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ON ELEMENTS OF CHANCE

ABSTRACT. One aspect of the utility of gambling may evidence itself in failures of idempotence, i.e., when all chance outcomes give rise to the same consequence the 'gamble' may not be indifferent to its common consequence. Under the assumption of segregation, such gambles can be expressed as the joint receipt of the common consequence and what we call 'an element of chance', namely, the same gamble with the common consequence replaced by the status quo. Generalizing, any gamble is indifferent to the joint receipt of its element of chance and a certain consequence, which is called the 'kernel equivalent' of the gamble. Under idempotence, the kernel equivalent equals the certainty equivalent. Conditions are reported (Theorem 4) that are sufficient for the kernel equivalents to have the kind of utility representation first discussed by Luce and Fishburn (1991), including being idempotent. This utility representation of the kernel equivalents together with the derived form of utility over joint receipts yields a utility representation of the original structure. Possible forms for the utility of an element of chance are developed.

KEY WORDS: Element of chance, Idempotence, Kernel equivalent, Rank-dependent utility, Utility of chance, Utility of gambling

INTRODUCTION

A series of papers, based on both gambles and a binary operation $\oplus$ of joint receipt, has resulted in a theory of utility different from the traditional ones based solely on preferences among gambles. For a summary, see Luce (2000). The purpose of this paper is to explore what happens when one of the most basic aspects, idempotence, is dropped.

1. STRUCTURES BASED ON JOINT RECEIPT

1.1. *Basic notations*

Throughout the paper the structure $\mathfrak{D} = \langle \mathcal{D}_2, e, \succsim, \oplus \rangle$, has the following primitives:

- $\mathcal{C}$ is a set of pure (i.e., certain) consequences, and $\mathcal{E}_{\mathbf{E}}$ is an algebra of chance events generated by a chance experiment[1] **E** with universal event E.

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The elements of $\mathcal{C}$ are mutually exclusive objects of value to the decision maker about which there is no uncertainty, e.g., sums of money, bills, new appliances, etc. Often these are called 'goods' and 'bads'. They are chosen so as to be unrelated to the chance experiments. Examples of chance experiments are: (1) Tossing a die, in which case $E = \{1, 2, 3, 4, 5, 6\}$. (2) Spinning a balanced pointer over a partitioned circle. (3) Or selecting a ball at random from an urn of 100 red and yellow balls of unknown composition except that there are at most 80 red balls and at most 60 yellow ones, in which case E is the set of all the balls.

- $\mathcal{B}_1$ is the union of $\mathcal{C}$ and the family of first-order binary gambles generated from $\mathcal{C}$ and $\mathcal{E}_\mathbf{E}$, i.e., if $x, y \in \mathcal{C}$, $C \in \mathcal{E}_\mathbf{E}$, then $(x, C; y)$ denotes the gamble in which the consequence to the decision maker when the experiment **E** is executed is x if the event C occurs and y if the event $E \backslash C$ occurs.

Suppose in the third example of a chance experiment, C = a red ball is chosen (which one is immaterial), x = \$100 and y = -\$50, then the gamble $(x, C; y) = (\$100, C; -\$50)$ means that the decision maker receives \$100 if the ball drawn is red and pays \$50 if it is yellow.

- $\mathcal{D}_1$ is the closure of $\mathcal{B}_1$ under the binary operation $\oplus$, where if $f, g \in \mathcal{B}_1$, then $f \oplus g$ is interpreted to mean that the decision maker receives both f and g. It is assumed that the experiments giving rise to f and g are independently realized.
- $\mathcal{B}_2$ and $\mathcal{D}_2$ are generated from $\mathcal{B}_1$ in an analogous fashion, resulting in compound gambles of the following general type $(g, C; h)$ where $g, h \in \mathcal{B}_1$.
- $\succsim$ is a preference weak order over $\mathcal{D}_2$.
- $e \in \mathcal{C}$ is (no change from) the status quo relative to which gains and losses are defined in terms of $\succsim$.

The element e simply is the consequence for which the decision maker perceives as effecting no change from the status quo. In experimental practice, this is usually interpreted to mean no material exchange between the respondent and the experimenter, although many have questioned the correctness of this interpretation. Many hold that the local status quo is context dependent on the set of alternatives confronting the decision maker.

- $\mathcal{C}^+ = \{x : x \in \mathcal{C} \text{ and } x \succsim e\}$, and $\mathcal{B}_i^+$ and $\mathcal{D}_i^+ = 1, 2$, are induced from the gains consequences, $\mathcal{C}^+$. Similar definitions hold for losses.

A standard but little discussed assumption of the published literature, whose elimination is the focus of this paper, is:

DEFINITION 1. *Idempotence:* for all $x \in \mathcal{C}$ and all $C \in \mathcal{E}_{\mathbf{E}}$,

$$(x, C; x) \sim x. \tag{1}$$

For the case where idempotence holds, the following is a slightly more restrictive version of Luce's (2000, Theorem 4.4.4) formulation of a result of Luce and Fishburn (1991, 1995).

1.2. *Segregation, RDU, and p-additivity*

THEOREM 1. *Suppose that $\mathfrak{D} = \langle D_2^+, e, \succsim, \oplus\rangle$ is a structure in which $\oplus$ over D_2^+ satisfies*[2] *commutativity and monotonicity, and e is an identity of $\oplus$. In the following statements it is assumed that: $U : \mathcal{C}, \mathcal{D}_2^+ \xrightarrow{onto} [0, k[$ preserves the order $\succsim$; $U(e) = 0$; and for experiment $\mathbf{E}$, $W : \mathcal{E}_{\mathbf{E}} \xrightarrow{onto} [0, 1]$ with $W(\emptyset) = 0$ and $W(E) = 1$. Then, any two of the following three statements imply the third:*

1. *Binary segregation: for $g, h \in \mathcal{B}_1^+$*

$$(g \oplus h, C; h) \sim (g, C; e) \oplus h. \tag{2}$$

2. *(U, W) forms a rank-dependent utility (RDU) representation over $\mathcal{B}_2^+$: for $g, h \in \mathcal{D}_1^+$*

$$U(g, C; h) = \begin{cases} U(g)W(C) + U(h)[1 - W(C)], & g \succsim h \\ U(g)[1 - W(\overline{C})] + U(h)W(\overline{C}), & g \prec h \end{cases} \tag{3}$$

3. *(U, W) forms the following representation: for a real constant δ with unit = 1/units of U and $f, g \in \mathcal{D}_2^+$,*

$$U(f \oplus g) = U(f) + U(g) - \delta U(f)U(g), \tag{4}$$

and

$$U(g, C; e) = U(g)W(C). \tag{5}$$

The form of Equation (4) is called *p-additive* because it is the only polynomial form with $U(e) = 0$ that is transformable into a non-negative, additive representation V over gains. In fact, when Equation (4) holds with $\delta \neq 0$, the transformation is $\kappa V(x) = -sgn(\delta)\ln[1 - \delta U(x)]$, $\kappa > 0$. Thus, under the three properties of the conclusion we see that $\oplus$ is not only commutative but also associative for gains. Moreover, if one sets $h = g$ in Equation (3) we have $U(g, C; g) = U(g)$ whence $(g, C; g) \sim g$, i.e. idempotence.

A similar result holds for losses. The case of mixed gains and losses is more complex (see below).

2. ELEMENTS OF CHANCE

2.1. *Dropping idempotence*

The main purpose of this paper is to ask what happens to the utility structure if we forego idempotence and suppose that there may be a perceived difference between x and $(x, C; x)$, where $C \neq \emptyset, E$.

For readers deeply indoctrinated in the Savage (1954) formulation of uncertain decisions, it is quite unnatural to formulate the idea of violations of idempotence. There is really no distinction available between a consequence and the act that assigns that consequence to every state of nature. And certainly the subjective expected utility representation implies idempotence. Within the present framework, it turns out that such violations are relatively easily handled.

