Structured Sequent Calculi for Combining Intuitionistic and Classical First-Order Logic ⋆

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Abstract. We define a sound and complete logic, called $\mathcal{FO}^{\supset}$, which extends classical first-order predicate logic with intuitionistic implication.
As expected, to allow the interpretation of intuitionistic implication, the semantics of $\mathcal{FO}^{\supset}$ is based on structures over a partially ordered set of worlds. In these structures, classical quantifiers and connectives (in particular, implication) are interpreted within one (involved) world. Consequently, the *forcing relation* between worlds and formulas, becomes *non-monotonic* with respect to the ordering on worlds. We study the effect of this lack of monotonicity in order to define the satisfaction relation and the *logical consequence relation* which it induces.
With regard to proof systems for $\mathcal{FO}^{\supset}$, we follow Gentzen's approach of *sequent calculi* (*cf.* [8]). However, to deal with the two different implications simultaneously, the sequent notion needs to be more *structured* than the traditional one. Specifically, in our approach, the antecedent is structured as a sequence of sets of formulas. We study how inference rules preserve soundness, defining a structured notion of logical consequence. Then, we give some general sufficient conditions for the *completeness* of this kind of sequent calculi and also provide a sound calculus which satisfies these conditions. By means of these two steps, the completeness of $\mathcal{FO}^{\supset}$ is proved in full detail. The proof follows Hintikka's set approach (*cf.* [11]), however, we define a more general procedure, called *back-saturation*, to saturate a set with respect to a sequence of sets.

1 Introduction

Combining different logical systems has become very common in several areas of computer science. In this paper, we introduce a logical system, called $\mathcal{FO}^{\supset}$, obtained by using a combination of classical and intuitionistic logic. Our original motivation for defining this logic was to provide logical foundations for *logic programming* (LP) languages which combine classical ($\rightarrow$) and intuitionistic ($\supset$) implication. A well-founded LP language should be entirely supported by an underlying logic. This means that the declarative semantics of the LP language is based on the logical model-theory, and at the same time, the operational semantics is supported by the logical proof-theory; *cf.* [15] for technical details. Sequent calculi provide a very natural and direct way to formalise the operational semantics of LP languages. It is well known that standard LP (Horn clauses programming) is well-founded in classical first-order predicate logic. However, as mentioned in [10]:

> *"If the two implications are considered altogether, the resulting semantics differs from that of both intuitionistic and classical logic."*

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Then, for the extension of LP languages with intuitionistic implication, our aim is to obtain complete sequent calculi for a sound extension of first-order predicate logic with intuitionistic implication. We want to make it clear that the concern of this paper is the logic $\mathcal{FO}^{\supset}$ itself, but not its application to LP. In fact, the latter is the main subject of [1].

The problem in combining Hilbert axiomatizations of classical and intuitionistic logic is shown in [2]:

> *"It cannot be attained by simply extending the union of both axiomatizations by so called interaction axioms"*

In fact, whenever both kinds of implicative connectives are mixed, some equivalences and axiom schemas are lost, and they collapse into classical logic. What they do to achieve a Hilbert-style axiomatization (and also a tableaux method) is to restrict some axiom schemas (and tableaux rules) to a special kind of formulas called persistent. The same restriction is used in the natural deduction system introduced in [12] which also combines classical and intuitionistic logic. Our proposal enables greater flexibility for developing deductions, by introducing structure in the sequents, since this structure makes unnecessary the above persistence-restriction. We will return to this matter in the last section, after $\mathcal{FO}^{\supset}$ has been presented.

$\mathcal{FO}^{\supset}$ syntax is first-order without any restriction, hence, we deal with variables, functions, predicates, connectives and quantifiers. For semantical differentiation of both implications, it suffices to consider standard Kripke structures, over a partially ordered set of worlds. Worlds have an associated classical first-order structure for the semantical interpretation of terms over its universe ([20, 21]). An intuitionistic implication $\varphi \supset \psi$ is satisfied in a world w if every world greater than w satisfying φ also satisfies ψ. By contrast, classical connectives and quantifiers are interpreted within one (involved) world. In particular, a world w satisfies $\varphi \to \psi$ if either w does not satisfy φ or satisfies ψ. As a consequence, the forcing relation between worlds and formulas becomes non-monotonic with respect to the ordering on worlds. This lack of monotonicity is taken into account to define the *satisfaction* and *logical consequence* relations. In particular, a Kripke model satisfies a formula whenever all its minimal worlds force it. This satisfaction relation avoids the collapse of both implications. Moreover, it induces (in the usual way) a logical consequence relation for $\mathcal{FO}^{\supset}$. With regard to sequent calculi for $\mathcal{FO}^{\supset}$ the central point is the nature of sequents. Gentzen's original notion (*cf.* [8]) considers a sequent to be a pair (Γ, Φ) where the *antecedent* Γ and the *consequent* Φ are finite sequences of formulas. It is usual (in classical logic) to consider sets, instead of sequences, because this prevents some extra inference rules for duplication of formulas (contraction rule), interchanges of formulas (interchange rule), and so on. In intuitionistic logic, the consequent usually consists of a single formula. To deal with classical and intuitionistic implications within the same logic, the antecedent requires more structure, in order to avoid the collapse of both implications. We introduce this idea by means of two simple examples. Let us consider the three propositional symbols p, q, r. It is obvious that $p \to q$ is semantically weaker than $p \supset q$. Therefore, we should not allow the derivation of the sequent with $p \to q$ as antecedent and $p \supset q$ as consequent. Roughly speaking, $p \to q$ is a "one world sentence", whereas $p \supset q$ is "about all greater worlds". This meaning suggests that the "$\supset$-in-the-right" rule, to split $p \supset q$ for putting p on the left and leaving q on the right, must be used with care about which worlds $p \to q$ and p are speaking about. In fact, the antecedent of the next sequent in our derivation tree has the sequence (but not the set) $\langle p \to q; p \rangle$; obviously with q as consequent. This sequent should not be derivable, since $p \to q$ is saying nothing

about greater worlds satisfying p. Moreover, to derive, for example, the valid (classical) sentence $(p \to q) \to ((q \to r) \to (p \to r))$, it becomes useful to relate $p \to q$, $q \to r$ and p to the same world in the antecedent sequence, taking r as the consequent. Thus sequences of sets arise as antecedents. Accordingly, in this paper, a sequent consists of a pair (Δ, χ) where the antecedent Δ is a (finite) sequence of (finite) sets of formulas and the consequent χ is a single formula. Hence, we say *structured sequent calculus* to emphasise its nature. For the study of structured deductive systems in a more general setting, see [5].
The rest of the paper is organised as follows: in Section 2 we give preliminary details about syntax and notation; in Section 3 we establish the model theory and the necessary semantical notions and properties; in Section 4 we present the soundness and completeness results for structured sequent calculi; finally, in Section 5, we summarise conclusions and related work.

