg B, \Delta \leq C}{\neg(A \vee B), \Delta \leq C}$$

$$\neg\wedge\text{-Agglomeration}\ \frac{\neg A, \Delta \leq C \qquad \neg B, \Gamma \leq C}{\neg(A \wedge B), \Delta, \Gamma \leq C}$$

$$\neg\neg\text{-Idempotence}\ \frac{}{\neg\neg A \leq A} \qquad \neg\text{-Introduction}\ \frac{A \approx B}{\neg A \approx \neg B}$$

$$\text{Def}(\approx)\ \frac{}{A \approx B \leftrightarrow A \leq B \wedge B \leq A}$$

As I've said, the agglomeration rules form a nice unified package. Once one has adopted these, there is also a strong abductive case for the idempotence rule. There is also such a case for ¬-introduction, since this allows us to generate Leibniz's Law. And Def(≈) is just our definition of ground-theoretic equivalence. In the next section we'll see how this system connects to Angell's system of analytic containment. This sheds more light on the connection between identification and weak full ground. It will also allow us to compare our system to the system Correia considers in [2] (Section 10).

7 Angell's system

In a series of publications dating from 1977, Angell developed various systems of analytic containment.[28] These systems model *analytic equivalence*. This is an identity-like notion. The logic of weak ground I have developed (LWG) is definitionally equivalent to a slightly extended version of Angell's system. This means it is equivalent to a version of Angell's system to which we add some definitions of new operators. A *definition* of an operator, O, is a rule of the form $\frac{}{\Phi \leftrightarrow \Psi}$ where Φ

[27] Correia [4] essentially rejects Leibniz's law since he rejects ¬-Intro. This seems to me the most defensible set of LL instances to reject.

[28] See [1] for the most developed version.

contains one occurrence of O and Ψ contains no occurrences of O. In this section I'll outline the relevant version of Angell's system. And I'll explain why this matters.

The extended version of Angell's system is the system AC*. Fine presents this in [14, 224]. In my presentation I just relabel some rules. The grammar and method of proof of this system are the same as that of LWG. AC* then corresponds to E1-E17 below, together with the rules of classical logic:[29]

THE SYSTEM AC*

$$\text{E1}\ \frac{A \approx B}{B \approx A} \qquad \text{E2}\ \frac{A \approx B \qquad B \approx C}{A \approx C} \qquad \text{E3}\ \frac{A \approx B}{\neg A \approx \neg B}$$

$$\text{E4}\ \frac{}{A \approx \neg\neg A} \qquad \text{E5}\ \frac{}{A \approx A \wedge A} \qquad \text{E6}\ \frac{}{A \approx A \vee A}$$

$$\text{E7}\ \frac{}{A \wedge B \approx B \wedge A} \qquad \text{E8}\ \frac{}{A \vee B \approx B \vee A}$$

$$\text{E9}\ \frac{A \approx B}{(A \wedge C) \approx (B \wedge C)} \qquad \text{E10}\ \frac{A \approx B}{A \vee C \approx B \vee C}$$

$$\text{E11}\ \frac{}{\neg(A \wedge B) \approx (\neg A \vee \neg B)} \qquad \text{E12}\ \frac{}{\neg(A \vee B) \approx (\neg A \wedge \neg B)}$$

$$\text{E13}\ \frac{}{(A \wedge B) \wedge C \approx A \wedge (B \wedge C)} \qquad \text{E14}\ \frac{}{(A \vee B) \vee C \approx A \vee (B \vee C)}$$

$$\text{E15}\ \frac{}{A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)} \qquad \text{E16}\ \frac{}{A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)}$$

$$\text{E17}\ \frac{A \approx B}{A \leftrightarrow B}$$

To get AC**, we just add the following definition of weak full ground:

$$\text{E18}\ \frac{}{\Delta \leq C \leftrightarrow C \approx C \vee \hat{\Delta}}$$

Here we allow Δ to be a list with any number of conjunctions in $\hat{\Delta}$ replaced with commas. I show in the Appendix A.3 that AC** is equivalent to LWG. So AC* is definitionally equivalent to LWG.

This equivalence matters for a few reasons. Primarily, it matters because it means E1–E17 is the complete truth-functional logic for identifications. At least, it does if LWG is the complete logic of weak full ground and ground-theoretic equivalence just is identification. This is a big deal: it matters what the correct logic of identification is. Secondarily, it allows the logic of weak full ground and that of identification to illuminate each other. We can see that the validity of the rules of LWG in some sense explain the validity of those of AC**, and vice versa. For instance, identification is associative with respect to disjunction in part because weak full ground obeys disjunction agglomeration. And weak full ground obeys disjunction agglomeration in part because of the associativity of identification.[30]

Tertiarily, this result illuminates the semantics of weak full ground. This is because Fine has presented a truthmaker semantics for AC*. Roughly, he takes $A \approx B$ to be

[29] Angell's system AC differs from this in that it lacks E17 and the rules for classical logic.

[30] Obviously, the relevant sense of explanation violates non-circularity. But there seems to me senses of explanation which violate non-circularity. If one denies this, one should think that *either* the validity of the rules for weak full ground explain those of identification, *or* vice versa (but not both). In this case, I prefer the latter view.

true *iff* every state which verifies A is a part of a state which verifies B, and vice versa.[31] He proves a version of this semantics is sound and complete for a subsystem of AC*. The subsystem is E1-E15 without classical logic. His semantics is also sound for AC* although Fine has no completeness proof. This means such a semantics will be sound and complete for the corresponding subsystem of LWG. This subsystem is LWG without IMP and classical logic. It will also be at least sound for the system LWG. This is some evidence that a truthmaker semantics will prove adequate for LWG.

8 The logic of strict ground

We have so far focused on the logic of weak full ground. I think LWG is the right logic for this notion. Let's now explore how this impacts the logic of strict and partial ground. I'll do this by presenting a definitional extension of the system LWG. To develop this logic, we first need to expand our grammar. To the vocabulary of LWG, we add three further ground-theoretic connectives: $\preceq$ (weak partial ground), $\prec$ (strict partial ground) and $<$ (strict full ground). We then replace the second clause of our definition of a well-formed formula with the following two clauses:

- If a is a basic formula and b is a basic formula, then $(a \preceq b)$ and $(a \prec b)$ are wffs
- If Δ is a list and b is a basic formula, then $(\Delta \leq b)$ and $(\Delta < b)$ are wffs

This expands the definition of a wff. It allows wffs which express strict and partial ground. We now add some definitions of these new connectives. Following Fine, I suggested that A strictly grounds B iff A weakly grounds B and B does not even help weakly ground A. Let $B \npreceq A$ just abbreviate $\neg(B \preceq A)$. Then this is well expressed by the following definitions:

$$\text{Def}(\prec)\ \frac{}{A \prec B \leftrightarrow A \preceq B \wedge B \npreceq A}$$

$$\text{Def}(<)\ \frac{}{A_1, A_2 \ldots < B \leftrightarrow ((A_1, A_2 \ldots \leq B) \wedge ((B \npreceq A_1) \wedge (B \npreceq A_2))) \ldots}$$

The first definition says that A strictly partially grounds B iff A weakly partially grounds B and B doesn't weakly partially ground A. The second definition says that Δ strictly fully grounds B iff Δ weakly fully grounds B and B grounds no member of Δ. We can proceed similarly with weak partial ground. The definition we end up with here is:

$$\text{Def}(\preceq)\ \frac{}{A \preceq B \leftrightarrow A, B \leq B}$$

This says that A weakly partially grounds B iff A, together with B itself, fully grounds B.

We reason to this definition as follows. We initially said that A partially grounds B iff A, perhaps together with some other propositions, fully grounded B. This suggests

[31] See [14, 204-10].

that, if we had sentential quantifiers in the language, we could define weak partial ground as follows:

$$\text{Def}(\preceq)^* \; \frac{}{A \preceq B \leftrightarrow \exists p(A, p \leq B)}$$

Def($\preceq$) encapsulates this definition without the need to include sentential quantifiers in the language. It does this because $\exists p(A, p \leq B)$ is equivalent to $A, B \leq B$. The standard rules for sentential quantification straightforwardly guarantee the left-right of this equivalency. The right-left is a consequence of these rules together with the following principle:

$$\text{Exchange} \; \frac{A, \Delta \leq C}{A, C \leq C}$$

The proof of Exchange is in the Appendix A.4. Call LWG, together with these three definitions, LWG$^+$. LWG$^+$ is an extension of LWG which can deal with strict and partial ground. This is the system I endorse for these notions.