Observe that if, as we shall assume, binary segregation, Equation (2), holds, then

$$(x, C; x) \sim x \oplus (e, C; e), \tag{6}$$

where $(e, C; e)$ simply means running the experiment with the set of outcomes partitioned as $\{C, \overline{C}\}$. So idempotence fails if and only if it fails for $(e, C; e)$, which it does if and only if the respondent has a strict preference between the status quo and running the experiment to see if C occurs or not. The preference relation of $(e, C; e)$ to e tells us something about the decision maker's attitude toward the chance experiment being run.

This argument is not quite correct because even if x is a gain, it may happen that $(e, C; e) \prec e$ and even that $(x, C; x) \prec e$. Thus, we are not really justified in using Equation (2) as stated. Rather, we must work with the entire structure of gains, losses, and mixed gains and losses and then give a more general definition of segregation (see Definition 2 below). In that structure we shall require that $\oplus$ be everywhere commutative and associative and, indeed, that it have an additive representation V — which is quite distinct from U (§ 3.3).

To state the generalization of Equation (2), it is convenient to define a concept of 'subtraction' by: for $x, y, z \in \mathcal{C}$

$$x \ominus y \sim z \Longleftrightarrow x \sim y \oplus z. \tag{7}$$

We shall assume the structure is sufficiently dense always to admit the solution z (see the hypothesis of Theorem 1 that the utility function is onto a real interval). This is satisfied if, for example, $\mathcal{C}$ includes all money amounts.

DEFINITION 2. For $x \succsim y$, *(binary) general segregation* holds if and only if

$$(x, C; y) \sim \begin{cases} (x \ominus y, C; e) \oplus y, & (x, C; y) \succ (e, C; e) \\ (e, C; y \ominus x) \oplus x, & (x, C; y) \prec (e, C; e) \end{cases}. \tag{8}$$

Note that if the structure is idempotent, this definition agrees with Definition 6.2.1 of Luce (2000). Moreover, for $x \succsim e$, it agrees with Equation (2). Thus, for all $x \in \mathcal{C}$ and independent of whether $(e, C; e)$ is seen as a gain or a loss, Equation (6) $(x, C; x) \sim x \oplus (e, C; e)$ remains true.

The 'gamble' $(e, C; e)$ is obviously special in that it has no consequences aside from the status quo; it just involves running the underlying experiment and focusing on whether or not C occurs.

DEFINITION 3. Each 'gamble' $(e, C; e)$ is called an *element of chance* and its utility $U(e, C; e)$ as *utility of chance.*

2.2. *Related literature*

This notion of an element of chance relates to a small literature on what is called the utility of gambling. As has been repeatedly noted, von Neumann & Morgenstern (1944) recognized the phenomenon, but felt that it could not be dealt with at the level of axiomatization of preferences over gambles. Subsequently, attempts to axiomatize choices so that a natural concept of utility of gambling appears were reported by Diecidue, Schmidt & Wakker (1999), Fishburn (1980) and Schmidt (1998). Conlisk (1993) provides a general review of the area and some proposals to deal with it. And Pope (1996/97, 1998, and many references there) has emphasized the failure of most utility theories to take note of something B. Pascal in 1670 first cited in his *Pensées*: that people distinguish between the pleasure or displeasure of chance (uncertainty) and the objective evaluation of the worth of the gamble from the perspective of its consequences. Indeed, she partitioned the analysis into three factors (Pope 1996/97, p. 44):

> "*Factor 1* the **stake** – the net wealth outcome, and
> "*Factor 2* the **context** – the pleasure of the game ...
> "*Factor 3* **chance** – a curiosity advantage in **not** knowing the net wealth outcome, and conversely a disadvantage in the form of boredom of playing it safe."

This is the first study where the concept is treated in terms of the joint receipt operation. We are, however, uncertain whether to class the element of chance $(e, C; e)$ as belonging to Factor 2 or 3. On the one hand, by itself it does seem to be the context of the gamble, and yet as we shall see in partitioning any gamble into a certain consequence and an element of chance, it seems to capture an aspect of Factor 3 as well. After an extended correspondence, Dr. Robin Pope concluded that, from her perspective, it concerns only an aspect of Factor 2.

2.3. *Two examples*

Pope (1991) offers an analysis of temporal aspects of even apparently static decision making, and uses that to explain why Savage's (1954) analysis of the Allais paradox in terms of his sure-thing principle is misleading. Recall, this principle asserts that when two alternatives have a common consequence, that term can be ignored and the decision is based on the remaining subgambles.

In terms of the present perspective, the difficulty in accepting Savage's analysis can be described as follows. He wished to invoke the argument that replaces a common consequence by another common one in pairs of gambles. To do so, he assumed that the certain alternative of \$1M is indifferent to a gamble of the form $(\$1M, C_1; \$1M, C_2; \$1M, C_3)$. However, using segregation and the fact that e is an identity of $\oplus$, this amounts to

$$\begin{aligned} \$1M \oplus e &\sim \$1M \\ &\sim (\$1M, C_1; \$1M, C_2; \$1M, C_3) \\ &\sim \$1M \oplus (e, C_1; e, C_2; e, C_3). \end{aligned}$$

So, by monotonicity of $\oplus$, $(e, C_1; e, C_2; e, C_3) \sim e$. But this need not be the case if running the experiment and seeing which of the three events occurs has some inherent utility, and so from the present perspective the Savage argument is flawed. Or put another way, the Allais paradox *may* be explained by properties of elements of chance even if the paradox is assumed not to be exhibited at a more idealized level (see the next section). Pope's (1991) account is different from this one.

In like manner, the general rank-dependent utility model exhibits a key property that Luce (1998) called *coalescing* and that others call 'combining' (Kahneman & Tversky 1979) and 'event splitting' (Starmer & Sugden 1993). It asserts that if two events of a gamble of order k have a common consequence, then the gamble can be reduced to one of order $k - 1$ in which the union of the two events replaces them. If the events in question are C_j and C_{j+1}, then the two elements of chance are

$$(e, C_1; ...; e, C_j; e, C_{j+1}...; e, C_k) \quad \text{and}$$
$$(e, C_1; ...; e, C_j \cup C_{j+1}...; e, C_k).$$

Even if coalescing holds in some deeper sense, which we develop in the next section on kernel equivalents, it need not hold in the observed data if the decision maker is not indifferent between these two elements of chance. This *may* be the explanation for the seeming compelling nature of coalescing as an axiom and its empirical failures in such papers as Birnbaum & Navarrete (1998) and Wu (1994).

As these two examples suggest, failures of idempotence *may* account for certain empirical failures, but great emphasis must be placed on the italicized word ‘may.’ We do not have any empirical information about elements of chance or any theory about elements of chance that verify any of the conjectures we are offering. So far, we merely have a possible way of taking utility of gambling into account, but no detailed examination of it. In Section 4 we will begin an exploration of possible forms for the utility of elements of chance, but what is reported is far from a finished theory.

3. KERNEL EQUIVALENTS OF GAMBLES

3.1. *Definition*

Now, let us turn to the general binary gamble $(x, C; y)$ and decompose it as follows.

DEFINITION 4. Suppose that $\langle D_1, e, \succsim, \oplus \rangle$ is a joint-receipt structure. For $x, y \in \mathcal{C}$ and $C \in \mathcal{E}_{\mathbf{E}}$, the *kernel equivalent* of $(x, C; y)$, $KE(x, C; y) \in \mathcal{C}$, is the solution of the indifference

$$(x, C; y) \sim KE(x, C; y) \oplus (e, C; e). \tag{9}$$

Clearly, we assume that such solutions exist, which they do in structures with representations onto intervals. From Equations (7) and (9), we may write

$$KE(x, C; y) \sim (x, C; y) \ominus (e, C; e).$$

It is important to note that, independent of the relations among x, y, and e, $KE(x, C; y)$ is in $\mathcal{C}$, i.e., it is a pure (i.e., certain) consequence. Of course, gamble solutions of Equation (9) may also exist but would not serve our purpose. Thus, any gamble is factored into the joint receipt of its kernel equivalent and the relevant element of chance. In a certain sense, $KE(x, C; y)$ is a pure certainty equivalent of the gamble $(x, C; y)$ in which the preference for the element of chance $(e, C; e)$ is ignored. Obviously, in the idempotent case $KE = CE$, where CE is the certainty equivalent defined for gamble f as $CE(f) \sim f$, where $CE(f) \in \mathcal{C}$.