2 Preliminaries

We consider *signatures* Σ consisting of countable (pairwise disjoint) sets VS_Σ of variable symbols, FS_Σ of function symbols, and PS_Σ of predicate symbols, with some specific arity for each function and predicate symbol. Function symbols of arity 0 are called *constant symbols*. We denote by $Term_\Sigma$ the set of all well-formed first-order Σ-terms, inductively defined by:

- A variable $x \in VS_\Sigma$ is a Σ-term
- If $f \in FS_\Sigma$ is n-ary and $t_1, \ldots, t_n \in Term_\Sigma$, then $f(t_1, \ldots, t_n)$ is a Σ-term.

The set $Form_\Sigma$ of all well-formed Σ-formulas contains the atomic formulas and the recursive composition of formulas by means of intuitionistic implication and classical connectives and quantifiers. For convenience, we consider the following atomic formulas:

- $p(t_1, \ldots, t_n)$, for n-ary $p \in PS_\Sigma$ and $t_1, \ldots, t_n \in Term_\Sigma$
- $\textsc{f}$ (falsehood).

In addition, we consider the classical connectives: negation ($\neg$) and implication ($\to$); the intuitionistic implication ($\supset$); and the classical universal quantifier $\forall$. The remaining connectives and quantifiers, i.e. conjunction ($\wedge$), disjunction ($\vee$), existential quantification ($\exists$), and intuitionistic negation ($\sim$), can be defined as abbreviations: $\varphi \wedge \psi$ for $\neg(\varphi \to \neg\psi)$, $\varphi \vee \psi$ for $\neg\varphi \to \psi$, $\exists x\varphi$ for $\neg\forall x\neg\varphi$, and $\sim \varphi$ for $\varphi \supset \textsc{f}$. Even the intuitionistic universal quantifier, namely $\check{\forall}$, can be expressed in $\mathcal{FO}^{\supset}$ by the formula $(\neg\textsc{f}) \supset \forall x\varphi$ as definition of $\check{\forall} x\varphi$.
Terms without variable symbols are called *closed terms*. By means of quantifiers, formulas have free and bound variables. We denote by $free(\varphi)$ the set of all free variables of the formula φ. We call *Σ-sentence* a Σ-formula without free variables.
The uppercase Greek letters Γ and Φ (possibly with sub- and superscripts) will be used as metavariables for *sets* of formulas, whereas Δ, Δ', Δ'', ... are reserved to be metavariables for *sequences of sets* of formulas.
We denote by $cons(t)$ the set of all constant symbols appearing in the term t. A very simple structural induction extends *cons* to formulas, sets of formulas and sequences of sets of formulas.

Structures (or models) are Kripke models of first-order intuitionistic logic ([20, 21]), that is, we consider worlds with universes and function symbols interpretations, as well as predicate symbols interpretations. Variables assignments and substitution notation are especially awkward for dealing with Kripke models. For that reason, to define the semantics of quantifiers, we follow the approach in [19] of *naming* all elements of the universe:

Definition 1. Given a set of elements (or universe) A, a signature Σ can be extended to the signature Σ_A, by adding a new constant symbol $\widehat{a}$ for each element $a \in A$. ■

In order to simplify substitution notation we write $\varphi(\overline{t})$ for the *simultaneous substitution instance* of the formula $\varphi(\overline{x})$ with terms $\overline{t}$ for variables $\overline{x}$. The notation $\varphi(\overline{x})$ is a meta-expression, which does not mean that $\overline{x}$ is the exhaustive list of free variables of φ, nor that all of them occur as free variables in φ.

3 Non-Monotonic Forcing and Logical Consequence

In this section we firstly establish the semantical structures for $\mathcal{FO}^{\supset}$, and then define the *forcing, satisfaction and logical consequence* relations. At the same time, we point out some interesting meta-logical properties, such as, for example, the monotonicity of terms evaluation and the substitution lemma, which are usual and useful in most of the well-known logics.

Semantical structures for $\mathcal{FO}^{\supset}$ are standard first-order Kripke structures with a partially ordered set of worlds ([20, 21]). Worlds have an associated classical first-order structure over a universe. We consider that the signature of every world includes names for individuals of its universe. We abbreviate by Σ_w the signature Σ_{A_w}, which extends Σ with a name $\widehat{a}$ for each $a \in A_w$ (see Definition 1).

Definition 2. A *Kripke Σ-structure* is a triple $\mathcal{K} = (W(\mathcal{K}), \preceq, \langle \mathcal{A}_w \rangle_{w \in W(\mathcal{K})})$ where
(a) $(W(\mathcal{K}), \preceq)$ is a partially ordered set (of worlds)
(b) Each $\mathcal{A}_w$ is a first-order Σ_w-structure defined by:
- A non-empty universe A_w
- A function $f^{\mathcal{A}_w} : (A_w)^n \to A_w$ for each n-ary $f \in FS_{\Sigma_w}$
- A set At_w of atomic Σ-formulas of the form $p(\widehat{a_1}, \dots, \widehat{a_n})$

such that for any pair of worlds $v \preceq w$:
1. $A_v \subseteq A_w$,
2. $At_v \subseteq At_w$, and
3. $f^{\mathcal{A}_w}(\overline{a}) = f^{\mathcal{A}_v}(\overline{a})$, for all n-ary $f \in FS_{\Sigma_v}$ and all $\overline{a} \in (A_v)^n$. ■

These Kripke structures allow us to interpret Σ_w-terms in worlds in a monotonic way.

Definition 3. Let w be a world of a Kripke Σ-structure and let $t \in Term_{\Sigma_w}$, its interpretation $t^{\mathcal{A}_w}$ (briefly t^w) is inductively defined as follows:
- $\widehat{a}^{\mathcal{A}_w} = a$ for any $a \in A_w$
- $(f(t_1, \dots, t_n))^{\mathcal{A}_w} = f^{\mathcal{A}_w}(t_1^{\mathcal{A}_w}, \dots, t_n^{\mathcal{A}_w})$. ■

Proposition 4. *For any $t \in Term_{\Sigma_v}$, any Kripke-structure $\mathcal{K}$ and any pair of worlds $v, w \in W(\mathcal{K})$ such that $v \preceq w$: $t^w = t^v \in A_v \subseteq A_w$.*

Proof: The usual proof by structural induction on t is suitable. ■

The satisfaction of sentences in worlds is handled by the following *forcing relation*:

Definition 5. Letting $\mathcal{K}$ be a Kripke Σ-structure, the binary *forcing relation* $\Vdash$ between worlds in $W(\mathcal{K})$ and Σ-formulas is inductively defined as follows:
$w \nVdash F$
$w \Vdash p(t_1, \dots, t_n)$ iff $p(\widehat{t_1^w}, \dots, \widehat{t_n^w}) \in At_w$
$w \Vdash \neg\varphi$ iff $w \nVdash \varphi$
$w \Vdash \varphi \to \psi$ iff $w \nVdash \varphi$ or $w \Vdash \psi$
$w \Vdash \varphi \supset \psi$ iff for all $v \in W(\mathcal{K})$ such that $v \succeq w$: if $v \Vdash \varphi$ then $v \Vdash \psi$
$w \Vdash \forall x \varphi(x)$ iff $w \Vdash \varphi(\widehat{a})$ for all $a \in A_w$.