It is useful to briefly note the connection between this system and the rules in Fine's pure logic of ground.[32] First, except for Identity and CUT, all the rules Fine takes as basic are derivable in LWG$^+$. Most importantly, I show in the Appendix A.4 that the following two rules are derivable in LWG$^+$:

$$\text{Subsumption}(\leq/\preceq) \; \frac{\Delta, A \leq B}{A \preceq B} \qquad\qquad \text{Trans } (\preceq/\preceq) \; \frac{A \preceq B \qquad B \preceq C}{A \preceq C}$$

I see this as an explanatory virtue of LWG$^+$. The validity of the rules of LWG$^+$ explains the validity of these rules (and Fine's other rules). In contrast, the rules Fine takes as basic do not entail, and so couldn't explain, those in LWG$^+$. Second, LWG$^+$ has only one undefined ground-theoretic primitive: weak full ground. Fine's logic has four such primitives. So LWG$^+$ is more ideologically parsimonious than Fine's system. It is less profligate when it comes to undefined primitives.

These virtues are connected in two ways to the agglomeration rules from Section 4. First, to prove the validity of Fine's rules with the above definitions requires said agglomeration rules. So the explanatory power of LWG$^+$ rest partly on these rules. Second, the definition of weak partial ground only flies if we've got the agglomeration rules.[33] This is because, on the intended notion of partial ground, if A, B, C fully ground D then A partially grounds D. The agglomeration rules guarantee that in this case $A, B \wedge C$ grounds D. So the rules of LWG$^+$ guarantee that A partially grounds D. But Fine's system (Section 3) provides no such guarantee. So it provides no way of proving –with the above definitions– that if A, B, C grounds D, then A partially grounds D. So the agglomeration rules enable the ideological parsimony of LWG$^+$. This seems to me another reason to endorse the agglomeration rules. In the rest of the section I will explore further the logic of strict ground LWG$^+$ generates.

[32]See [11] and [13, 54-57]. He's got more rules than are in the system presented in Section 3 since he has rules governing strict ground.

[33]The definitions of strict partial and strict full ground are tenable, though.

8.1 Introduction rules for strict ground

In this section I will outline the introduction rules for strict full ground generated by LWG$^+$. This should give the reader a better feel for the logic this system induces. The elegant feature of LWG$^+$ here is that these are all derived from the rules we just introduced. As should be obvious, no non-trivial introduction rule will be derivable for double negation since A and ¬¬A are ground-theoretically equivalent. But we can derive rules for all the other operators. Recall that $A \npreceq B$ is just an abbreviation of $\neg(A \preceq B)$. Then these derived rules are as follows:

$$\vee\text{-I}(<)_1\ \dfrac{B \npreceq A}{A < A \vee B} \qquad \vee\text{-I}(<)_2\ \dfrac{A \npreceq B}{B < A \vee B}$$

$$\wedge\text{-I}(<)\ \dfrac{A \npreceq B \qquad B \npreceq A}{A, B < A \wedge B}$$

$$\neg\vee\text{-I}(<)_1\ \dfrac{\neg B \npreceq \neg A}{\neg A < \neg(A \wedge B)} \qquad \neg\vee\text{-I}(<)_2\ \dfrac{\neg A \npreceq \neg B}{\neg B < \neg(A \vee B)}$$

$$\neg\wedge\text{-I}(<)\ \dfrac{\neg A \npreceq \neg B \qquad \neg B \npreceq \neg A}{\neg A, \neg B < \neg(A \wedge B)}$$

Here is the proof of $\vee$-I$(<)_1$:

Proof of $\vee$-I$(<)_1$.

$$\dfrac{\dfrac{}{A \leq A \vee B}\,\vee\text{-I}_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \leq A \vee B}\,\vee\text{-I}_2 \qquad [A \vee B \preceq A]}{\vdots}\,\text{Sub}(\leq/\preceq), \text{Trans}(\preceq/\preceq)}{B \preceq A} \qquad B \npreceq A}{A \vee B \npreceq A}\,\text{Reductio}}{A < A \vee B}\,\text{Def}(<)$$

I leave the rest of the derivations in the Appendix A.4. These make up a general set of introduction rules for the impure logic of strict ground. One can think of these rules as restrictions on the introduction rules we have for weak ground. So, suppose we have some introduction rule for weak ground. Then we have the corresponding rule for strict ground *provided* that certain ground-theoretic relationships between the propositions it concerns don't obtain.

When can we actually apply these rules? When we know the restriction is satisfied. That is, we can use these rules when we know some propositions don't weakly partially ground other propositions. Fortunately, we have plenty such knowledge. I know that Socrates' existence doesn't even weakly partially ground Jupiter's existence. So I can apply $\vee$-I$(<)_1$ to derive that *Socrates exists* strictly grounds *Socrates or Jupiter exist*. So I think we can often use these rules to work out what strictly grounds what.

This completes my discussion of the system LWG$^+$ itself. LWG$^+$ allows us to elegantly deal with strict, weak, partial and full ground in a single system. I think it is the most plausible logic of ground. In the rest of the paper we will see how this connects to other systems in the literature. I'll show it's antipathetic towards the system Fine develops for strict ground in 'Guide to Ground.' But it's closely connected to that which Correia develops in 'Grounding and Truth-functions.' This shows what rules can be considered valid if we adopt LWG^+.

9 Fine's strict logic of ground

9.1 Fine's introduction rules

In this section, I will detail the connections between LWG^+ and the logic Fine presents in [13]. This has been by far the most influential logic of ground. Fine's elimination rules allow us to infer statements of weak ground from strict ground. His introduction rules allow us to infer statements of strict ground from no premises. We will see that LWG^+ is incompatible with almost all these rules.

Let's start with the introduction rules. The introduction rules Fine lays down are the following:

$$\vee\text{-I}\ \dfrac{}{A < A \vee B} \qquad\qquad \vee\text{-I}\ \dfrac{}{B < A \vee B}$$

$$\wedge\text{-I}\ \dfrac{}{A, B < A \wedge B}$$

$$\neg\neg\text{-I}\ \dfrac{}{A < \neg\neg A}$$

$$\neg\vee\text{-I}\ \dfrac{}{\neg A < \neg(A \wedge B)} \qquad\qquad \neg\vee\text{-I}\ \dfrac{}{\neg B < \neg(A \wedge B)}$$

These are like the rules LWG^+ generates, just *sans* restriction. LWG^+ cannot consistently be enriched with these rules. That's because of this lack of restriction. The simplest proofs of this take advantage of the idempotences. For instance, here's a proof that adding Fine's $\vee$-I rules generates inconsistency:

$$\dfrac{\vdots}{A \preceq A \wedge A \npreceq A}\quad \text{from:}\quad \text{Def}(\prec)\ \dfrac{\text{Subsumption}(</\prec)\ \dfrac{\text{Trans}\ \dfrac{\vee\text{-Idemp}\ \dfrac{}{A \vee A \leq A} \qquad \vee\text{-I}\ \dfrac{}{A < A \vee A}}{A < A}}{A \prec A}}{\vdots}$$

The other proofs are in the Appendix A.5. So is the proof of the 'Subsumption($</\prec$)' step. So LWG^+ is incompatible with all Fine's introduction rules.