3.2. *Testing properties in the non-idempotent case*

Testing an axiom in the non-idempotent case can sometimes be a bit unusual. It is unchanged when the same chance event occurs on each side of a preference relation, as in the case of consequence monotonicity where, provided $C \neq \emptyset$, the condition is

$$\begin{aligned} x \succsim y &\Leftrightarrow (x, C; z) \succsim (y, C; z) \\ &\Leftrightarrow KE(x, C; z) \oplus (e, C; e) \succsim KE(y, C; z) \oplus (e, C; e) \\ &\Leftrightarrow KE(x, C; z) \succsim KE(y, C; z). \end{aligned}$$

In this case whatever pleasure or displeasure running the experiment gives, it is exactly the same on both sides of the property.

But matters are different when, as in the following definition of a standard sequence

$$(x_i, C; e) \sim (x_{i+1}, D; e),$$

different chance events are involved on the two sides. If the concept in question is to hold at the level of kernel equivalents, then observe that by the definition of KE

$$\begin{aligned} &KE(x_i, C; e) \sim KE(x_{i+1}, D; e) \\ &\quad \Leftrightarrow (x_i, C; e) \ominus (e, C; e) \sim (x_{i+1}, D; e) \ominus (e, D; e). \end{aligned}$$

By the associativity and commutativity of $\oplus$, this last statement is equivalent to

$$(x_i, C; e) \oplus (e, D; e) \sim (x_{i+1}, D; e) \oplus (e, C; e). \tag{10}$$

So this would be the correct definition of a standard sequence to use experimentally.

This example typifies a common principle, which we may call a *balanced experimental design*, that we will encounter when we examine what is involved for kernel equivalents to satisfy the usual RDU theory. It has an important experimental implication. For example, to construct a KE-standard sequence empirically, as in Equation (10), or test later properties such as Equation (17), in a laboratory, one would have to arrange for two independent realizations of the underlying ‘experiment.’ For example, suppose the experiment entails a pinwheel in which a needle is spun over a partitioned circle to determine whether C or $\overline{C}$ occurs. Then to test Equation (10) we would have to have a second pinwheel partitioned $\{D, \overline{D}\}$. Whether one were realizing the left or the right side of the preference relation in Equation (10) both wheels would be spun. On the left side, no consequences would be attached to the D-pinwheel and on the right side none to the C-pinwheel. If, for example, a respondent expresses a preference for the left side, then on running the experiment he or she will find out what would have happened had the right side been chosen.

Exactly this type of balanced design has been carried out by Mellers, Schwartz, Ho & Ritov (1997) and Mellers, Schwartz & Ritov (1999). Substantial differences are observed between seeing the outcomes of running only the experiment underlying the gamble chosen and seeing the outcomes of running both experiments (see Figs. 4 and 8 of the latter paper). No one has yet attempted to test the several axioms for kernel equivalents using a design where all chance experiments involved in a choice are actually conducted for the respondent.

In a sense, then, the thrust of this paper for experimentalists is that if one believes that there are elements of chance different from the status quo, i.e., that idempotence fails empirically, then the experiments should be designed so as to get at the kernel equivalents of the properties being tested.

3.3. *Elements of chance and kernel equivalents for general gambles*

It is important to recognize that the major ideas involved are in fact not restricted to binary gambles. For general finite gambles, idempotence, element of chance, and the kernel equivalent are defined for $x_i \in \mathcal{C}$, $x_1 \succsim x_2 \succsim \ldots \succsim x_n$, respectively, as

$$\begin{aligned}
(x, C_1; x, C_2; \ldots; x, C_n) &\sim x, \\
(x, C_1; x, C_2; \ldots; x, C_n) &\sim x \oplus (e, C_1; e, C_2; \ldots; e, C_n), \\
(x_1, C_1; x_2, C_2; \ldots; x_n, C_n) &\sim KE(x_1, C_1; x_2, C_2; \ldots; x_n, C_n) \\
&\quad \oplus (e, C_1; e, C_2; \ldots; e, C_n).
\end{aligned}$$

General segregation is defined as: If $(x_1, C_1; ...; x_n, C_n) \succsim (e, C_1; ...; e, C_n)$, then

$$(x_1, C_1; ...; x_n, C_n) \sim (x_1 \ominus x_n, C_1; ...; e, C_n) \oplus x_n, ,$$

If $(x_1, C_1; ...; x_n, C_n) \prec (e, C_1; ...; e, C_n)$, then

$$(x_1, C_1; ...; x_n, C_n) \sim (e, C_1; ...; x_n \ominus x_1, C_n) \oplus x_1$$

In the following, for simplicity we restrict attention without further notice to binary gambles.

3.4. *Utility representations of kernel equivalents*

PROPOSITION 2. *Suppose that $\mathfrak{D} = \langle \mathcal{D}_1, e, \succsim, \oplus \rangle$ is a structure for which consequence monotonicity holds for gambles; commutativity, associativity, and monotonicity hold for $\oplus$; and kernel equivalents exist for all gambles. Then, the following hold:*

1. *Consequence monotonicity holds for gambles if and only if it holds for kernel equivalents.*
2. *If general segregation holds, then the kernel equivalents are idempotent.*
3. *For gains and losses separately, general segregation holds in $\mathfrak{D}$ if and only if the kernel equivalents satisfy segregation.*
4. *Suppose that the kernel equivalents satisfy the three properties of Theorem 1, then, for $x, y \in \mathcal{C}^+$ and $(e, C; e) \succsim e$,*

$$U(x, C; y) = \begin{cases} U(x)W'(C) + U(y)[1 - W'(C)] \\ \qquad +U(e, C; e)[1 - \delta U(y)], \; x \succsim y \\ U(x)[1 - W'(\overline{C})] + U(y)W'(\overline{C}) \\ \qquad +U(e, C; e)[1 - \delta U(x)], \; x \prec y \end{cases}, \tag{11}$$

where

$$W'(C) = W(C)[1 - \delta U(e, C; e)] \tag{12}$$

and δ is the constant of Equation (4).

All proofs are in Appendix A.

Note that the weight $W'(C)$, Equation (12), depends on δ and U, in addition to C, but not on either of the consequences. The term added to ordinary rank-dependent utility depends on the element of chance, the constant δ, and the utility of the lesser consequence. If the latter is e, then it depends only on $(e, C; e)$. This is similar to, but different from, the representations of Diecidue et al. (1999), which has an additive term that depends on the probability of winning and the amount to win, and Fishburn (1980), who has an additive term that depends on the probability of winning. Schmidt (1998) arrives at different utility functions for a certain consequence depending on whether it is part of a gamble or alone, and so it is quite different from the other models.

Clearly, we need to understand the utility of chance $U(e, C; e)$. A beginning, but no more than that, is given in Section 4.