For a set of formulas Φ, $w \Vdash \Phi$ means that $w \Vdash \varphi$ for all $\varphi \in \Phi$. ■

It is obvious that $\Vdash$ is relative to $\mathcal{K}$, hence we will write $\Vdash_{\mathcal{K}}$ whenever the simplified notation $\Vdash$ may be ambiguous.

$\mathcal{FO}^{\supset}$ satisfies the following meta-logical property:

Lemma 6. (Substitution Lemma) *For any Kripke Σ-structure $\mathcal{K}$, any $w \in W(\mathcal{K})$, any closed Σ-terms t_1, t_2 and any Σ-formula φ with $free(\varphi) \subseteq \{x\}$:*

$$\textit{If } t_1^w = t_2^w \textit{ and } w \Vdash \varphi(t_1), \textit{ then } w \Vdash \varphi(t_2).$$

Proof: A very easy structural induction on φ. ■

By Definition 2, it is obvious that the forcing relation $\Vdash$ behaves monotonically for atomic sentences, therefore we say that $\mathcal{FO}^{\supset}$ is *atomically monotonic*. However, in contrast with (pure) intuitionistic logic, monotonicity can not be extended to arbitrary sentences. This happens because $\mathcal{FO}^{\supset}$ gives a non-intuitionistic semantics to negation, classical implication and universal quantification. The following example illustrates this behaviour.

Example 7. Consider the following Kripke structure $\mathcal{K}$ with $W(\mathcal{K}) = \{u, v, w\}$ such that $u \preceq v \preceq w$, $\mathcal{A}_u$ with universe $\{a\}$, $\mathcal{A}_v$ with universe $\{a, b\}$, $\mathcal{A}_w$ with universe $\{a, b, c\}$, $At_u = \emptyset$, $At_v = \{r(\widehat{a}), r(\widehat{b})\}$, and $At_w = \{r(\widehat{a}), r(\widehat{b})\}$. In such a model, the world u forces $\neg r(\widehat{a})$ and also $r(\widehat{a}) \to q(\widehat{a})$, but v does not force either. Moreover, v forces $\forall x r(x)$, but w does not. ■

Now, our aim is to define a satisfaction relation between Kripke structures and sentences, from which a logical consequence relation between sets of sentences and sentences may be induced in the usual way.

Definition 8. Let $\models\!\!\!\!\approx$ be a satisfaction relation between Σ-structures and Σ-sentences, then the induced logical consequence relation, denoted by the same symbol $\models\!\!\!\!\approx$, is defined as follows.
For every set of Σ-sentences $\Gamma \cup \{\varphi\}$:
$\Gamma \models\!\!\!\!\approx \varphi$ iff for all Kripke Σ-structure $\mathcal{K}$: $\mathcal{K} \models\!\!\!\!\approx \Gamma \implies \mathcal{K} \models\!\!\!\!\approx \varphi$. ■

It should be noted at the outset that we can not define the satisfaction relation as it is commonly done in well-known logics with possible worlds semantics, like intuitionistic logic (*cf.* [20, 21]) and modal logics (*cf.* [4]). The reason is that, under this usual notion (recalled in Definition 9), both implications collapse into the intuitionistic one.

Definition 9. A Kripke structure $\mathcal{K}$ *satisfies* (or is a *model* of) a sentence φ if and only if $w \Vdash \varphi$ for all $w \in W(\mathcal{K})$. We call this relation *global satisfaction* and it is denoted by $\mathcal{K} \models_G \varphi$. ■

It is easy to observe that, for logics with monotonic forcing relation (e.g. intuitionistic logic), the fact to be forced in all worlds is equivalent to the fact to be forced in all minimal worlds. Hence, for this kind of logics, the logical consequence induced (in the sense of Definition 8) from global satisfaction ($\models_G$) is equivalent to the one induced from the following *minimal worlds satisfaction relation*:

Definition 10.
(a) We say that a world $w \in W(\mathcal{K})$ is *minimal* if and only if there does not exist $v \in W(\mathcal{K})$ such that $v \prec w$.
(b) A Kripke Σ-structure $\mathcal{K}$ *satisfies* (or is a *model* of) a Σ-sentence φ ($\mathcal{K} \models \varphi$) if and only if $w \Vdash \varphi$ for each minimal world $w \in W(\mathcal{K})$.
(c) For sets of sentences: $\mathcal{K} \models \Gamma$ if and only if $\mathcal{K} \models \varphi$ for every $\varphi \in \Gamma$. ■

We adopt, for $\mathcal{FO}^{\supset}$, the satisfaction relation $\models$ of Definition 10(b) and the logical consequence relation (also denoted by $\models$) induced from it, in the sense of Definition 8. As a matter of fact, $\models$ has a direct (instead of induced) equivalent definition, which is based on the *local* (in contrast with global) point of view. We are going to define the local relation $\models_L$ and then we prove that it is equivalent to $\models$ and stronger than $\models_G$.

Definition 11. $\Gamma \models_L \varphi$ if and only if for every Kripke structure $\mathcal{K}$ and for every world $w \in W(\mathcal{K})$, if $w \Vdash \Gamma$ then $w \Vdash \varphi$. ■

Proposition 12.
(i) *$\models$ and $\models_L$ are equivalent logical consequence relations.*
(ii) *$\models_L$ is stronger than $\models_G$.*
(iii) *If the forcing relation is monotonic, then $\models_L$ and $\models_G$ are equivalent logical consequence relations.*

Proof:
(i) It is trivial that $\models_L$ is stronger than $\models$. To prove the converse, let us suppose that there exist $\mathcal{K}$ and $w \in W(\mathcal{K})$ such that $w \Vdash \Gamma$ and $w \nVdash \varphi$. Then, we can define

$$\mathcal{K}^w = (\{v | v \in W(\mathcal{K}), v \succeq w\}, \preceq, \langle \mathcal{A}_v \rangle_{v \in W(\mathcal{K}), v \succeq w})$$

$\mathcal{K}^w$ is a Kripke structure with a unique minimal world w such that $w \Vdash \Gamma$ and $w \nVdash \varphi$. Therefore, $\mathcal{K}^w \Vdash \Gamma$ and $\mathcal{K}^w \nVdash \varphi$.
(ii) is trivial.
(iii) Suppose that $\Gamma \models_G \varphi$ and consider any $\mathcal{K}$ and $w \in W(\mathcal{K})$ such that $w \Vdash_{\mathcal{K}} \Gamma$. Now, consider $\mathcal{K}^w$ as above. By forcing monotonicity $w' \Vdash_{\mathcal{K}^w} \Gamma$ holds for all $w' \in W(\mathcal{K}^w)$. Then, $w' \Vdash_{\mathcal{K}^w} \varphi$ holds for all $w' \in W(\mathcal{K}^w)$. So, $w \Vdash_{\mathcal{K}} \varphi$. ■

Overall, the $\mathcal{FO}^{\supset}$ logical consequence $\models$ has been induced from the minimal world satisfaction relation of Definition 10(b), however, the equivalent formulation given by Definition 11 and Proposition 12(i) can be considered when convenient.