Fine's introduction rules for strict ground are widely endorsed. So one might wonder whether this provides good reason to reject LWG^+. I think it does not. Instead, I think that LWG^+ generates precisely the intuitively compelling instances of Fine's rules. Consider Fine's rules for conjunction, disjunction and their negated counterparts. These rules have many instances in which A and B stand in no ground-theoretic connection. LWG^+ validates all such instances of these rules. It will, for example, generate the result that *it's raining* strictly grounds *it's either raining or snowing*. These seem to me the intuitively compelling instances of Fine's rules. In contrast, the instances of these rules which LWG^+ fails to generate are those in which A is a weak partial ground of B, or vice versa. The most obvious such cases are cases of idempotence. LWG^+ does not, for example, generate the result that *it's raining* strictly grounds *it's raining or it's raining*. But these are not intuitively compelling instances of Fine's rule. The requisite grounding connections do not obviously hold in these cases. So rejecting them poses no obvious problem for LWG^+. My own view is that in such cases intuition gives out. Abductive considerations should win the day.

The situation is not as cut-and-dried for double negation. LWG^+ generates no instances of the double negation rule. This makes it more plausible to reject the idempotence rule for double negation. As I've previously stressed, this rule is a bit peripheral to the agglomeration rules. So one could do this without rejecting these rules. I prefer not to do this, because I think that the abductive case for $\neg\neg$-Idempotence is strong. And I don't strongly intuit that Fine's double negation rule is valid. So I reject all of Fine's rules. But a reader with different intuitions might well endorse a subsystem of LWG^+ together with Fine's double negation rule. They'll then endorse a marginally different system to that which I think is best. In the next section, we will turn to another important part of Fine's system: his elimination rules.

9.2 Fine's elimination rules

Fine's elimination rules allow us to infer statements of weak ground from those of strict ground. He suggests, for instance, that if Δ strictly grounds $A \vee B$, then either Δ weakly grounds A or Δ weakly grounds B or Δ is made up of sentences some of which ground A and some of which ground B. As I show in this section, one cannot tenably add Fine's elimination rules to LWG^+.

Let's begin by articulating Fine's elimination rules precisely. To do this, we extend the grammar so as to allow full ground to take pluralities on its right-hand side. We add the following clauses to our definition of a *well-formed formula*:

- If a is a basic formula and Γ is a list, then $(a \leq \Gamma)$ is a wff
- If Δ and Γ are lists, then $(\Delta \leq \Gamma)$ is a wff

Here $(\Delta \leq \Gamma)$ should be read as: there is some decomposition of Δ into families of sentences, $\Delta_1, \Delta_2 \ldots$ and some decomposition of Γ into $A, B \ldots$ such that $(\Delta_1 \leq A), (\Delta_2 \leq B) \ldots$. We also change the proof theory. Before we allowed only single-premise conclusions. Now we allow derivations which have many conclusions. They take the form of a tree with multiple roots:

$$\frac{\Phi_1; \Phi_2; \ldots}{\Psi_1; \Psi_2 \ldots}$$

Here the semi-colon below the line indicates that the conclusion should be read disjunctively: if a tree with $\Psi_1; \Psi_2 \ldots$ as its root is valid, then so is one with $\Psi_1 \vee \Psi_2 \ldots$ at its root.

The elimination rules Fine then proposes are as follows:

$$\vee\text{-E}\ \frac{\Delta < A \vee B}{\Delta \leq A;\ \Delta \leq B;\ \Delta \leq A, B} \qquad \wedge\text{-E}\ \frac{\Delta < A \wedge B}{\Delta \leq A, B}$$

$$\neg\neg\text{-E}\ \frac{\Delta < \neg\neg A}{\Delta \leq A}$$

$$\neg\vee\text{-E}\ \frac{\Delta < \neg(A \vee B)}{\Delta \leq \neg A;\ \Delta \leq \neg B;\ \Delta \leq \neg A, \neg B} \qquad \neg\wedge\text{-}E\ \frac{\Delta < \neg(A \wedge B)}{\Delta \leq \neg A, \neg B}$$

Essentially, these capture the view that Δ strictly grounds a logically complex proposition only if it weakly grounds something which, by the corresponding Finean introduction rule for ground (and his pure logic), provably grounds that proposition. This makes it clear that these rules will lead to problems in LWG$^+$. In LWG$^+$ we have more introduction rules for ground. So we can prove the obtaining of certain grounding relations without recourse to Fine's introduction rules.

In fact, only ¬¬-E is tenable in LWG$^+$.[34] If we assume some propositions *don't* ground one another, then the rest of these elimination rules must fail. There are many such assumptions we could make. I'll just consider how one set of assumptions is, in LWG$^+$, inconsistent with ∨-E. This make it clear that ∧-E, ¬∨-E and ¬∧-E are also untenable. As I've said, the relevant assumptions involve lack of grounding connections. Namely, suppose that for any three propositions, A, B, C, the following hold:

$$C \not\leq A \vee B \qquad A \not\leq B \vee C \qquad B \not\leq A$$

These obviously hold for some propositions. Let A be *apples exist*, let B be *balloons exist* and C be *candles exist*. The existence of candles doesn't even partially ground the existence of apples or balloons. The existence of apples doesn't ground the existence of balloons or candles. The existence of balloons doesn't ground the existence of apples. With these in hand inconsistency follows from the associativity of disjunction and Fine's disjunction elimination rule. Here's the argument: $C \not\leq A \vee B$ and associativity entail that $(A \vee B)$ strictly grounds $(A \vee (B \vee C))$. It then follows via Fine's disjunction elimination rule that either $A \vee B$ weakly grounds A or $A \vee B$ weakly grounds $B \vee C$. But if $A \vee B$ weakly grounds A, then B weakly grounds A. This conflicts with the assumption that $B \not\leq A$. Meanwhile, if $A \vee B$ weakly grounds $B \vee C$, then A weakly grounds $B \vee C$. This conflicts with the assumption that $A \not\leq B \vee C$. So we have a contradiction.

The prospects of adding simple elimination rules to LWG$^+$ are dim.[35] One might wonder how much of a cost this is to LWG$^+$. I think it is not much of a cost, for two reasons. First, the problem with the elimination rules arises from far less than the full strength of LWG$^+$. The contradiction above derives from associativity alone. This seems to be on much firmer ground than the other rules. Second, I think Fine's elimination rules are independently implausible. This is because they rule out live possibilities.

[34] This is valid. In LWG$^+$, $\neg\neg A$ grounds A and so by the transitivity of ground anything which grounds $\neg\neg A$ must also ground A.

[35] Because LWG$^+$ doesn't have a 1-1 correspondence between introduction rules and the logical form of grounded propositions.

Here's an example of a live possibility these rules rule out: in the modal logic of ground, we might think that $\Box A$ strictly grounds A.[36] This channels the common idea that the necessity of a proposition is a satisfactory explanation of that proposition's truth. So, for instance, that there *must* be something (rather than nothing) would explain why there *is* something (rather than nothing). But now suppose A is a necessary logically complex proposition, e.g. $P \vee \neg P$. It seems implausible that $\Box(P \vee \neg P)$ either weakly grounds P or weakly grounds $\neg P$. It seems especially implausible when P is contingent. So Fine's rules exclude this plausible principle in the modal logic of ground.

Another issue of this sort concerns the putative mind-dependence of certain truths. One way to spell out the claim that the truths of some domain are mind-dependent is to say they're strictly grounded in certain mental states. For instance, one might think all mathematical truths are strictly grounded in knowledge of those truths. On this view, *2+2=4* is grounded in the fact that it's known that *2+2=4*. But now consider a logically complex mathematical truth like CH∨¬CH. Plausibly, that CH∨¬CH is known does not ground either CH or ¬CH. So Fine's elimination rules would rule out this thesis in the philosophy of mathematics. Similar examples exist in other putatively mind-dependent domains, such as ethics and aesthetics. So the omission of Fine's elimination rules seems to me a negligible cost to the system.[37] This completes our comparison between LWG^+ and Fine's logic of strict ground. As we've seen, the rules of LWG^+ can't be squared with those of Fine's logic. If one accepts the rules of LWG^+, one must reject almost all of Fine's rules.