We turn next to cases of mixed gains and losses, either because $x \succsim e \succsim y$ or $(e, C; e) \prec e$ or both. Part 4 of Proposition 2 states only the case where everything is a gain. In the mixed cases matters become significantly more complex. Part of the complexity results from the fact (see Corollary to Theorem 4.4.4, Luce, 2000) that for gains alone there are three distinct relations between U and the additive representation V of $\oplus$ depending upon the sign of δ :

- If $\delta = 0$, then for some $\alpha > 0$,

$$U = \alpha V. \tag{13}$$

- If $\delta > 0$, then U is subadditive, i.e., $U(f \oplus g) < U(f) + U(g)$, is bounded by $1/\delta$, and for some $\kappa > 0$

$$\delta U(f) = 1 - e^{-\kappa V(f)}, \quad \delta, \kappa > 0, \tag{14}$$

- If $\delta < 0$, then U is superadditive, unbounded, and for some $\kappa > 0$

$$|\delta|\, U(f) = e^{\kappa V(f)} - 1, \quad -\delta, \kappa > 0, \tag{15}$$

These can be described, respectively, as proportional, concave, and convex. Observe that these are statements relative to V, not, for example, to money.

There is a similar result in the domain of losses with constants δ' and κ'. For simplicity assume that $\kappa' = \kappa$. It is important when dealing with the mixed case to distinguish between the weights used when an event leads to a gain and those when it leads to a loss. We do so by writing W^+ for gains and W^- for losses.

Thus, when gains and losses are both involved, there are 9 possible pairings of utility types. (When there are only gains or only losses, only 3 cases arise.) The most common, although far from universal pairing, is concave gains and convex losses. Depending upon which pairing we assume, somewhat different formulas result. For example, with $x \succsim e \succsim y$ and $x \oplus y \succsim e$, the concave gains and convex losses case results in

$$U(x \oplus y) = \frac{U(x) + \left|\frac{\delta'}{\delta}\right| U(y)}{1 + |\delta'|\, U(y)}.$$

Formulas for the other cases are presented in Chapter 7 of Luce (2000).[3] Applying this case to Equation (9) on the assumptions $x \succsim e \succsim y$, $(x, C; y) \succsim e$, $KE(x, C; y) \succsim e$, and $(e, C; e) \prec e$, we have

$$\begin{aligned} U(x, C; y) &= U\left[KE(x, C; y) \oplus (e, C; e)\right] \\ &= \frac{U\left[KE(x, C; y)\right] + \left|\frac{\delta'}{\delta}\right| U(e, C; e)}{1 + |\delta'|\, U(e, C; e)} \end{aligned}$$

The formula for $U\left[KE(x, C; y)\right]$ is itself complex for the same reasons plus one additional one, namely, the link that is assumed to hold in the mixed case between $\oplus$ and gambles. Two cases have been investigated. One is general segregation. The second is a nonrational assumption called duplex decomposition which, however, has considerable empirical support. Assuming the former, then in the case of concave gains and convex losses it can be shown (Theorem 7.3.4, Luce, 2000) that in the case $KE(x, C; y) \succsim e$,

$$U\left[KE(x, C; y)\right] = U(x)W^+(C) + \frac{\left|\frac{\delta'}{\delta}\right| U(y)}{1 - |\delta'|\, U(y)}[1 - W^+(C)].$$

Putting the last two displays together gives the final formula for $U(x, C; y)$.

This is just one of the total of 150 possible cases (see Appendix B for this count). We know how to work out any one of interest, but it certainly is not worth reporting all of the formulas.

A brief comment on the non-uniqueness of the representation is needed. It arises despite the fact that everyone is presumed to satisfy the same behavioral axioms with the exception of the choice between general segregation and duplex decomposition in the mixed case and, of course, whether $(e, C; e) \succsim$ or $\precsim e$. Were we to confine our attention to just one of these, then the number drops from 150 to 96. The merit of such flexibility is that it allows for substantial individual differences in behavior, and yet it arises only from differences in the shapes of utility functions in the two domains. It certainly cautions against any simple averaging of data over individual respondents, and possibly gives additional support for the appropriateness of direct tests of individual qualitative axioms.

3.5. *Joint-receipt decomposition*

As stated, Theorem 1 has a weakness that is inherited when it is applied to kernel equivalents, as in Part 4 of Proposition 2. How does one know that it is possible for Equations (4) and (5) to hold simultaneously with the same utility function? Luce (1996) posed and answered this question, which is summarized here for the kernel equivalents (Luce 2000, Theorems 4.4.5 and 4.4.6). The answer has three parts:

- The structure of joint receipts is assumed to have an additive representation, i.e., it forms an Archimedean ordered group. The existence of a p-additive function $U^{(1)}$, Equation (4), follows immediately.
- The existence of a separable representation $U^{(2)}W^{(2)}$, which has to be justified. We take it up in Section 3.6.
- And the following property, called *joint-receipt decomposition*, must hold for the KE functions: for each $x, y \in \mathcal{C}^+$ and $C \in \mathcal{E}_{\mathbf{E}}$, there exists $D = D(x, C) \in \mathcal{E}_{\mathbf{E}}$ such that

  $$KE(x \oplus y, C; e) \sim KE(x, C; e) \oplus KE(y, D; e). \quad (16)$$

  Note that D is independent of y. A similar requirement holds for losses.

The conclusion is that given the existence of the two representations, joint-receipt decomposition is both necessary and sufficient for there to exist a single utility function U and weighting function W satisfying both Equations (4) and (5).

It is desirable to translate Equation (16) into an equivalent property of $\mathfrak{D}$.

PROPOSITION 3. *Suppose the conditions of Proposition 2 hold. Then, for* $x, y \in \mathcal{C}^+$, $C \in \mathcal{E}_{\mathbf{E}}$, *Equation (16) is equivalent to the existence of* $D = D(x, C) \in E_{\mathbf{E}}$ *such that*

$$(x \oplus y, C; e) \oplus (e, D; e) \sim (x, C; e) \oplus (y, D; e). \tag{17}$$

We continue to call Equation (17) *joint-receipt decomposition* because it reduces to that concept in the idempotent case. This, of course, is an example of what was described in Section 3.2 as a balanced experimental design.

Equation (17) is the first of a number of examples of how one generalizes concepts from the idempotent case to the non-idempotent one. The principle in each case is to make sure that the same elements of chance appear on both sides of either $\succsim$ or $\sim$. This is how we arrived at Equation (17). In effect, doing so allows cancellation of any impacts of attitudes about running the experiment.

3.6. *Axiomatization of kernel equivalents that have an RDU representation*

In arriving at a separable representation of the kernel equivalents $KE(x, C; e)$ one needs to satisfy the axioms of conjoint measurement. One of the crucial ones is the Thomson condition which has been shown (Luce 2000, Proposition 3.5.1) to be implied by KE satisfying in addition to consequence monotonicity the property of *status-quo event commutativity*, i.e., for $C, D \in \mathcal{E}_{\mathbf{E}}$ and $x \in \mathcal{C}^+$,

$$KE[KE(x, C; e), D; e] \sim KE[KE(x, D; e), C; e]. \tag{18}$$

So, we need a condition in $\mathfrak{D}$ that is equivalent to this. Assuming commutativity, associativity and general segregation it is:

Status-quo event commutativity: For all $C, D \in \mathcal{E}_{\mathbf{E}}$ and $x \in \mathcal{C}^+$,

$$((x, C; e), D; (e, C; e)) \sim ((x, D; e), C; (e, D; e)) \tag{19}$$

In the idempotent case, this reduces to the usual definition, which is the reason for using the same name.

So we have the Thompson condition, but the remaining axioms of additive conjoint measurement are also needed to get the separable representation. To that end we define three concepts in $\mathfrak{D}$.

Order Independence of Events: For all $x, y \in \mathcal{C}^+$, $C, D \in \mathcal{E}_{\mathbf{E}}$

$$\begin{aligned}&(x, C; e) \oplus (e, D; e) \succsim (x, D; e) \oplus (e, C; e)\\&\quad \Leftrightarrow (y, C; e) \oplus (e, D; e) \succsim (y, D; e) \oplus (e, C; e).\end{aligned} \tag{20}$$

This means that the induced order $\succsim_{\mathbf{E}}$,

$$C \succsim_{\mathbf{E}} D \Leftrightarrow (x, C; e) \oplus (e, D; e) \succsim (x, D; e) \oplus (e, C; e), \tag{21}$$

is well defined.