It is easy to see that $\models$ is monotonic with respect to set inclusion:

Fact 13. If $\Gamma \subseteq \Gamma'$ and $\Gamma \models \varphi$, then $\Gamma' \models \varphi$. ■

4 Structured Sequent Calculi and Completeness

As explained in Section 1, we consider structured sequents. Specifically, a sequent consists of a pair (Δ, χ) (written as $\Delta \mapsto \chi$) where the antecedent Δ is a *(finite) sequence of (finite) sets of formulas* and the consequent χ is a single formula. In order to simplify sequent notation we reserve Γ and Φ (possibly with sub- and superscripts) for *sets* of formulas, and Δ, Δ', Δ'', ... for *sequences of sets* of formulas; the semicolon sign $(;)$ will represent the infix operation for concatenation of sequences; $\Gamma \cup \{\varphi\}$ will be abbreviated by Γ, φ; and a set Γ is identified with the sequence consisting of this unique set. Thus, the semicolon sign $(;)$ is used to split a sequence into its components (sets of formulas) and the comma sign $(,)$ splits sets of formulas into its elements. For instance, $\Delta; \Gamma, \varphi; \Delta'$ denotes the sequence beginning with the sequence of sets Δ, followed by the set $\Gamma \cup \{\varphi\}$, and ending with the sequence of sets Δ'. With an abuse of notation we will write $\varphi \in \Delta$ to mean that φ belongs to some of the sets of formulas in Δ.

In this section we firstly define the notion of proof in a structured sequent calculus. Secondly, we explain what is required for soundness of this kind of calculi. For this purpose a structured consequence notion is defined. Then, we give general sufficient conditions for completeness, which have a dual advantage. On the one hand, different calculi for different purposes could be obtained from the general conditions. On the other hand, these conditions make the completeness proof easier. Lastly, we provide a sound and complete structured sequent calculus for $\mathcal{FO}^{\supset}$.

Definition 14. A *proof* of a sequent $\Delta \mapsto \chi$ in a calculus C (or *C-proof*) is a finite tree, constructed using inference rules from C, whose root is the sequent $\Delta \mapsto \chi$ and whose leaves are axioms (or initial sequents) in the calculus C. When there exists a *C-proof* of the sequent $\Delta \mapsto \chi$, we write $\Delta \vdash_C \chi$. Moreover, $\vdash_C$ is extended to (finite) sequences of infinite sets as follows. For Δ which is a sequence $\Gamma_0; \ldots; \Gamma_n$ of (possibly infinite) sets of Σ-formulas and φ a Σ-formula, $\Delta \vdash_C \varphi$ holds if and only if there exists a sequence Δ' of finite sets $\Gamma_0'; \ldots; \Gamma_n'$ such that $\Gamma_i' \subseteq \Gamma_i$ for each $i \in \{1..n\}$ and there exists a C-proof of the sequent $\Delta' \mapsto \varphi$. ■

Therefore, $\vdash_C$ is the *derivability relation* induced by the calculus C. Notice that every set in the antecedent of a sequent must be finite, whereas in the derivability relation, sets can be infinite. A trivial consequence of definition 14 is the following fact, which provides the *thinning rule* as meta-rule:

Fact 15. If $\Delta; \Gamma; \Delta' \vdash_C \chi$ and $\Gamma \subseteq \Gamma'$, then $\Delta; \Gamma'; \Delta' \vdash_C \chi$. ■

Now, we define a relation $\models^*$ between sequences of sets of formulas and formulas, such that its restriction to a single set of formulas coincides with the logical consequence relation $\models$.

Definition 16. Let Δ be a sequence $\Gamma_0; \Gamma_1; \ldots; \Gamma_n$ of sets of Σ-sentences and χ a Σ-sentence, then we say that $\Delta \models^* \chi$ if and only if for all Kripke Σ-structure $\mathcal{K}$ and for all sequence of worlds $u_0, u_1, \ldots u_n \in W(\mathcal{K})$ such that $u_{i-1} \preceq u_i$ (for $i = 1, \ldots, n$):

$$u_i \Vdash \Gamma_i \text{ (for all } i = 0, \ldots, n) \Longrightarrow u_n \Vdash \chi. \quad ■$$

Reflecting on the direct (local) definition of the logical consequence relation (Definition 11 and Proposition 12), it is obvious that:

Proposition 17. *For every set of Σ-sentences $\Phi \cup \{\chi\}$: $\Phi \models^* \chi$ iff $\Phi \models \chi$.* ■

The soundness of a calculus C is warranted whenever each proof rule of C preserves $\models^*$. In other words, let us consider the following schema of proof rule:

$$\frac{\Delta_1 \mapsto \chi_1 \quad \ldots \quad \Delta_n \mapsto \chi_n}{\Delta \mapsto \chi}$$

then, because of finiteness of sets in a C-proof and by induction on its length, it suffices that each proof rule of C (with the above scheme) satisfies:

$$\Delta_i \models^* \chi_i \text{ (for all } i = 1, \ldots, n) \Longrightarrow \Delta \models^* \chi.$$

Notice that the monotonicity of $\models$ with respect to set inclusion (Fact 13) can be generalized to $\models^*$ in the following sense:

Fact 18. If $\Gamma \subseteq \Gamma'$ and $\Delta; \Gamma; \Delta' \models^* \varphi$, then $\Delta; \Gamma'; \Delta' \models^* \varphi$. ■

However, $\models^*$ is non-monotonic with respect to sub-sequence relations, in general. For instance, $\Delta; \Gamma \models^* \varphi$ could not hold, although $\Delta \models^* \varphi$ holds.

As a first step to achieve completeness, we provide general sufficient conditions for $\vdash_C$, which are formulated on the basis of Hintikka sets (*cf.* [11]). We have to saturate sets which are involved in a sequence. Therefore, we must take into account previous sets in the sequence. For this purpose, we introduce the notion of *back-saturated set* with respect to a sequence of sets Δ. Because of technical reasons, back-saturation is also made with respect to a set of auxiliary constant symbols.

Definition 19. Let Δ be a sequence of sets of Σ-sentences and let AC be a countable set of new auxiliary constants. We say that a set Γ of $\Sigma \cup AC$-sentences is *back-saturated* with respect to (AC, Δ) if and only if it satisfies the following conditions:

1. $F \notin \Gamma$
2. if A is atomic and $A \in \Delta; \Gamma$, then $\neg A \notin \Gamma$
3. $\varphi \to \psi \in \Gamma \Longrightarrow \neg\varphi \in \Gamma$ or $\psi \in \Gamma$
4. $\varphi \supset \psi \in \Delta; \Gamma \Longrightarrow \neg\varphi \in \Gamma$ or $\psi \in \Gamma$
5. $\forall x \varphi(x) \in \Gamma \Longrightarrow \varphi(t) \in \Gamma$ for all closed $t \in Term_{\Sigma \cup AC}$
6. $\neg(\varphi \to \psi) \in \Gamma \Longrightarrow \varphi \in \Gamma$ and $\neg\psi \in \Gamma$
7. $\neg\forall x \varphi(x) \in \Gamma \Longrightarrow \neg\varphi(c) \in \Gamma$ for some $c \in AC$.