10 Correia's worldly logic of ground

I want to do one final thing. I want to show that LWG^+ is definitionally equivalent to the system Correia considers in [2]. Let's start by outlining Correia's system. He calls this system **G**. It's meant to model an identification-like notion and factive strict full ground. It goes like this. We begin with E1-E17 from Section 7 and the rules of classical logic.[38] In other words, we start with the system AC*. Then we add strict full factive ground ($<_f$) to the vocabulary and the following clause to the definition of a well-formed formula:

– If Δ is a list and c is a basic formula, then $\Delta <_f c$ is a wff

Finally we add rules governing strict full factive ground. Correia considers a bunch of rules. But they're easy to summarize. Let $\Delta \not\approx \Delta \vee (B \wedge \Delta)$ mean that no item in

[36]The modal logic of ground is an entirely unexplored area which I suspect will repay further study. Another promising thought is that possibility is grounded in actuality, so A grounds $\Diamond$A. I don't think that these are obviously valid, but they are live possibilities.

[37]My counter-examples here concern disjunction. It's less obvious whether there are counter-examples involving conjunctions (apart from those which arise in LWG^+). But I suspect there are: I suppose some conjunctions might, for instance, be grounded in laws of nature even though none of their conjuncts are grounded in those laws. For instance, it might be that *It's a law that all negatively charged things attract* grounds the conjunction of all instances of the law, without grounding any particular conjunct.

[38]Correia actually formulates this system slightly differently. It is straightforward to show his formulation is equivalent to E1-E17.

Δ, C, is equivalent to $C \vee (B \wedge C)$. The rules Correia considers are equivalent to the following rule:[39]

$$\text{Def}(<_f/\approx)\ \frac{}{\Delta <_f B \leftrightarrow \hat{\Delta} \wedge (B \approx B \vee \hat{\Delta}) \wedge (\Delta \not\approx \Delta \vee (B \wedge \Delta))}$$

This completes Correia's system **G**. As I said, Correia considers this system in [2]. But he doesn't exactly endorse it. He only includes Def($<_f/\approx$) in the system because it makes giving a semantics easier.[40] Yet the connection between this system and my own is interesting nonetheless. I want to show that **G** and LWG$^+$ are definitionally equivalent.

To do this we need to add some definitions to both systems. We'll then show that the resulting systems are equivalent. To **G** we add E18 as a definition of weak full ground. We also add the definitions of weak partial ground, strict partial ground and strict full ground from Section 8. Call this system **G***. To LWG$^+$ we just add a definition of strict full factive ground. We could add Def($<_f/\approx$). But this wouldn't be very illuminating. So instead I'll add:

$$\text{Def}(<_f/<)\ \frac{}{\Delta <_f B \leftrightarrow (\hat{\Delta} \wedge (\Delta < B))}$$

This just says that Δ is a factive strict full ground for B iff all the members of Δ are true and Δ is a non-factive strict full ground for B. Call this system LWG^{++}.

Let's use DEFs to stand for the definitions of weak partial, strict partial and strict full ground. It's useful to think of the two systems as follows:

$$\begin{aligned} \mathbf{G}^* &= \text{AC}^{**} + \text{DEFs} + \text{Def}(<_f/\approx) \\ \text{LWG}^{++} &= \text{LWG} + \text{DEFs} + \text{Def}(<_f/<) \end{aligned}$$

In other words, **G*** is AC** plus some definitions of non-factive notions of ground and an extra definition of factive ground. LWG^{++} is LWG plus those same definitions but a different definition of factive ground.

I want to show that these systems are equivalent. I show that AC** and LWG are equivalent in the Appendix A.3. LWG^{++} and **G*** share the definitions of non-factive ground. So we just need to show Def($<_f/<$) is valid in **G*** and Def($<_f/\approx$) is valid in LWG^{++}. To show this, it's enough to show that in both systems: $\Delta < B$ iff $(B \approx B \vee \hat{\Delta})$ and $(\Delta \not\approx \Delta \vee (B \wedge \Delta))$. It'll then follow that these two rules are themselves equivalent. How do we show that? Well both systems contain the definition of strict full ground. So, in both systems, $(\Delta < B)$ iff $(\Delta \leq B)$ and B

[39] Correia shows that these rules entail this axiom in [2, 270-71]. That this axiom entails these rules, given his rules for $\approx$, is implicit in his completeness proof at [2, 278].

[40] See [2, 272]. At the time, he actually thought this definition could be counter-exampled. In particular, he pointed out that it implies that, if the existence of sets are grounded in their members, then the fact that {Socrates} exists just is the disjunction of the fact that Socrates exists and another fact [2, 272]. He found this implausible. So he rejected the left-right of his definition. But his views have changed since. In [7, 15-19], he and Alexander Skiles point out that {Socrates} existence could be the disjunction of *itself* and Socrates' existence. This seems a lot less implausible than the thought that {Socrates} existence was the disjunction of Socrates' existence and some third totally novel fact. This treatment of the counter-example is exactly that generated by the Reduction Theorem ($\approx/\leq$) and our definitions. In fact, given the rules of LWG$^+$ this circumstance is routine: whenever A strictly grounds B, B just is A or B.

doesn't weakly partially ground any item in Δ. So it suffices to show that in both systems: *(a)* ($\Delta \leq B$) iff ($B \approx B \vee \hat{\Delta}$), and; *(b)* B doesn't weakly partially ground any item in Δ iff $\Delta \not\approx \Delta \vee (B \wedge \Delta)$. The Reduction Theorem ($\preceq/\approx$) is valid in both systems. This gets us *(a)*. To get *(b)*, we rely on the following principle:

$$\text{Def}(\preceq/\approx)\ \frac{}{B \preceq C \leftrightarrow C \approx C \vee (B \wedge C)}$$

This is a definition of weak partial ground in terms of identification. It's also valid in both systems. From this *(b)* follows straightforwardly. For $\Delta \not\approx \Delta \vee (B \wedge \Delta)$ just says that no item, C, in Δ is equivalent to $C \vee (B \wedge C)$. And, by Def($\preceq/\approx$), this is true iff B doesn't weakly partially ground any C in Δ. So, in both systems, B doesn't weakly partially ground any item in Δ iff no C in Δ is equivalent to $C \vee (B \wedge C)$. So LWG^{++} and $\mathbf{G}^*$ are equivalent. So LWG^{+} and **G** are definitionally equivalent. That's what I wanted to show.

Why does this matter? It puts the rules of **G** on a firmer footing. It is otherwise not clear why we should accept the rules of **G**. In particular it's unclear why we should accept the definition of factive strict full ground. Why should we think that Δ factively grounds B just in case all the members of Δ are true, B is equivalent to $(B \vee \hat{\Delta})$ and no item, C, in Δ is equivalent to $C \vee (B \wedge C)$? The truth of this doesn't exactly leap off the page. The above result means the validity of the rules of LWG^{++} can explain why it is true. This definition is correct, roughly, because the second conjunct picks out weak full ground, the third conjunct picks out weak partial ground, and the definition of factive ground in LWG^{++} is right. More generally, I think this result shows that Correia got it right with his system **G**. This system is adequate for modelling identification and strict full factive ground. To model other notions we can use LWG^{+}.

11 Concluding remarks

This completes my discussion of the logic of ground. I've suggested that various pressures should lead us to strengthen Fine's logic of weak full ground. Once we do we should end up with a system closely connected to some others in the literature. I think this system generates a lot of intuitive validities. But perhaps its most attractive features are aesthetic. It allows us to smoothly derive the logic of strict and partial ground from that of weak full ground. It allows us to cleanly pick out the grounding relations logically complex propositions stand in. So it's elegant. I think elegance is a guide to theory-choice. So I myself think this is the best logic of ground.

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Appendix

In this appendix, I will prove many of the claims I've made in the main text. The proof of some claim in the main text will fall under the section heading in which that claim was made.

A.1 What do logically complex propositions ground?