Standard Sequences: For $C, D \in \mathcal{E}_{\mathbf{E}}$ with $C \succ_{\mathbf{E}} D$, $x_i \in \mathcal{C}^+$, the i from a consecutive sequence of integers, form a *consequence standard sequence* if and only if

$$(x_i, C; e) \oplus (e, D; e) \sim (x_{i+1}, D; e) \oplus (e, C; e).$$

And for $x, y \in \mathcal{C}^+$, $x \succ y$, $C_i \in \mathcal{E}_{\mathbf{E}}$, the i from a consecutive sequence of integers, form an *event standard sequence* if and only if

$$(x, C_i; e) \oplus (e, C_{i+1}; e) \sim (y, C_{i+1}; e) \oplus (e, C_i; e).$$

Restricted Solvability: For each $\overline{x}, y, \underline{x} \in \mathcal{C}^+$and $C, D \in \mathcal{E}_{\mathbf{E}}$, if

$$\begin{aligned}(\overline{x}, C; e) \oplus (e, D; e) &\succ (y, D; e) \oplus (e, C; e)\\&\succ (\underline{x}, C; e) \oplus (e, D; e),\end{aligned}$$

then there exists $x \in \mathcal{C}^+$ such that

$$(x, C; e) \oplus (e, D; e) \sim (y, D; e) \oplus (e, C; e).$$

And for each $x, y \in \mathcal{C}^+$, $\overline{C}, D, \underline{C} \in \mathcal{E}_{\mathbf{E}}$, if

$$\begin{aligned}&(x, \overline{C}; e) \oplus (e, D; e) \oplus (e, \underline{C}; e)\\&\quad \succ (y, D; e) \oplus (e, \overline{C}; e) \oplus (e, \underline{C}; e)\\&\quad \succ (x, \underline{C}; e) \oplus (e, \overline{C}; e) \oplus (e, D; e),\end{aligned}$$

then there exist $C \in \mathcal{E}_{\mathbf{E}}$ such that

$$(x, C; e) \oplus (e, D; e) \sim (y, D; e) \oplus (e, C; e).$$

Note that these definitions reduce to the usual ones in the idempotent case.

Summarizing:

THEOREM 4. *Suppose that* $\mathfrak{D} = \langle \mathcal{D}, e, \succsim, \oplus \rangle$ *is a structure for which consequence monotonicity holds for gambles; commutativity, associativity, and monotonicity hold for* $\oplus$*; and kernel equivalents exist for all binary gambles. Then, for the gains kernel equivalents to satisfy parts 1 and 3 of Theorem 1, and therefore the rank-dependent form of part 2, the following conditions 1–5 about* $\mathfrak{D}$ *are necessary and 1–7 are sufficient:*

1. *There is a p-additive representation, Equation (4), of* $\mathfrak{D}$ *onto the real numbers.*
2. *General segregation, Equation (8), is satisfied.*
3. *Status-quo event commutativity, Equation (19), is satisfied.*
4. *Joint-receipt decomposition, Equation (17), is satisfied.*
5. *(Archimedeanness) Every bounded standard sequence is finite.*
6. *The induced order* $\succsim_{\mathbf{E}}$*, Equation (21), is dense.*
7. *Restricted solvability is satisfied.*

COROLLARY. *If the following property, called monotonicity of event inclusion, holds*

$$x \succ y,\, C \subseteq D \Rightarrow (x, C; y) \oplus (e, D; e) \precsim (x, D; y) \oplus (e, C; e),$$

then for $C \subseteq D$, $W(C) \leq W(D)$.

4. FORMS OF $U(e, C; e)$ SYMMETRIC IN $W(C)$ AND $W(\overline{C})$

4.1. *Basic strategy*

We turn now to the question of possible forms for $U(e, C; e)$. Several observations guide our approach.

Given that $e \sim (e, E; e) \sim (e, \emptyset; e)$ and that $e \oplus e \sim e$, we see that $U(e, E; e) = U(e, \emptyset; e) = U(e) = 0$. So whatever general form $U(e, C; e)$ has, these two boundary conditions must be met. To get expressions for $U(e, C; e)$, $C \neq E, \emptyset$, we will seek equations of the form

$$KE(g) \oplus (e, C; e) \sim KE(h). \tag{22}$$

The motivation is that if we choose g and h appropriately, then we can calculate $U(e, C; e)$. Specifically, assuming U is p-additive for kernel equivalents and that everything is either a gain or a loss, we see that

$$U(e, C; e) = \frac{U[KE(h)] - U[KE(g)]}{1 - \delta U[KE(g)]} \tag{23}$$

Because the only thing that we know about the events are the weights assigned to them, we explore the hypothesis that $U(e, C; e)$ depends on $W(C)$ and $W(\overline{C})$. Moreover, if we assume that elements of chance satisfy complementarity, i.e.

$$(e, C; e) \sim (e, \overline{C}; e), \tag{24}$$

then the roles of $W(C)$ and $W(\overline{C})$ are symmetric.

There are two major possibilities for this dependence that we know how to get at using RDU applied to kernel equivalents, namely, $W(C)W(\overline{C})$ and $W(C) + W(\overline{C}) - \delta k W(C)W(\overline{C})$. The relevant equations giving rise to these two forms are

$$U\left(KE[KE(z, C; e), \overline{C}; e]\right) = U(z)W(C)W(\overline{C}), \tag{25}$$

and

$$\begin{aligned} &U[KE(z, C; e) \oplus KE(z, \overline{C}; e)] \\ &\quad = U(z)[W(C) + W(\overline{C}) - \delta U(z)W(C)W(\overline{C})]. \end{aligned} \tag{26}$$

Either of these or a constant can play the role of either g or h in Equation (22), which leads us to examine 6 cases. The cases where g and h in Equation (23) both satisfy Equation (25) or both satisfy Equation (26) yield results very similar to those of the 6 we report, and, of course, the results when both g and h are constant are obvious. One converts the KE expressions of Equations (25) and (26) to the corresponding qualitative ones in the structure $\mathcal{D}$ by adjoining $(e, C; e) \oplus (e, \overline{C}, e) \sim (e, C; e) \oplus (e, C, e)$ [see Equation (24)]. Assuming associativity, commutativity, and segregation yields, respectively,

$$((z, C; e), \overline{C}; (e, C; e)) \quad \text{and} \quad (z, C; e) \oplus (z, \overline{C}; e). \tag{27}$$

We write down the behavioral form and the corresponding expression for each of the 6 cases following the convention of calling the z-value of the g gamble in Equation (22) z and that of the h gamble z', and we let $k = U(z)$ and $k' = U(z')$. The proofs, which follow the above outline, are quite simple. The first and third are given in Appendix A and the others are left to the reader.

4.2. *Dependence on* $W(C) + W(\overline{C}) - \delta k W(C) W(\overline{C})$ *and/or* $W(C)W(\overline{C})$

The following proposition concerns, first, two forms that depend on $W(C)W(\overline{C})$, then two that depend on $W(C)+W(\overline{C})-\delta k W(C)W(\overline{C})$, and finally two that depend on both of these.

PROPOSITION 5. *Assume that the kernel equivalents satisfy the three conditions stated in the conclusion of Theorem 1 and that elements of chance satisfy complementarity. In the following statements, we assume there exist* $z, z' \in \mathcal{C}^+$ *independent of* $C \neq E, \emptyset$ *meeting the asserted condition, and we let* $k = U(z)$ *and* $k' = U(z')$.