The first two conditions constitute the so-called *atomic coherence property* of Γ. ■

Now, we will show that, for calculi satisfying the conditions in Figure 1, any set of sentences, in the antecedent of a sequent, can be back-saturated preserving non-derivability. This is the crucial lemma for completeness.

Lemma 20. *Let $\Delta; \Gamma; \Delta'$ be a sequence of sets of Σ-sentences, χ a Σ-sentence, and AC be a countable set of new auxiliary constants. If $\Delta; \Gamma; \Delta' \nvdash_C \chi$ and $\vdash_C$ satisfies the conditions of Figure 1, then there exists a set Γ^* such that:*
(i) *Γ^* is back-saturated with respect to (AC, Δ),*
(ii) *$\Gamma \subseteq \Gamma^*$ and*
(iii) *$\Delta; \Gamma^*; \Delta' \nvdash_C \chi$.*

Proof: We will build Γ^*, starting with Γ, by iteration of a procedure which adds sentences. At each iteration step $k \in I\!N$ a set Γ_k of sentences is built. To do that we enumerate the set AC by $\{c_0, c_1, \ldots, c_n, \ldots\ldots\}$; the set of all closed $\Sigma \cup AC$-terms by $\{t_0, t_1, \ldots, t_n, \ldots\ldots\}$; and the set of all $\Sigma \cup AC$-sentences by $\{\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots\ldots\}$. The procedure initializes $\Gamma_0 := \Gamma$. Then, for any $k \geq 1$ it obtains Γ_k from Γ_{k-1} in the following way.
(a) If k is odd, it takes the least pair $(i, j) \in I\!N^2$ (in lexicographic order) such that $\gamma_i \in \Gamma_{k-1}$ have the form $\forall x\varphi(x)$ and $\varphi(t_j) \notin \Gamma_{k-1}$. Then, it makes $\Gamma_k := \Gamma_{k-1} \cup \{\varphi(t_j)\}$.
(b) If k is even, it takes the least $i \in I\!N$ such that γ_i has not been treated yet (in the step k) and one of the following two facts holds:

- **(b1)** $\gamma_i \in \Delta; \Gamma_{k-1}$ and it has the form $\varphi \supset \psi$
- **(b2)** $\gamma_i \in \Gamma_{k-1}$ and its form is either $\varphi \rightarrow \psi$ or $\neg(\varphi \rightarrow \psi)$ or $\neg\forall x\varphi(x)$.

Then, it obtains Γ_k, depending on the case, as follows:

- **(b1)** If $\Delta; \Gamma_{k-1}, \neg\varphi; \Delta' \not\vdash_C \chi$ then $\Gamma_k := \Gamma_{k-1} \cup \{\neg\varphi\}$ else $\Gamma_k := \Gamma_{k-1} \cup \{\psi\}$
- **(b2)** γ_i is $\varphi \rightarrow \psi$: If $\Delta; \Gamma_{k-1}, \neg\varphi; \Delta' \not\vdash_C \chi$ then $\Gamma_k := \Gamma_{k-1} \cup \{\neg\varphi\}$ else $\Gamma_k := \Gamma_{k-1} \cup \{\psi\}$
  γ_i is $\neg(\varphi \rightarrow \psi)$: $\Gamma_k := \Gamma_{k-1} \cup \{\varphi, \neg\psi\}$
  γ_i is $\neg\forall x\varphi(x)$: It takes the least j such that $c_j \in AC \setminus cons(\Delta; \Gamma_{k-1}; \Delta'; \chi)$ and then it makes $\Gamma_k := \Gamma_{k-1} \cup \{\neg\varphi(c_j)\}$.

If Γ_k is back-saturated with respect to (AC, Δ), the procedure stops at this step k for which $\Gamma^* = \Gamma_k$. Otherwise we approximate

$$\Gamma^* = \bigcup_{k \in \mathcal{N}} \Gamma_k.$$

By construction $\Gamma \subseteq \Gamma^*$. To finish the proof, we must justify that $\Delta; \Gamma^*; \Delta' \not\vdash_C \chi$ and Γ^* is back-saturated with respect to (AC, Δ).
We will show that $\Delta; \Gamma_k; \Delta' \not\vdash_C \chi$, for all $k \in I\!N$ (by induction on k). For $k = 0$ the assertion holds by hypothesis. Now, consider any $k \in I\!N$, $\Delta; \Gamma_{k-1}; \Delta' \not\vdash_C \chi$ holds by the induction hypothesis. For each possible case, in the extension of Γ_{k-1} to Γ_k, there are some conditions in Figure 1 which ensure that $\Delta; \Gamma_k; \Delta' \not\vdash_C \chi$. The case (a) works because of condition 4, (b1) because of condition 3 and the case (b2), because of conditions 2, 6, and 8.
Finally, the atomic coherence of Γ^* results from $\Delta; \Gamma^*; \Delta' \not\vdash_C \chi$ because of conditions 1, 5 and 10. The remaining back-saturation properties of Γ^* hold by construction. ■

Now, we will prove that conditions of Figure 1 are, in fact, sufficient for completeness.

Lemma 21. *If the derivability relation $\vdash_C$ of a calculus C satisfies the conditions of Figure 1, then for any set of Σ-sentences $\Phi \cup \{\chi\}$: $\Phi \models \chi \Longrightarrow \Phi \vdash_C \chi$*

Proof: Suppose that $\Phi \not\vdash_C \chi$. We will prove that $\Phi \not\models \chi$ by showing the existence of a counter-model $\mathcal{K}$ with worlds in the set $I\!N^*$ of sequences of natural numbers, ordered by $u \leq v$ if and only if u is an initial segment of v. [1] This structure $\mathcal{K}$ will be such that $\epsilon \Vdash \Phi$ and $\epsilon \nVdash \chi$. Let us consider the existence of a countable family of disjoint sets of new auxiliary constants $\langle AC_i \rangle_{i \in I\!N}$. With each sequence $s \in I\!N^*$ we associate the set $AC_{\#s}$ and the signature

$$\Sigma_s \equiv \Sigma \cup \bigcup_{n \leq \#s} AC_n.$$

[1] ϵ will denote the empty sequence, $\cdot$ is the infix function for adding a natural number to a sequence and $\#$ is the length function over sequences.