A.1.1 Disjunction

I first prove commutativity, associativity, supplementation and DeMorgan(1) hold for disjunction:

Proof of Commutativity-$\vee$.

$$\vee\text{-A}\ \dfrac{\vee\text{-I}_2\ \dfrac{}{A \leq B \vee A} \qquad \vee\text{-I}_1\ \dfrac{}{B \leq B \vee A}}{A \vee B \leq B \vee A}$$

Proof of Associativity-$\vee$.

$$\vee\text{-A}\ \dfrac{\vee\text{-A}\ \dfrac{\vee\text{-I}_1\ \dfrac{}{A \leq A \vee (B \vee C)} \qquad \dfrac{\vee\text{-I}_1, \vee\text{-I}_2, \text{Trans}\ \dfrac{}{\vdots}}{B \leq A \vee (B \vee C)}}{(A \vee B) \leq A \vee (B \vee C)} \qquad \dfrac{\vee\text{-I}_2, \text{Trans}\ \dfrac{}{\vdots}}{C \leq A \vee (B \vee C)}}{(A \vee B) \vee C \leq A \vee (B \vee C)}$$

Proof of Supplementation-$\vee$.

$$\vee\text{-A}\ \dfrac{\text{Trans}\ \dfrac{A \leq B \qquad \vee\text{-I}_1\ \dfrac{}{B \leq B \vee D}}{A \leq B \vee D} \qquad \text{Trans}\ \dfrac{C \leq D \qquad \vee\text{-I}_1\ \dfrac{}{C \leq C \vee D}}{C \leq B \vee D}}{A \vee C \leq B \vee D}$$

Proof of DeMorgan(1)-$\vee\wedge$.

$$\vee\text{-A}\ \dfrac{\neg\wedge\text{-I}_1\ \dfrac{}{\neg A \leq \neg(A \wedge B)} \qquad \neg\wedge\text{-I}_2\ \dfrac{}{\neg B \leq \neg(A \wedge B)}}{\neg A \vee \neg B \leq \neg(A \wedge B)}$$

I then prove the distributivity principles follow from $\vee$-Agglomeration. Note that when I label a step ‘R’ I indicate I’m taking advantage of the repetition invariance of lists (so A is the same list as A, A. . .) to re-iterate the same lemma.

Proof of $\wedge\vee$*D1.*

$$\wedge\text{-A}\ \dfrac{\text{Permutation}\ \dfrac{\text{Repetition}\ \dfrac{\vee\text{-A}\ \dfrac{\dfrac{\wedge\text{-I}, \vee\text{-I}_1, \text{Trans, Permutation}\ \dfrac{}{\vdots}}{B, A \leq (A \wedge B) \vee (A \wedge C)} \qquad \dfrac{\wedge\text{-I}, \vee\text{-I}_2, \text{Trans, Permutation}\ \dfrac{}{\vdots}}{C, A \leq (A \wedge B) \vee (A \wedge C)}}{B \vee C, A, A \leq (A \wedge B) \vee (A \wedge C)}}{B \vee C, A \leq (A \wedge B) \vee (A \wedge C)}}{A, B \vee C \leq (A \wedge B) \vee (A \wedge C)}}{A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)}$$

Proof of $\wedge\vee$*D2.*

$$\vee\text{-A}\ \dfrac{\wedge\text{-A}\ \dfrac{\dfrac{\begin{array}{c}\text{Identity}, \vee\text{-I}_1, \wedge\text{-I, CUT —}\\ \vdots\end{array}}{A, B \leq A \wedge (B \vee C)}}{A \wedge B \leq A \wedge (B \vee C)} \qquad \wedge\text{-A}\ \dfrac{\dfrac{\begin{array}{c}\text{Identity}, \vee\text{-I}_2, \wedge\text{-I, CUT —}\\ \vdots\end{array}}{A, C \leq A \wedge (B \vee C)}}{A \wedge C \leq A \wedge (B \vee C)}}{(A \wedge B) \vee (A \wedge C) \leq A \wedge (B \vee C)}$$

Proof of $\vee\wedge$*D1.*

$$\vee\text{-A}\ \dfrac{\dfrac{\begin{array}{c}\vee\text{-I}_1, \wedge\text{-I, CUT, R —}\\ \vdots\end{array}}{A \leq (A \vee B) \wedge (A \vee C)} \qquad \wedge\text{-A}\ \dfrac{\dfrac{\begin{array}{c}\vee\text{-I}_2, \wedge\text{-I, CUT —}\\ \vdots\end{array}}{B, C \leq (A \vee B) \wedge (A \vee C)}}{B \wedge C \leq (A \vee B) \wedge (A \vee C)}}{A \vee (B \wedge C) \leq (A \vee B) \wedge (A \vee C)}$$

Proof of $\vee\wedge$*D2.*

$$\wedge\text{-A}\ \dfrac{\text{Permutation}\ \dfrac{\vee\text{-A}\ \dfrac{\vee\text{-I}_1\ \dfrac{}{A \leq A \vee (B \wedge C)} \qquad \text{Permutation}\ \dfrac{\vee\text{-A}\ \dfrac{\vee\text{-I}_1\ \dfrac{}{A \leq A \vee (B \wedge C)} \qquad \dfrac{\begin{array}{c}\wedge\text{-I}, \vee\text{-I}_2, \text{Trans —}\\ \vdots\end{array}}{B, C \leq A \vee (B \wedge C)}}{A \vee B, C \leq A \vee (B \wedge C)}}{C, A \vee B \leq A \vee (B \wedge C)}}{A \vee C, A \vee B \leq A \vee (B \wedge C)}}{A \vee B, A \vee C \leq A \vee (B \wedge C)}}{(A \vee B) \wedge (A \vee C) \leq A \vee (B \wedge C)}$$

Note how Weak $\vee$-Agglomeration could be used in all but the last proof. I now use $\vee\wedge$D2 to derive $\vee$-Agglomeration from Weak $\vee$-Agglomeration. In the below proof 'Id' stands for 'Identity.'

$$\dfrac{\text{Id}\ \dfrac{}{\hat{\Delta} \wedge \hat{\Gamma} \leq \hat{\Delta} \wedge \hat{\Gamma}} \quad \text{Id}\ \dfrac{}{A \leq A} \quad \text{Weak } \vee\text{-A}\ \dfrac{\text{Identity}\ \dfrac{}{B \leq B} \quad \text{Identity}\ \dfrac{}{B \leq B}}{B \vee B \leq B} \quad \dfrac{\begin{array}{c}\wedge\text{-A}\ \dfrac{\text{Perm}\ \dfrac{\text{Amalg}\ \dfrac{A, \Delta \leq C \qquad B, \Gamma \leq C}{A, \Delta, B, \Gamma \leq C}}{\Delta, \Gamma, A, B \leq C}}{}\\ \vdots\end{array}}{\hat{\Delta} \wedge \hat{\Gamma}, A, B \leq C}}{\hat{\Delta} \wedge \hat{\Gamma}, A, B \vee B \leq C}$$

$$\dfrac{\begin{array}{c}\vee\wedge\text{D2, Identity, CUT}\ \dfrac{\wedge\text{-A}\ \dfrac{\text{Permutation}\ \dfrac{\hat{\Delta} \wedge \hat{\Gamma}, A, B \vee B \leq C}{A, B \vee B, \hat{\Delta} \wedge \hat{\Gamma} \leq C}}{(A \wedge (B \vee B)), \hat{\Delta} \wedge \hat{\Gamma} \leq C}}{}\\ \vdots\end{array}}{(A \vee B) \wedge (A \vee B), \hat{\Delta} \wedge \hat{\Gamma} \leq C}$$

$$\text{Repetition}\ \dfrac{\dfrac{\begin{array}{c}\wedge\text{-I, CUT}\ \dfrac{(A \vee B) \wedge (A \vee B), \hat{\Delta} \wedge \hat{\Gamma} \leq C}{}\\ \vdots\end{array}}{A \vee B, A \vee B, \Delta, \Gamma \leq C}}{A \vee B, \Delta, \Gamma \leq C}$$

Note that we could replace Weak $\vee$-Agglomeration with $\vee$-Idempotence in this proof. So this also shows that, in the presence of Fine's rules, $\vee$-Agglomeration derives from $\vee$-Idempotence, $\vee\wedge$D2 and $\wedge$-Agglomeration.