1. *The indifference*

$$((z, C; e), \overline{C}; (e, C; e)) \sim (z', C, z')$$

is equivalent to

$$U(e, C; e) = \frac{k' - kW(C)W(\overline{C})}{1 - k\delta W(C)W(\overline{C})}.$$

2. *The indifference*

$$((z', C; e), \overline{C}; (e, C; e)) \sim (e, C; e) \oplus (e, C; e) \oplus (z, C; z)$$

is equivalent to

$$U(e, C; e) = \frac{k' W(C)W(\overline{C}) - k}{1 - \delta k}.$$

3. The indifference

$$(z, C; e) \oplus (z, \overline{C}; e) \sim (z', C; z')$$

is equivalent to

$$U(e, C; e) = \frac{k' - k[W(C) + W(\overline{C}) - \delta k W(C) W(\overline{C})]}{1 - \delta k[W(C) + W(\overline{C}) - \delta k W(C) W(\overline{C})]}.$$

4. The indifference

$$(z', C; e) \oplus (z', \overline{C}; e) \sim (e, C; e) \oplus (e, C; e) \oplus (z, C; z)$$

is equivalent to

$$U(e, C; e) = \frac{k'[W(C) + W(\overline{C}) - \delta k' W(C) W(\overline{C})] - k}{1 - \delta k}.$$

5. The indifference

$$(z, C; e) \oplus (z, \overline{C}; e) \oplus (e, C; e) \sim ((z', C; e), \overline{C}; (e, C; e))$$

is equivalent to

$$U(e, C; e) = \frac{k' W(C) W(\overline{C}) - k[W(C) + W(\overline{C}) - \delta k W(C) W(\overline{C})]}{1 - \delta k[W(C) + W(\overline{C}) - \delta k W(C) W(\overline{C})]}.$$

6. The indifference

$$(z', C; e) \oplus (z', \overline{C}; e) \sim ((z, C; e), \overline{C}; (e, C; e)) \oplus (e, C; e)$$

is equivalent to

$$U(e, C; e) = \frac{k'[W(C) + W(\overline{C}) - \delta k' W(C) W(\overline{C})] - k W(C) W(\overline{C})}{1 - \delta k W(C) W(\overline{C})}.$$

It seems plausible that $U(e, C; e)$ should be continuous as $W(C) \to 0$ or 1. From this assumption and Proposition 5 we see that we have the following expressions for continuous $U(e, C; e)$:

COROLLARY. *Suppose Proposition 5 holds. If $U(e, C; e)$ is continuous as $W(C) \to 0$ or 1, then the following restrictions obtain with the numbers referring to the relevant parts of the statement of Proposition 5:*

1 & 6. $z' = e$ *and* $U(e, C; e) = \frac{kW(C)W\overline{C})}{1-\delta kW(C)W\overline{C})}$.
2 & 5. $z = e$ *and* $U(e, C; e) = k'W(C)W\overline{C})$.
3. $z' = z$ *and* $U(e, C; e) = k\frac{1-W(C)-W(\overline{C})+\delta kW(C)W\overline{C})}{1-\delta k[W(C)+W(\overline{C})-\delta kW(C)W\overline{C})]}$.
4. $z' = z$ *and* $U(e, C; e) = \frac{k}{1-\delta k}\left[W(C) + W(\overline{C}) - \delta kW(C)\right.$ $\left.W\overline{C}) - 1\right]$.

Without these restrictions, the structure is discontinuous as either $W(C) \to 0$ or $\to 1$. This possibility evidences a major difference between certainty and even the smallest deviation from it. Informally, the utility of gambling is heightened by the surprise factor of a small chance, and lessened by more probable events up to the point of being equally likely.

Figures 1–4 each provide the six formulas for four different, but related, sets of parameters. The parameter β arises because it is assumed that $W(p) = p^{\beta}$. It is clear that a considerable range of possible symmetric forms can arise from these formulas, and that any one formula itself exhibit a wide range.

Although we have made no systematic attempt to see how well each model can mimic the others, our guess is that attempting to

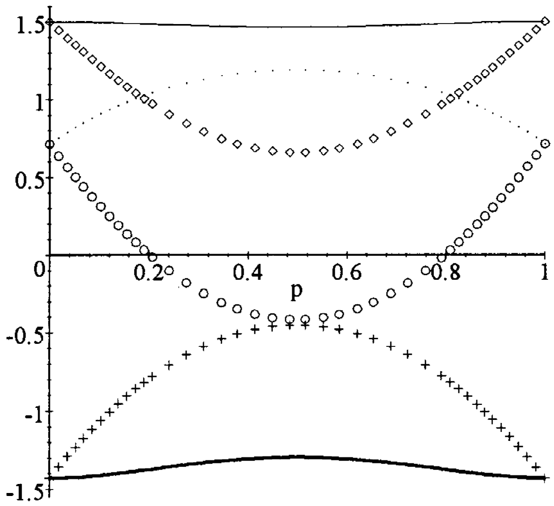

Figure 1. Plots of the various expressions for $U(e, p; e)$ for the parameters: $k = 1, \delta = 0.3, k' = 1.5, \beta = 2$. Solid: Proposition 5.1. Dotted: Proposition 5.2. Point: Proposition 5.3. Circle: Proposition 5.4. Cross: Proposition 5.5. Diamond: Proposition 5.6.

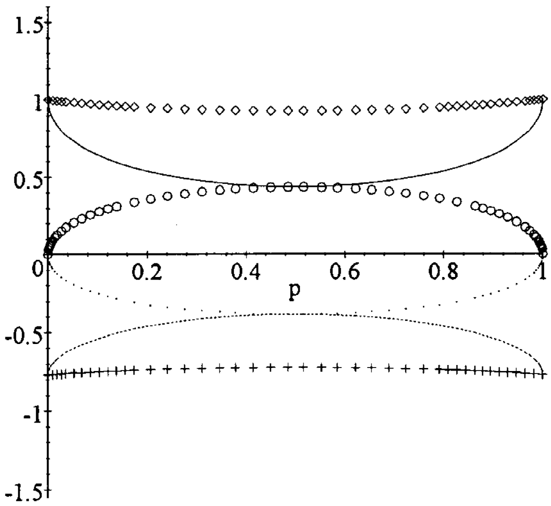

Figure 2. Plots of the various expressions for $U(e, p; e)$ for the parameters: $k = 1, \delta = -0.3, k' = 1, \beta = 1/2$. Solid: Proposition 5.1. Dotted: Proposition 5.2. Point: Proposition 5.3. Circle: Proposition 5.4. Cross: Proposition 5.5. Diamond: Proposition 5.6.

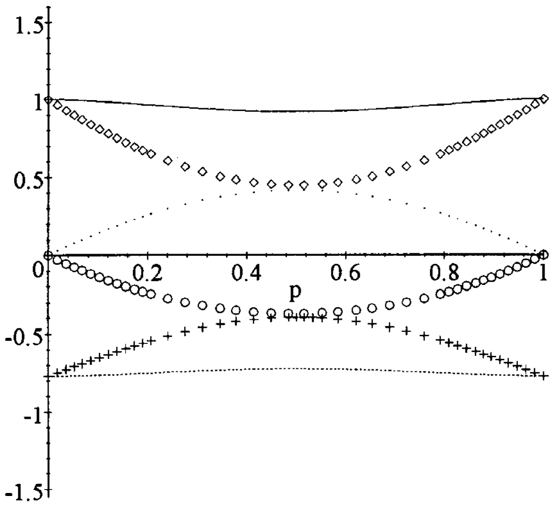

Figure 3. Plots of the various expressions for $U(e, p; e)$ for the parameters: $k = 1, \delta = -0.3, k' = 1/2, \beta = 2$. Solid: Proposition 5.1. Dotted: Proposition 5.2. Point: Proposition 5.3. Circle: Proposition 5.4. Cross: Proposition 5.5. Diamond: Proposition 5.6.