1. $A \in \Delta \Longrightarrow \Delta \vdash_C A$
2. $\Delta; \Gamma, \neg\varphi; \Delta' \vdash_C \chi$ and $\Delta; \Gamma, \psi; \Delta' \vdash_C \chi \Longrightarrow \Delta; \Gamma, \varphi \to \psi; \Delta' \vdash_C \chi$
3. $\Delta; \Gamma; \Delta'; \Gamma', \neg\varphi; \Delta'' \vdash_C \chi$ and $\Delta; \Gamma; \Delta'; \Gamma', \psi; \Delta'' \vdash_C \chi \Longrightarrow$ $\Delta; \Gamma, \varphi \supset \psi; \Delta'; \Gamma'; \Delta'' \vdash_C \chi$
4. $\Delta; \Gamma, \varphi(t); \Delta' \vdash_C \chi \Longrightarrow \Delta; \Gamma, \forall x\varphi(x); \Delta' \vdash_C \chi$
5. $\Delta; \Gamma \vdash_C A \Longrightarrow \Delta; \Gamma, \neg A; \Delta' \vdash_C \chi$
6. $\Delta; \Gamma, \varphi, \neg\psi; \Delta' \vdash_C \chi \Longrightarrow \Delta; \Gamma, \neg(\varphi \to \psi); \Delta' \vdash_C \chi$
7. $\Delta; \Gamma; \varphi \vdash_C \psi \Longrightarrow \Delta; \Gamma, \neg(\varphi \supset \psi); \Delta' \vdash_C \chi$
8. $\Delta; \Gamma, \neg\varphi(c); \Delta' \vdash_C \chi \Longrightarrow \Delta; \Gamma, \neg\forall x\varphi(x); \Delta' \vdash_C \chi$
9. $\Delta; \Gamma, \neg\varphi \vdash_C \textsc{f} \Longrightarrow \Delta; \Gamma \vdash_C \varphi$
10. $\Delta; \Gamma, \textsc{f}; \Delta' \vdash_C \chi$

where A is atomic, $t \in Term_{\Sigma \cup AC}$ and $c \in AC \setminus cons(\Delta; \Gamma, \varphi(x); \Delta')$.

Fig. 1. Sufficient conditions for completeness

Now, we inductively associate a set Γ_s of Σ-formulas with each $s = \langle n_0, n_1, \ldots, n_k\rangle \in \mathbb{N}^*$. Let Δ_s denote the sequence $\Gamma_\epsilon; \Gamma_{\langle n_0\rangle}; \Gamma_{\langle n_0,n_1\rangle}; \ldots; \Gamma_{\langle n_0,n_1,\ldots,n_k\rangle}$ and $\Delta_{<s}$ the sequence $\Gamma_\epsilon; \ldots; \Gamma_{\langle n_0,n_1,\ldots,n_{k-1}\rangle}$.
The collection $\{\Gamma_s | s \in \mathbb{N}^*\}$ is defined to satisfy that each Γ_s is back-saturated with respect to $(AC_{\#s}, \Delta_{<s})$, $\Phi \cup \{\neg\chi\} \subseteq \Gamma_\epsilon$, and $\Delta_s \not\vdash_C \textsc{f}$.
As basis step, we define Γ_ϵ as the back-saturated set with respect to AC_0 and the empty sequence of sets such that $\Phi \cup \{\neg\chi\} \subseteq \Gamma_\epsilon$ and $\Gamma_\epsilon \not\vdash_C \textsc{f}$. Γ_ϵ exists because $\Phi, \neg\chi \not\vdash_C \textsc{f}$ holds by condition 9. As inductive step, we define $\Gamma_{s\cdot j}$ for $s \in \mathbb{N}^*$ and $j \in \mathbb{N}$, provided that $\Delta_s \not\vdash_C \textsc{f}$. In order to do this, we consider an enumeration $\{\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots\ldots\}$ of all sentences of Γ_s of the form $\neg(\varphi_1 \supset \varphi_2)$. Then, let γ_j be $\neg(\varphi \supset \psi)$. By conditions 7 and 9, we have that $\Delta_{<s}; \Gamma_s; \varphi, \neg\psi \not\vdash_C \textsc{f}$. So, there exists $\Gamma_{s\cdot j} \supseteq \{\varphi, \neg\psi\}$ back-saturated with respect to $(AC_{\#s+1}, \Delta_s)$ such that $\Delta_s; \Gamma_{s\cdot j} \not\vdash_C \textsc{f}$.
Now, by means of $\{\Gamma_s \mid s \in \mathbb{N}^*\}$, we can define $\mathcal{K} = (\mathbb{N}^*, \leq, \langle \mathcal{A}_s\rangle_{s\in\mathbb{N}^*})$ as follows:

- $A_s = \{t \mid t \in Term_{\Sigma_s}$ and t is closed$\}$
- $f^{\mathcal{A}_s}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
- $At_s = \{p(\widehat{t_1}, \ldots, \widehat{t_n}) \mid p(t_1, \ldots, t_n) \in \Delta_s\}$.

It is easy to see that $\mathcal{K}$ is a Kripke structure and also that $\widehat{t}^s = t = t^s$ holds for any $s \in \mathbb{N}^*$. To finish the proof we have to check that $\epsilon \Vdash \Phi$ and $\epsilon \not\Vdash \chi$. Hence, since $\Phi \cup \{\neg\chi\} \subseteq \Gamma_\epsilon$, it suffices to show that $\eta \in \Gamma_s \implies s \Vdash \eta$ holds for any $s \in \mathbb{N}^*$ and any Σ-sentence η. We prove this fact by induction on η, using Definition 19, since each Γ_s is back-saturated with respect to $(AC_{\#s}, \Delta_{<s})$:

- $p(t_1, \ldots, t_n) \in \Gamma_s \Longrightarrow p(\widehat{t_1}, \ldots, \widehat{t_n}) \in At_s \Longrightarrow s \Vdash p(t_1, \ldots, t_n)$
- $\neg\varphi$ requires induction on φ:
  - $\neg p(t_1, \ldots, t_n) \in \Gamma_s \Longrightarrow p(t_1, \ldots, t_n) \notin \Delta_s \Longrightarrow p(\widehat{t_1}, \ldots, \widehat{t_n}) \notin At_s$ $\Longrightarrow s \not\Vdash p(t_1, \ldots, t_n) \Longrightarrow s \Vdash \neg p(t_1, \ldots, t_n)$
  - $\neg(\varphi \to \psi) \in \Gamma_s \Longrightarrow \varphi \in \Gamma_s$ or $\neg\psi \in \Gamma_s \Longrightarrow s \Vdash \varphi$ or $s \not\Vdash \psi \Longrightarrow s \Vdash \varphi \to \psi$
  - $\neg(\varphi \supset \psi) \in \Gamma_s \Longrightarrow \varphi, \neg\psi \in \Gamma_{s\cdot j}$ for some $j \in \mathbb{N} \Longrightarrow s \cdot j \Vdash \varphi$ and $s \cdot j \not\Vdash \psi$ $\Longrightarrow s \not\Vdash \varphi \supset \psi \Longrightarrow s \Vdash \neg(\varphi \supset \psi)$
  - $\neg\forall x\varphi(x) \in \Gamma_s \Longrightarrow \neg\varphi(c) \in \Gamma_s$ for some $c \in AC_{\#s} \Longrightarrow s \Vdash \neg\varphi(c)$ for some $c \in AC_{\#s} \Longrightarrow$ (by $c^s = \widehat{c}^s$ and Lemma 6) $s \Vdash \neg\varphi(\widehat{c})$ for some $c \in AC_{\#s} \Longrightarrow$ $s \Vdash \neg\forall x\varphi(x)$
- $\varphi \to \psi \in \Gamma_s \Longrightarrow \neg\varphi \in \Gamma_s$ or $\psi \in \Gamma_s \Longrightarrow s \Vdash \neg\varphi$ or $s \Vdash \psi \Longrightarrow s \Vdash \varphi \to \psi$