A.1.2 Double negation

Here I just prove the $\wedge\vee$ equivalencies. We get two of these equivalencies with the agglomeration rules (and the $\neg$-Introduction rules) alone. However, in getting the other equivalencies $\neg\neg$-Idempotence becomes useful. Here are the proofs:

$$\neg\wedge\text{-A}\ \dfrac{\text{Trans}\ \dfrac{\neg\neg\text{-Idemp}\ \dfrac{}{\neg\neg A \leq A} \qquad \vee\text{-I}_1\ \dfrac{}{A \leq A \vee B}}{\neg\neg A \leq A \vee B} \qquad \dfrac{\neg\neg\text{-Idemp, }\vee\text{-I}_2\text{, Trans} \text{ — } \\ \vdots}{\neg\neg B \leq A \vee B}}{\neg(\neg A \wedge \neg B) \leq A \vee B}$$

$$\neg\vee\text{-A}\ \dfrac{\text{CUT}\ \dfrac{\neg\neg\text{-Idemp}\ \dfrac{}{\neg\neg A \leq A} \qquad \neg\neg\text{-Idemp}\ \dfrac{}{\neg\neg B \leq B} \qquad \wedge\text{-I}\ \dfrac{}{A, B \leq A \wedge B}}{\neg\neg A, \neg\neg B \leq A \wedge B}}{\neg(\neg A \vee \neg B) \leq A \wedge B}$$

A.2 Ground-theoretic equivalence

Proof of LL To prove LL we show it holds in two cases: one in which C contains no ground-theoretic operators, and one in which C contains some ground-theoretic operators. This exhausts the cases. Let's begin by considering the first case. In this case, C is a truth-functional formula. That means it is either atomic, or contains only truth-functional operators. Observe that IMP establishes that if $A \approx B$, then $A \leftrightarrow B$. It follows that A and B are inter-substitutable in any truth-functional formula *salva veritate*, and so the relevant instance of LL must hold. Now let's consider the second case, in which C contains some instance of a ground-theoretic connective. To prove that LL holds in this case, we take advantage of the following lemma:

$$\textbf{Substitution}: \quad A \approx B \rightarrow C \approx D$$

Where D is the result of just swapping As for Bs in C. This just says that when A and B are ground-theoretically equivalent, then any formula, C, is ground-theoretically equivalent to the result of swapping As for Bs in that formula. One can prove this by induction on the complexity of formulas. The base case –where C contains no logical operators– is trivial. For the inductive step, in the case of conjunction we use $\wedge$-agglomeration, for disjunction we use $\vee$-agglomeration and for negation we use $\neg$-Introduction.

We then use **Substitution** to prove that LL holds in the ground-theoretic case. The case where C contains only instances of $\approx$ follows directly from **Substitution**, so we're left with just cases in which C is of the form $\Delta \leq E$. There are two kinds of LL instances in these cases. Firstly, we might form D by replacing As for Bs in E. Secondly, we might form D by replacing As for Bs in Δ. Suppose we replace As for Bs in E. Call the resultant formula $E^{[A/B]}$. By **Substitution**, $E \approx E^{[A/B]}$. So, by Def($\approx$) and transitivity, $\Delta \leq E^{[A/B]}$. So this first kind of LL instances hold. Now suppose we replace As for Bs in Δ. To replace As for Bs in Δ, we need to replaces As for Bs in some member (or members), $F_1 ... F_n$, of Δ. Label the formula(s) we do this to $F_1^{[A/B]} ... F_n^{[A/B]}$ and label the list of all the other formulas Γ. The resultant list can thus be written: $F_1^{[A/B]} ... F_n^{[A/B]}, \Gamma$. By **Substitution**, $F_k \approx F_k^{[A/B]}$ for every F_k in $F_1 \ldots F_n$. Since for every G in Γ, $G \leq G$, we can therefore apply Def($\approx$) and CUT to derive $F_1^{[A/B]} ... F_n^{[A/B]}, \Gamma \leq E$. So this second kind of LL instance holds. So LL holds when C is of the form $\Delta \leq E$. So LL holds for every ground-theoretic C. Since LL holds for every non-ground theoretic C, it follows that LL holds generally. □

Proof of $\vee\wedge$*D2* We prove $\vee\wedge$D2 follows from Weak $\vee$-Agglomeration in the presence of LL, the DeMorgan(2) rules and $\neg\neg$-Idempotence. A full natural deduction proof of this is unwieldy. But it suffices to note that the following ground-theoretic equivalences are provable:

$$
\begin{array}{lll}
1 & (A \vee B) \wedge (A \vee C) & \approx \neg\neg((A \vee B) \wedge (A \vee C)) \\
2 & & \approx \neg(\neg(A \vee B) \vee \neg(A \vee C)) \\
3 & & \approx \neg((\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)) \\
4 & & \approx \neg(\neg A \wedge (\neg B \vee \neg C)) \\
5 & & \approx \neg(\neg A \wedge \neg(B \wedge C)) \\
6 & & \approx \neg\neg(A \vee (B \wedge C) \\
7 & & \approx A \vee (B \wedge C)
\end{array}
$$

Line 1 and 7 rely on $\neg\neg$-Introduction and $\neg\neg$-I. Line 2, 3, 5 and 6 rely on the DeMorgan rules together with LL. Line 4 relies on $\wedge\vee$D1, $\wedge\vee$D2 and LL. As I show above $\wedge\vee$D1 and $\wedge\vee$D2 are both consequence of Weak $\vee$-Agglomeration. So this shows that $\vee\wedge$D2 can be derived from Weak $\vee$-Agglomeration in the presence of the other rules. Given the proof in the Appendix A.1 this shows that $\vee$-Agglomeration can, in this context, be derived from Weak $\vee$-Agglomeration. □

A.3 Angell's system

I here prove AC** and LWG are equivalent. We first prove that all the basic rules of AC** are rules in LWG. E1 is trivial, E2 follows from Trans and E3 is just $\neg$-Introduction. E4, E5 and E6 follow from idempotence. E7 and E8 follow from commutativity. E9 and E10 follow from supplementation. E11 and E12 follow from the DeMorgan laws. E13 and E14 follow from associativity. E15 and E16 follow from distributivity. E17 follows from IMP and E18 follows from the reduction theorem ($</\approx$).

We now prove all basic rules in LWG are rules in AC**. When I use the definition of weak full ground, E18, together with classical logic I will move directly between formulas of equivalence and those of weak full ground. I will skip indicating the compression of the steps with vertical dots. Here are the proofs:

Proof of disjunction and conjunction introduction.

We begin with the proof of $\vee$-I$_1$:

$$
\text{E18}\;\dfrac{\text{E2}\;\dfrac{\dfrac{\text{E14, E2}\;\dfrac{\text{E10}\;\dfrac{\text{E6}\;\dfrac{}{A \approx A \vee A}}{A \vee B \approx (A \vee A) \vee B}}{\vdots}}{A \vee B \approx A \vee (A \vee B)} \qquad \text{E8}\;\dfrac{}{A \vee (A \vee B) \approx (A \vee B) \vee A}}{A \vee B \approx (A \vee B) \vee A}}{A \leq A \vee B}
$$

The proof of the other disjunction rule is essentially the same. The proof of $\wedge$-I relies just on E6 and E18.

Proof of negation rules.