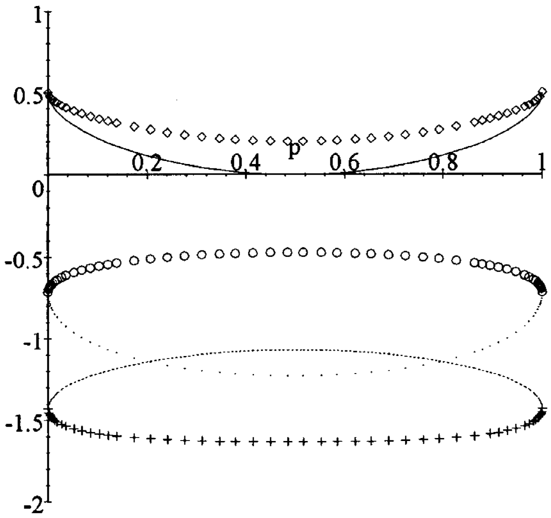

Figure 4. Plots of the various expressions for $U(e, p; e)$ for the parameters: $k = 1, \delta = 0.3, k' = 1/2, \beta = 2$. Solid: Proposition 5.1. Dotted: Proposition 5.2. Point: Proposition 5.3. Circle: Proposition 5.4. Cross: Proposition 5.5. Diamond: Proposition 5.6.

select among them using estimated functions will not be easy. Moreover, we know of no principled argument on which to select among them In particular, rationality arguments do not seem to apply. However, it appears that the simplest cases to test empirically are the first and third cases of Proposition 5. Much additional work is called for on trying to get a better understanding of the utility of elements of chance.

5. CONCLUSIONS

The utility of gambling in the context of choosing among uncertain alternatives has been an elusive concept. We explored one aspect, namely, relaxing the assumption that gambles are idempotent — the so-called constant acts of Savage (1954). By assuming segregation, which was a key building block of the theory described by Luce (2000), the non-idempotent binary case devolves just to understanding $(e, C; e)$. Recognizing this fact led us to decompose any binary gamble into the joint receipt of a pure consequence, called the kernel equivalent of the gamble, and $(e, C; e)$, called an element of chance. It was then shown that the kernel equivalents are idempotent and satisfy segregation, and if they satisfy the standard idempotent theory, then the form of $U[KE(x, C; y)]$ is determined. From that and the p-additive behavior of $U(x \oplus y)$ we arrived at formulas for $U(x, C; y)$. These expressions are fairly simple when everything involved is seen as gains (or losses). They become significantly more complex when mixed gains and losses are involved. In total, including pure gains and losses, there are 150 different cases. Any individual is, presumably, described by only one of the cases.

We also showed at the level of the non-idempotent structure the qualitative properties corresponding to those used to axiomatize the idempotent structure of kernel equivalents. The basic principle is to make sure that the same experiments are realized on two sides of a preference or an indifference.

The entire theory generalizes easily to the analogous rank-dependent representation of gambles with finitely many consequences.

We do not have any adequate understanding of the utility of elements of chance, $U(e, C; e)$.

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APPENDIX A: PROOFS

PROPOSITION 2

1. For $x, y, z \in \mathcal{C}$, $C \in \mathcal{E}_{\mathbf{E}} \backslash \{\emptyset, E\}$, using the monotonicity of $\oplus$,

$$\begin{aligned} x \succsim y &\Leftrightarrow (x, C; z) \succsim (y, C; z) \\ &\Leftrightarrow KE(x, C; z) \oplus (e, C; e) \succsim KE(y, C; z) \oplus (e, C; e) \\ &\Leftrightarrow KE(x, C; z) \succsim KE(y, C; z). \end{aligned}$$

2. Suppose $x \succsim e$, and so, by consequence monotonicity, $(x, C; x) \succsim (e, C; e)$. Using Definition 2, segregation of the gambles, and commutativity of $\oplus$,

$$KE(x, C; x) \oplus (e, C; e) \sim (x, C; x) \sim x \oplus (e, C; e).$$

By the monotonicity of $\oplus$, $KE(x, C; x) \sim x$, proving idempotence.

3. Let $x, y \in \mathcal{C}$, $x \succsim y$. Observe that:

$$\begin{aligned}(x, C; y) \succsim (e, C; e) &\Leftrightarrow KE(x, C; y) \oplus (e, C; e) \succsim (e, C; e)\\ &\Leftrightarrow KE(x, C; y) \succsim e.\end{aligned}$$

So the conditions of general segregation agree in the two structures.

Next, we observe that by associativity and commutativity,

$$\begin{aligned}&[KE(x \ominus y, C; e) \oplus y] \oplus (e, C; e)\\ &\quad\sim \left[KE(x \ominus y, C; e) \oplus (e, C; e)\right] \oplus y\\ &\quad\sim (x \ominus y, C; e) \oplus y\end{aligned}$$

and by Definition 2,

$$KE(x, C; y) \oplus (e, C; e) \sim (x, C; y).$$

Thus, by the transitivity of $\sim$ and the monotonicity of $\oplus$,

$$\begin{aligned}&KE(x \ominus y, C; e) \oplus y \sim KE(x, C; y)\\ &\quad\Leftrightarrow (x \ominus y, C; e) \oplus y \sim (x, C; y).\end{aligned}$$

4. To establish Equation 11 we first note that by consequence monotonicity for gambles, for $x, y \in \mathcal{C}^+$, $(x, C; y) \succsim (e, C; e)$, and so $KE(x, C; y) \succsim e$. By hypothesis $(e, C; e) \succsim e$. So applying Equation (4) to Equation (9) and using the rank-dependent form of Equation (3), we have for $x \succsim y \succsim e$,

$$\begin{aligned}U(x, C; y) &= U[K(x, C; y)][1 - \delta U(e, C; e)] + U(e, C; e)\\ &= (U(x)W(C) + U(y)[1 - W(C)])\\ &\quad [1 - \delta U(e, C; e)] + U(e, C; e)\\ &= U(x)W(C)[1 - \delta U(e, C; e)]\\ &\quad + U(y)\,(1 - W(C)[1 - \delta U(e, C; e)])\\ &\quad + U(e, C; e)[1 - \delta U(y)],\end{aligned}$$

whence the conclusion. The case $e \precsim x \precsim y$ is similar.

PROPOSITION 3

Using consequence monotonicity, associativity, and commutativity freely,

$$\begin{aligned} &KE(x \oplus y, C; e) \sim KE(x, C; e) \oplus KE(y, D; e) \\ &\quad \Leftrightarrow KE(x \oplus y, C; e) \oplus (e, C; e) \oplus (e, D; e) \\ &\qquad \sim KE(x, C; e) \oplus KE(y, D; e) \oplus (e, C; e) \oplus (e, D; e) \\ &\quad \Leftrightarrow (x \oplus y, C; e) \oplus (e, D; e) \sim (x, C; e) \oplus (y, D; e). \end{aligned}$$

THEOREM 4

The conditions of Theorem 4 justify all the conditions required for parts 1 and 3 of Theorem 1 to hold except for the existence of a separable representation. By Theorem 3.5.3 of Luce (2000), we know that we must show for the domain of kernel equivalents of the form $KE(x, C; e)$ that $\succsim$ on $\mathcal{C}^+$ and $\succsim_{\mathbf{E}}$ are dense; that consequence monotonicity, restricted solvability, and status-quo event commutativity hold, and that they are Archimedean. Moreover, we must show the images of the representation are, respectively, a real interval $[0, k[$ and $[0, 1]$ for the utility and weighting functions.

The density of $\succsim$ is assured by 1 and that of $\succsim_{\mathbf{E}}$ is postulated in 6.

Consequence monotonicity follows from that of $\mathfrak{D}$.

Note that

$$(x, C; e) \oplus (e, D; e) \sim KE(x, C; e) \oplus (e, C; e) \oplus (e, D; e),$$

and so the several defined properties hold for the kernel equivalents because $(e, C; e) \oplus (e, D; e)$ appears on both sides and so can be cancelled. Thus, in the usual sense, the kernel equivalents satisfy Archimedeanness and restricted solvability.