- $\varphi \supset \psi \in \Gamma_s \Longrightarrow \varphi \supset \psi \in \Delta_w$ for all $w \geq s \Longrightarrow \neg\varphi \in \Gamma_w$ or $\psi \in \Gamma_w$ for all $w \geq s$ $\Longrightarrow w \Vdash \neg\varphi$ or $w \Vdash \psi$ for all $w \geq s \Longrightarrow s \Vdash \varphi \supset \psi$
- $\forall x\varphi(x) \in \Gamma_s \Longrightarrow \varphi(t) \in \Gamma_s$ for all closed $t \in Term_{\Sigma_s} \Longrightarrow s \Vdash \varphi(t)$ for all closed $t \in Term_{\Sigma_s} \Longrightarrow$ (by $t^s = \widehat{t}^s$ and Lemma 6) $s \Vdash \varphi(\widehat{t})$ for all $t \in A_s \Longrightarrow s \Vdash \forall x\varphi(x)$. $\blacksquare$

The conditions of Figure 1 could be rewritten as inference rules, changing $\vdash_C$ by $\Mapsto$. By Fact 15 and Lemma 21, this is a sound and complete calculus for $\mathcal{FO}^{\supset}$, which is cut-free, but it is not very natural. In Figure 2, we give a more natural calculus, with two inference rules for introducing each connective and quantifier on the left and on the right, respectively. Notice that the differences between the rules for both implications reflect that $\supset$ can be used for any "ulterior" set in the sequence, whereas $\rightarrow$ only is valid for the set containing it. We also would like to remark that by looking at the antecedent as a set of formulas, the rules in Figure 2 become well-known inference rules in both classical and intuitionistic sequent calculi. In particular, viewed in this manner, the rules for both implications collapse into a single pair of rules.
The structural rules $(Init)$, (Abs), (Cas) and (RaA) respectively allow us to built initial sequents, to put any consequent in the place of falsehood, to reason by distinction of the two cases given by the *law of the excluded middle* (classical negation), and to make *reductio ad absurdum* reasoning. Notice also that by a combination of $(\neg L)$ and (Cas) this calculus provides a derived cut-rule (see Figure 3). Besides, it is easy to check that, using the abbreviations for the rest of connectives and quantifiers ($\wedge$, $\vee$, $\exists, \sim$), the expected inference rules can be derived for them.

Structural Rules

$(Init)$ $\Delta \Mapsto A$ if $A \in \Delta$ and A is atomic(incl. F)

$$(Abs)\ \frac{\Delta \Mapsto F}{\Delta \Mapsto \chi}$$

$$(Cas)\ \frac{\Delta; \Gamma, \varphi; \Delta' \Mapsto \chi \quad \Delta; \Gamma, \neg\varphi; \Delta' \Mapsto \chi}{\Delta; \Gamma; \Delta' \Mapsto \chi} \qquad (RaA)\ \frac{\Delta; \Gamma, \neg\varphi \Mapsto F}{\Delta; \Gamma \Mapsto \varphi}$$

Connective Rules

$$(\neg L)\ \frac{\Delta; \Gamma \Mapsto \varphi}{\Delta; \Gamma, \neg\varphi; \Delta \Mapsto \chi} \qquad (R\neg)\ \frac{\Delta; \Gamma, \varphi \Mapsto F}{\Delta; \Gamma \Mapsto \neg\varphi}$$

$$(\rightarrow L)\ \frac{\Delta; \Gamma \Mapsto \varphi \quad \Delta; \Gamma, \psi; \Delta' \Mapsto \chi}{\Delta; \Gamma, \varphi \rightarrow \psi; \Delta' \Mapsto \chi} \qquad (R \rightarrow)\ \frac{\Delta; \Gamma, \varphi \Mapsto \psi}{\Delta; \Gamma \Mapsto \varphi \rightarrow \psi}$$

$$(\supset L)\ \frac{\Delta; \Gamma; \Delta'; \Gamma' \Mapsto \varphi \quad \Delta; \Gamma; \Delta'; \Gamma', \psi; \Delta'' \Mapsto \chi}{\Delta; \Gamma, \varphi \supset \psi; \Delta'; \Gamma'; \Delta'' \Mapsto \chi} \qquad (R \supset)\ \frac{\Delta; \{\varphi\} \Mapsto \psi}{\Delta \Mapsto \varphi \supset \psi}$$

Quantifier Rules

$$(\forall L)\ \frac{\Delta; \Gamma, \varphi(t); \Delta' \Mapsto \chi}{\Delta; \Gamma, \forall x\varphi; \Delta' \Mapsto \chi} \qquad (R\forall)\ \frac{\Delta \Mapsto \varphi(c)}{\Delta \Mapsto \forall x\varphi}$$

(t is a term in $(\forall L)$, and c is a new constant which does not appear in the lower sequent of $(R\forall)$).

Fig. 2. A sound and complete calculus for $\mathcal{FO}^{\supset}$

Now, we will show that the set of inference rules in the Figure 2 constitutes a sound and complete sequent calculus for $\mathcal{FO}^{\supset}$.

Firstly, the soundness of this calculus can be proved easily but requires a long proof. The proof consists in checking that each inference rule is sound with respect to (or preserves) the relation $\models^*$ (Definition 16). As an example, we show that $(\rightarrow L)$ is sound. Let Δ be some sequence of sets $\Gamma_0; \ldots; \Gamma_n$ and let Δ' be another sequence $\Phi_0; \ldots; \Phi_m$. Suppose that there exists a Kripke structure $\mathcal{K}$ and a sequence of worlds in $W(\mathcal{K})$: $u_0 \preceq \ldots \preceq u_n \preceq w \preceq v_0 \preceq \ldots \preceq v_m$ such that:

- $u_i \Vdash \Gamma_i$ for all $i = 1, \ldots, n$
- $w \Vdash \Gamma \cup \{\varphi \rightarrow \psi\}$
- $v_j \Vdash \Phi_j$ for all $j = 1, \ldots, m$
- $v_m \nVdash \chi$

Since $w \Vdash \varphi \rightarrow \psi$, then $w \nVdash \varphi$ or $w \Vdash \psi$. The first case means that $\Delta; \Gamma \not\models^* \varphi$ and in the second case it turns out that $\Delta; \Gamma, \psi; \Delta' \not\models^* \psi$. Therefore, if $\Delta; \Gamma \models^* \varphi$ and $\Delta; \Gamma, \psi; \Delta' \models^* \chi$, then $\Delta; \Gamma, \varphi \rightarrow \psi; \Delta' \models^* \chi$.