Here are proofs of one of the negated disjunction rules, and the negated conjunction rule:

$$
\text{E18}\;\dfrac{\text{E8, E2}\;\dfrac{\text{E11, E2}\;\dfrac{\text{E3}\;\dfrac{\text{E13, E2}\;\dfrac{\text{E9}\;\dfrac{\text{E5}\;\dfrac{}{A \approx A \wedge A}}{A \wedge B \approx (A \wedge A) \wedge B}}{\vdots}}{\neg(A \wedge B) \approx \neg(A \wedge (A \wedge B))} \text{ from } A \wedge B \approx (A \wedge (A \wedge B))}{\vdots \quad \neg(A \wedge B) \approx \neg A \vee \neg(A \wedge B)}}{\vdots \quad \neg(A \wedge B) \approx \neg(A \wedge B) \vee \neg A}}{\neg A \leq \neg(A \wedge B)}
$$

$$
\text{E18}\;\dfrac{\text{E8, E2}\;\dfrac{\text{E6, E2}\;\dfrac{\text{E10}\;\dfrac{\text{E12}\;\dfrac{}{\neg(A \vee B) \approx (\neg A \wedge \neg B)}}{\neg(A \vee B) \vee \neg(A \vee B) \approx (\neg A \wedge \neg B) \vee \neg(A \vee B)}}{\vdots \quad \neg(A \vee B) \approx (\neg A \wedge \neg B) \vee \neg(A \vee B)}}{\vdots \quad \neg(A \vee B) \approx \neg(A \vee B) \vee (\neg A \wedge \neg B)}}{\neg A, \neg B \leq \neg(A \vee B)}
$$

The proof of $\neg$-I and $\neg\neg$-idempotence are as follows:

$$
\text{E18}\;\dfrac{\text{E8, E2}\;\dfrac{\text{E6, E2}\;\dfrac{\text{E10}\;\dfrac{\text{E1}\;\dfrac{\text{E4}\;\dfrac{}{A \approx \neg\neg A}}{\neg\neg A \approx A}}{\neg\neg A \vee \neg\neg A \approx A \vee \neg\neg A}}{\vdots \quad \neg\neg A \approx A \vee \neg\neg A}}{\vdots \quad \neg\neg A \approx \neg\neg A \vee A}}{A \leq \neg\neg A}
$$

$$
\text{E18}\;\dfrac{\text{E8, E2}\;\dfrac{\text{E6, E2}\;\dfrac{\text{E10}\;\dfrac{\text{E4}\;\dfrac{}{A \approx \neg\neg A}}{A \vee A \approx \neg\neg A \vee A}}{\vdots \quad A \approx \neg\neg A \vee A}}{\vdots \quad A \approx A \vee \neg\neg A}}{\neg\neg A \leq A}
$$

Meanwhile, $\neg$-Introduction is just an instance of E3.

Proof of agglomeration rules.

In the interests of readability, I'll prove slightly simplified versions of the agglomeration rules. These are simplified in that they omit the arbitrary lists (Δ, Γ) following the sentences which we agglomeration into conjunctions or disjunctions. These can be easily added.

The proofs of the conjunction and disjunction agglomerations rules are as follows:

$$\text{E18}\,\dfrac{\text{E18}\,\dfrac{A, B \leq C}{C \approx C \vee (A \wedge B)}}{A \wedge B \leq C}$$

$$\text{E14, E2}\,\dfrac{\text{E2}\,\dfrac{\text{E18}\,\dfrac{B \leq C}{C \approx C \vee B} \qquad \text{E10}\,\dfrac{\text{E18}\,\dfrac{A \leq C}{C \approx C \vee A}}{C \vee B \approx (C \vee A) \vee B}}{C \approx (C \vee A) \vee B}}{\vdots}$$

$$\text{E18}\,\dfrac{\overline{C \approx C \vee (A \vee B)}}{A \vee B \leq C}$$

Note that in the proof of $\wedge$-agglomeration, in the second application of E18 we take advantage of the fact that removing any number of $\wedge$ operators from $(A \wedge B)$ can be implemented by removing none at all.

We now prove the agglomeration rules for negated conjunction and disjunction respectively:

$$\text{E10, E8, E2}\,\dfrac{\text{E18}\,\dfrac{\neg A, \neg B \leq C}{C \approx C \vee (\neg A \wedge \neg B)} \qquad \text{E12, E1}\,\dfrac{}{\neg A \wedge \neg B \approx \neg(A \vee B)}}{\vdots}$$

$$\text{E18}\,\dfrac{\overline{C \approx C \vee \neg(A \vee B)}}{\neg(A \vee B) \leq C}$$

$$\text{E10, E2}\,\dfrac{\text{E18}\,\dfrac{\neg B \leq C}{C \approx C \vee \neg B} \qquad \text{E18}\,\dfrac{\neg A \leq C}{C \approx C \vee \neg A}}{\vdots}$$

$$\text{E14, E2}\,\dfrac{\overline{C \approx (C \vee \neg A) \vee \neg B}}{\vdots}$$

$$\text{E10, E8, E2}\,\dfrac{\overline{C \approx C \vee (\neg A \vee \neg B)} \qquad \text{E11}\,\dfrac{}{\neg(A \wedge B) \approx (\neg A \vee \neg B)}}{\vdots}$$

$$\text{E18}\,\dfrac{\overline{C \approx C \vee \neg(A \wedge B)}}{\neg(A \wedge B) \leq C}$$

Proof of pure logic rules.

We have left the pure logic to last. Identity is just an instance of E13. IMP follows from E17, together with classical logic. CUT is somewhat more difficult. To prove CUT, we first prove $\leq$ obeys a transitivity principle:

$$\dfrac{\dfrac{\text{E10, E1, E2}\ \dfrac{\text{E18}\ \dfrac{\Delta \leq B}{B \approx B \vee \hat{\Delta}} \quad \text{E18}\ \dfrac{B \leq C}{C \approx C \vee B}}{\vdots}}{C \approx C \vee (B \vee \hat{\Delta})} \quad \dfrac{\text{E14, E1}\ \dfrac{}{\vdots}}{C \vee (B \vee \hat{\Delta}) \approx (C \vee B) \vee \hat{\Delta}}}{\text{E2}\ \ C \approx (C \vee B) \vee \hat{\Delta}} \quad \text{E18}\ \dfrac{B \leq C}{C \approx C \vee B}$$ 1034

$$\text{E10, E1, E2}\ \dfrac{\dfrac{\vdots}{C \approx C \vee \hat{\Delta}}}{\text{E18}\ \ \Delta \leq C}$$

We now prove that $\wedge$-Supplementation is valid in AC**. A full natural deduction proof of this is very unwieldy, so I begin by noting that, given $A_1 \wedge A_2 \approx (A_1 \vee \hat{\Delta}_1) \wedge (A_2 \vee \hat{\Delta}_2)$, the following are provable in AC**:

$$\begin{array}{lll}
1 & A_1 \wedge A_2 & \approx (A_1 \vee \hat{\Delta}_1) \wedge (A_2 \vee \hat{\Delta}_2) \\
2 & & \approx ((A_1 \vee \hat{\Delta}_1) \wedge A_2) \vee ((A_1 \vee \hat{\Delta}_1) \wedge \hat{\Delta}_2) \\
3 & & \approx (((A_1 \wedge A_2) \vee (\hat{\Delta}_1 \wedge A_2)) \vee ((A_1 \wedge \hat{\Delta}_2) \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2))) \\
4 & & \approx (((A_1 \wedge A_2) \vee (\hat{\Delta}_1 \wedge A_2)) \vee (A_1 \wedge \hat{\Delta}_2))) \vee ((\hat{\Delta}_1 \wedge \hat{\Delta}_1) \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2)) \\
5 & & \approx (((A_1 \wedge A_2) \vee (\hat{\Delta}_1 \wedge A_2)) \vee ((A_1 \wedge \hat{\Delta}_2) \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2))) \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2) \\
6 & & \approx (A_1 \wedge A_2) \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2)
\end{array} \qquad (1)$$

Line 2 and 3 both apply distributivity (E15 and E16). Line 4 applies Idempotence (E5). Line 5 applies associativity (E13). Line 6 uses the third line in substitutions like occur in the above proofs. I've labelled this inference (1). Given this, the following tree is valid:

$$\dfrac{\text{E9, E7, E2}\ \dfrac{\text{E18}\ \dfrac{\hat{\Delta}_1 \leq A_1}{A_1 \approx A_1 \vee \hat{\Delta}_1} \quad \text{E18}\ \dfrac{\hat{\Delta}_2 \leq A_2}{A_2 \approx A_2 \vee \hat{\Delta}_2} \quad \dfrac{\text{E5, E1, E2}\ \dfrac{}{\vdots}}{A_1 \wedge A_2 \approx A_1 \wedge A_2}}{\vdots}}{\text{(1)}\ \dfrac{A_1 \wedge A_2 \approx (A_1 \vee \hat{\Delta}_1) \wedge (A_2 \vee \hat{\Delta}_2)}{\text{E18}\ \dfrac{A_1 \wedge A_2 \approx (A_1 \wedge A_2) \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2)}{\hat{\Delta}_1 \wedge \hat{\Delta}_2 \leq A_1 \wedge A_2}}}$$