We prove KE satisfies status-quo event commutativity if and only if $\mathfrak{D}$ satisfies the same property. Noting that $KE(x, C; e) \oplus (e, C; e) = (x, C; e) \succsim e$ and using commutativity, associativity, and general segregation freely,

$$\begin{aligned}
&KE[KE(x, C; e), D; e] \sim KE[KE(x, D; e), C; e] \\
&\quad \Leftrightarrow KE[KE(x, C; e), D; e] \oplus (e, C; e) \oplus (e, D; e) \\
&\qquad \sim KE[KE(x, D; e), C; e] \oplus (e, C; e) \oplus (e, D; e) \\
&\quad \Leftrightarrow (KE(x, C; e), D; e) \oplus (e, C; e) \sim (KE(x, D; e), C; e) \\
&\qquad\quad \oplus (e, D; e) \\
&\quad \Leftrightarrow (KE(x, C; e) \oplus (e, C; e), D; (e, C; e)) \\
&\qquad \sim (KE(x, D; e) \oplus (e, D; e), C; (e, D; e)) \\
&\quad \Leftrightarrow ((x, C; e), D; (e, C; e)) \sim ((x, D; e), C; (e, D; e)).
\end{aligned}$$

Therefore Theorem 3.5.3 of Luce (2000) applies, and so there is a representation $U''W''$ with dense images. Because $x \sim (x, E; e)$, assumption 1 implies that the mapping U'' is onto an interval of the real numbers. To show that W'' is onto [0, 1], choose any $r \in [0, 1]$. Since U'' is onto a real interval, we may select $x \succsim y$ such that $\frac{U''(y)}{U''(x)} = r$. Then because $e \sim (e, E; e) \sim (e, \emptyset; e)$ and $y \sim (y, E; e)$,

$$\begin{aligned}
&x \succ y \succ e \\
&\quad \Leftrightarrow (x, E; e) \oplus (e, E; e) \oplus (e, \emptyset; e) \succ (y, E; e) \\
&\quad \oplus (e, E; e) \oplus (e, \emptyset; e) \succ (x, \emptyset; e) \oplus (e, E; e) \oplus (e, E; e)
\end{aligned}$$

So, restricted solvability implies there exists C such that

$$(x, C; e) \oplus (e, E; e) \sim (y, E; e) \oplus (e, C; e).$$

By the definition of KE and using monotonicity of $\oplus$,

$$KE(x, C; e) \sim KE(y, E; e) \sim y.$$

Thus, $U''(x)W''(C) = U''(y) = rU''(x)$, whence $W''(C) = r$.

PROPOSITION 5.1

Assume $z, z' \succsim e$ and set $k = U(z)$, $k' = U(z')$.

$$
\begin{aligned}
&((z, C; e), \overline{C}; (e, C; e)) \sim (z', C; z') && \text{(Condition)}\\
&\Leftrightarrow (KE(z, C; e) \oplus (e, C; e), \overline{C}; (e, C; e)) \sim (z', C; z') && \text{(Def. 4)}\\
&\Leftrightarrow (KE(z, C; e), \overline{C}; e) \oplus (e, C; e) \sim z' \oplus (e, C; e) \succsim (e, C; e)\\
& && \text{(Def. 2 \& idempotence of kernel equivalents)}\\
&\Leftrightarrow (KE(z, C; e), \overline{C}; e) \sim z' && \text{(Monotonicity of } \oplus)\\
&\Leftrightarrow KE(KE(z, C; e), \overline{C}; e) \oplus (e, \overline{C}; e) \sim z' && \text{(Def. 4)}\\
&\Leftrightarrow U(e, \overline{C}; e)\left(1 - \delta U[KE(KE(z, C; e), \overline{C}; e)]\right)\\
&\quad + U[KE(KE(z, C; e), \overline{C}; e))] = U(z') && \text{(Prop. 2)}\\
&\Leftrightarrow U(e, \overline{C}; e)\left[1 - \delta U(z)W(C)W(\overline{C})\right] + U(z)W(C)W(\overline{C}) = U(z') && \text{(Eq. (25))}\\
&\Leftrightarrow U(e, C; e) = U(e, \overline{C}; e) = \frac{k' - kW(C)W(\overline{C})}{1 - \delta kW(C)W(\overline{C})}. && \text{(Algebra \& Eq. (24))}
\end{aligned}
$$

PROPOSITION 5.3

Using monotonicity and associativity of $\oplus$, idempotence of kernel equivalents, and kernel equivalents satisfy complementarity, we have

$$
\begin{aligned}
&(z', C; z') \sim (z, C; e) \oplus (z, \overline{C}; e)\\
&\Leftrightarrow (e, \overline{C}; e) \oplus z' \sim KE(z, C; e) \oplus (e, C; e) \oplus KE(z, \overline{C}; e) \oplus (e, \overline{C}; e) && \text{(Def. 4)}\\
&\Leftrightarrow z' \sim (e, C; e) \oplus KE((z, C; e) \oplus KE(z, \overline{C}; e)\\
&\Leftrightarrow U(z') \sim U(e, C; e)\left(1 - \delta U\left[KE((z, C; e) \oplus KE(z, \overline{C}; e)\right]\right)\\
&\quad + U\left[KE((z, C; e) \oplus KE(z, \overline{C}; e)\right] && \text{(Theorem 1)}\\
&\Leftrightarrow U(e, C; e) = \frac{k' - U\left[KE((z, C; e) \oplus KE(z, \overline{C}; e)\right]}{1 - \delta U\left[KE((z, C; e) \oplus KE(z, \overline{C}; e)\right]} && \text{(Algebra)}\\
&\Leftrightarrow U(e, C; e) = \frac{k' - k[W(C) + W(\overline{C}) - \delta kW(C)W(\overline{C})]}{1 - \delta k[W(C) + W(\overline{C}) - \delta kW(C)W(\overline{C})]}. && \text{(Eq. (26))}
\end{aligned}
$$

APPENDIX B: COUNTING CASES

Let $x+$ mean $x \succsim e$. Two observations:

- If $x+$, $y+$, then by consequence monotonicity $(x, C; y) \succsim (e, C; y)$, and so $KE(x, C; y)+$.

– If $KE(x, C; y)+$ and $(e, C; e)+$, then, by Def. 3, $(x, C; y)+$. The same holds with $+$ replaced by $-$.

Thus, for gains alone, these rule out all cases of $KE(x, C; y)-$ and also the case of $KE(x, C; y)+$, $(e, C; e)+$, $(x, C; y)-$, which we abbreviate $++-$. So there are just three viable combinations: $+++$, $+-+$, $+--$. The first involves 3 subcases depending on the sign of δ, either $+$, 0, $-$; whereas the next two involve both gains and losses and so there are 9 subcases. This is a total of 21 cases for gains. A similar calculation for losses yields again 21.

For the mixed case, the second observation reduces the 8 combinations to 6: $+++$, $+-+$, $+--$, $---$, $-++$, $-+-$. Each of these cases has 9 subcases because $KE(x, C; y)$ involves both gains and losses, and so depends on the form of both gains and losses. In addition, depending on how $\oplus$ links to gambles in the mixed cases, there are additional cases. So far there are two such proposals: general segregation and duplex decomposition, hence there are $6 \times 9 \times 2 = 108$ mixed cases.

Thus, the total is $21 + 21 + 108 = 150$.

NOTES

1. The term 'experiment' is used in the sense of statistics, namely, a source of chance outcomes, not in the sense of experimental science.
2. Each of the following conditions s hould, technically, be prefixed with 'weak' because they are defined in terms of $\sim$ rather than $=$. The abuse of terminology in omitting the adjective should not cause confusion.
3. The statement of part (ii) of Proposition 7.4.6 of Luce (2000) is incorrect. An errata is available either from http://aris.ss.uci.edu/cogsci/personnel/luce/Errata2.PDF or from rdluce@uci.edu.

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