And so we have proven $\wedge$-Supplementation. Here's the proof of CUT in the three premise case:

$$\text{Trans}\ \dfrac{\wedge\text{-I}\ \dfrac{}{\hat{\Delta}_1, \hat{\Delta}_2 \leq \hat{\Delta}_1 \wedge \hat{\Delta}_2} \quad \text{Trans}\ \dfrac{\wedge\text{-Supplementation}\ \dfrac{\wedge\text{-A}\ \dfrac{\Delta_1 \leq A_1}{\hat{\Delta}_1 \leq A_1} \quad \wedge\text{-A}\ \dfrac{\Delta_2 \leq A_2}{\hat{\Delta}_2 \leq A_2}}{\hat{\Delta}_1 \wedge \hat{\Delta}_2 \leq A_1 \wedge A_2} \quad \wedge\text{-A}\ \dfrac{A_1, A_2 \leq C}{A_1 \wedge A_2 \leq C}}{\hat{\Delta}_1 \wedge \hat{\Delta}_2 \leq C}}{\text{E18}\ \dfrac{\hat{\Delta}_1, \hat{\Delta}_2 \leq C}{\text{E18}\ \dfrac{C \approx C \vee (\hat{\Delta}_1 \wedge \hat{\Delta}_2)}{\Delta_1, \Delta_2 \leq C}}}$$

Finally, let's consider Def($\approx$). The left-right of this follows straightforwardly from E18 and E1. The right-left follows from E18, E1 and E8 and E2. So all the basic rules of LWG are valid in AC**. So AC** and LWG are equivalent.

A.4 The logic of strict ground

Let's start by proving Exchange. Begin by observing that if we assume $B \approx B \vee (A \wedge C)$, then we can infer $B \approx ((B \wedge A) \vee B)$. This is because, given $B \approx B \vee (A \wedge C)$, the following are ground-theoretic equivalencies:

$$
\begin{array}{lll}
1 & B & \approx B \vee (A \wedge C) \\
2 & & \approx (B \vee A) \wedge (B \vee C) \\
3 & & \approx ((B \vee A) \wedge (B \vee A)) \wedge (B \vee C) \\
4 & & \approx (B \vee A) \wedge ((B \vee A) \wedge (B \vee C)) \\
5 & & \approx (B \vee A) \wedge B \\
6 & & \approx (B \vee A) \wedge (B \vee B) \\
7 & & \approx B \vee (B \wedge A)
\end{array}
\tag{2}
$$

The natural deduction proof of this is long and hard to read, so I will omit it. The important line is line five, which we obtain from the previous line via $B \approx ((B \vee A) \wedge (B \vee C))$ and Leibniz's law. I'll label the inference associated with these equivalencies (2).

This allows us to prove Exchange via reasoning by cases. There are two cases. One where Δ has no members, and one where it has Δ has n members. The first case follows from Amalgamation and identity. The second case is more complex:

$$
\text{Permutation}\;\dfrac{\text{Trans}\;\dfrac{\dfrac{}{C, A \leq (C \wedge A)} \qquad \text{RT}(\approx/\leq)\;\dfrac{(2)\;\dfrac{\text{RT}(\approx/\leq)\;\dfrac{\wedge\text{-A}\;\dfrac{\dfrac{n-1 \text{ applications of } \wedge\text{-A}\;\dfrac{\Delta, A \leq C}{\vdots}}{\hat{\Delta}, A \leq C}}{\hat{\Delta} \wedge A \leq C}}{C \approx C \vee (\hat{\Delta} \wedge A)}}{C \approx C \vee (C \wedge A)}}{(C \wedge A) \leq C}}{C, A \leq C}}{A, C \leq C}
$$

So, in both cases Exchange holds and so the principle is proven. Here, as above, I omit indicating the compression of steps when using $(\approx/\leq)$ and the classical rules. I will do the same with the definitions below.

Let's now prove that Subsumption$(\leq/\preceq)$ and Transitivity $(\preceq/\preceq)$ hold in LWG^{+}.

Proofs of Subsumption and Transitivity.

$$
\text{Def}(\preceq)\;\dfrac{\text{Exchange}\;\dfrac{A, \Delta \leq B}{A, B \leq B}}{A \preceq B}
\qquad\qquad
\text{Def}(\preceq)\;\dfrac{\text{Exchange}\;\dfrac{\text{CUT}\;\dfrac{\text{Def}(\preceq)\;\dfrac{A \preceq B}{A, B \leq B} \qquad \dfrac{}{C \leq C} \qquad \text{Def}(\preceq)\;\dfrac{B \preceq C}{B, C \leq C}}{A, B, C \leq C}}{A, C \leq C}}{A \prec C}
$$

I will prove one other of Fine's subsumption rules is valid below, and from all this it is quite easy to show the rest of Fine's rules are valid.

A.4.1 Introduction rules for strict ground

Let's now prove the introduction rules for strict ground. The $\neg\wedge$ rules are proved in the same way as the proof of $\vee$ rules, but use $\neg\wedge\text{-I}_1$ and $\neg\wedge\text{-I}_2$. Here's the proof of the conjunction rule:

$$\text{Def}(<)\,\dfrac{\wedge\text{-I}\,\dfrac{}{A, B \leq A \wedge B} \qquad \text{Reductio}\,\dfrac{\dfrac{\text{Def}(\leq)\,\dfrac{\text{Exchange}\,\dfrac{\text{Permutation}\,\dfrac{\wedge\text{-I}\,\dfrac{}{A, B \leq A \wedge B}}{B, A \leq A \wedge B}}{B, A \wedge B \leq A \wedge B}}{B \preceq A \wedge B} \quad [A \wedge B \preceq A]}{B \preceq A} \quad B \not\preceq A}{A \wedge B \not\preceq A} \qquad \dfrac{\dfrac{A \not\preceq B}{\vdots}}{A \wedge B \not\preceq B}}{A, B < A \wedge B}$$

The $\neg\vee$ rule is proved in a similar way.

A.5 Fine's logic of strict ground

A.5.1 Fine's introduction rules

I prove that Fine's introduction rules are inconsistent with LWG$^+$. We begin by proving subsumption:

Proof of Subsumption(</$\prec$*).*

$$\text{Def}(\prec)\,\dfrac{\text{Sub}(\leq/\preceq)\,\dfrac{\text{Def}(<)\,\dfrac{\Delta, A < B}{\Delta, A \leq B}}{A \preceq B} \qquad \text{Reductio}\,\dfrac{\text{Def}(\prec)\,\dfrac{\text{Trans}(\preceq/\prec)\,\dfrac{\text{Sub}(\leq/\preceq)\,\dfrac{\text{Def}(<)\,\dfrac{\Delta, A < B}{\Delta, A \leq B}}{A \preceq B} \quad [B \prec A]}{A \prec A}}{A \preceq A \wedge A \not\preceq A}}{B \not\preceq A}}{A \prec B}$$

We can now prove the $\neg\neg$ introduction rule and the $\wedge$ introduction rule cannot be consistently added to LWG:

$$\text{Def}(\prec)\,\dfrac{\text{Sub}(</\prec)\,\dfrac{\dfrac{\wedge\text{-Idemp}\,\dfrac{}{A \wedge A \leq A} \qquad \text{R}\,\dfrac{\wedge\text{-I}\,\dfrac{}{A, A < A \wedge A}}{\text{A ¡ } A \wedge A}}{A < A}}{A \prec A}}{A \preceq A \wedge A \not\preceq A}$$

$$\text{Def}(\prec)\,\dfrac{\text{Sub}(</\prec)\,\dfrac{\text{Trans}\,\dfrac{\neg\neg - I\,\dfrac{}{A < \neg\neg A} \qquad \neg\text{-Idemp}\,\dfrac{}{\neg\neg A \leq A}}{A < A}}{A \prec A}}{A \preceq A \wedge A \not\preceq A}$$

Proofs for the negated conjunction and disjunction rules are the same, except rely on different idempotence principles. Any remaining proofs are available upon request.

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