Secondly, we will prove completeness by checking that the derivability relation induced by the calculus of Figure 2 satisfies the conditions of Figure 1. In order to do it easier, we introduce the derived inference rules of Figure 3. These rules provide cut, contraposition (in the last set of the antecedent), derivation of assumptions (of the last set of the antecedent), and contradiction respectively.

$(Cut)\ \dfrac{\Delta; \Gamma \mapsto \varphi \quad \Delta; \Gamma, \varphi; \Delta' \mapsto \chi}{\Delta; \Gamma; \Delta' \mapsto \chi}$	$(Ctp)\ \dfrac{\Delta; \Gamma, \neg\chi \mapsto \varphi}{\Delta; \Gamma, \neg\varphi \mapsto \chi}$
$(Ass)\ \Delta; \Gamma, \varphi \mapsto \varphi$	$(Ctd)\ \Delta; \Gamma, \varphi, \neg\varphi; \Delta' \mapsto \chi$

Fig. 3. Derived inference rules

(Cut) is derived by using $(\neg L)$ and (Cas), and (Ctp) by $(\neg L)$ and (RaA). (Ass) can be derived by induction on φ. In the basic case, it suffices to use $(Init)$. For the induction step it is enough to use the induction hypothesis together with the corresponding rules $(R\odot)$ and $(\odot L)$ in each case of binary connective or quantifier $\odot$. Finally, (Ctd) comes from $(\neg L)$ and (Ass).

Theorem 22. (Completeness) *The calculus of Figure 2 is complete for $\mathcal{FO}^{\supset}$.*

Proof: By Lemma 21, it is enough to check that the calculus satisfies the conditions of Figure 1. Some of the conditions are directly obtained from an inference rule. This is the case for condition 1 by $(Init)$, for 4 by $(\forall L)$, for 5 by $(\neg L)$ and for 9 by (RaA). Conditions 7 and 10 are also very easy to check; condition 7 holds by rules $(R \supset)$ and $(\neg L)$ and 10 by $(Init)$ and (Abs).
From now on, the thinning meta-rule (see Fact 15) is often implicitly assumed.
For both conditions 2 and 3 we use the same scheme of proof. We firstly apply (Cas) with φ and $\neg\varphi$. Then, the sequent with φ is a premise of the condition. For the other sequent we use (Cut) with $\neg\varphi$ as cut-formula. We obtain the other premise of the

condition and also a sequent which is easily derived by $(R\neg)$ and $(\rightarrow L)$. The resulting leaves are (Ass) and (Ctd) sequents.
A proof for condition 6 can be similarly obtained beginning with two consecutive applications of (Cut) with cut-formulas φ and $\neg\psi$ on the set Γ.
For condition 8, let us suppose that $\Delta;\Gamma,\neg\varphi(c);\Delta' \Rightarrow \chi$ is a derivable sequent. That is, we are assuming that it is composed of finite sets. With an abuse of notation, we use the same names for the finite sets in the sequent as for its (possibly infinite) extensions in the derivability condition 8. We apply finitely many times $(R \rightarrow)$ and $(R \supset)$ to put, one by one, the formulas of Δ' in the consequent. In this way we obtain a sequent of the form $\Delta;\Gamma,\neg\varphi(c) \Rightarrow \eta(\Delta',\chi)$, where $\eta(\Delta',\chi)$ is the above explained implicative formula combining both implications. Now, by (Ctp), $(R\forall)$ (c does not appear anywhere in the sequent), and again using (Ctp) we have a proof of the sequent (S_1) $\Delta;\Gamma,\neg\forall x\varphi(x) \Rightarrow \eta(\Delta',\chi)$. Further, by systematic application of $(\rightarrow L)$ and $(\supset L)$ to $\eta(\Delta',\chi)$, we prove the sequent (S_2) $\Delta;\Gamma,\neg\forall x\varphi(x),\eta(\Delta',\chi);\Delta' \Rightarrow \chi$. Applying (Cut) to (S_1) and (S_2) we derive the sequent $\Delta;\Gamma,\neg\forall x\varphi(x);\Delta' \Rightarrow \chi$. By thinning meta-rule (Fact 15) the corresponding derivability relation is obtained. ■

5 Conclusions and Related Work

We have shown that structured sequents enable the sequent calculi approach as proof method for a logic combining classical and intuitionistic implication. We have given semantical foundations for a logic which achieves such combination in the (unrestricted) first-order case. Some design aspects of the logic $\mathcal{FO}^{\supset}$ have been influenced and inspired by its original motivation in the area of logic programming. In particular, antecedents consisting of sequences of sets arose as a tool to formalize the operational semantics of LP languages combining classical and intuitionistic implication (see [10, 1]). We have improved these structured sequents to combine both whole logics. We have provided general sufficient conditions for completeness of sequent calculi dealing with this kind of sequents, and also a sound and complete calculus. The completeness proof is based on a procedure for saturating sets with respect to sequences of sets.
The well-known modal logic $S4$ (introduced in [14]) is closely related to intuitionistic logic, by a translation of intuitionistic formulas into $S4$ formulas ([3, 4]). Similarly, $\mathcal{FO}^{\supset}$ can be translated into $S4$. The $S4$ connective $\Box$ allows one to translate an intuitionistic implication $\varphi \supset \psi$ into $\Box(\varphi \rightarrow \psi)$. For atomic monotonicity, atomic formulas $p(\bar{t})$ also have to be translated into $\Box p(\bar{t})$. This transformational approach enables logical (model-theoretic) foundations for LP languages combining both implications ([9]), and also provides a useful connection between the *semantical aspects* $\mathcal{FO}^{\supset}$ and $S4$. On the contrary, this connection is not so useful for relating its *proof-theoretical aspects*. Structured sequents calculi provide a new *deduction style* allowing proofs which can not be translated to $S4$. Roughly speaking, structured sequent calculi are more flexible than traditional $S4$ proof methods. In other words, a probable sequent has more (essentially different) proofs in $\mathcal{FO}^{\supset}$ than its translation has in $S4$. For instance, consider the following $S4$-rule:

$$(R\Box)\ \frac{\Phi^{\#} \Rightarrow \chi}{\Phi \Rightarrow \Box\chi} \quad \text{where } \Phi^{\#} = \{\Box\varphi \mid \Box\varphi \in \Phi\}\ ^{2}$$

[2] $S4$ sequents are symmetric, but this is not relevant for our discussion.

The indispensable non-$\Box$-formulas in the antecedent of a sequent with a $\Box$-consequent must be used before one application of the $(R\Box)$ rule removes them. The semantical reason is that $\Box$-formulas are properties of "every greater world" and the rule removes the "one world" assumptions. In $\mathcal{FO}^{\supset}$, the structure of the antecedent allows us to preserve all its "information", even in deduction steps dealing with a "every greater world" consequent. A remarkable consequence is that $\mathcal{FO}^{\supset}$ is more suitable (than $S4$) for goal-directed proofs. A *goal-directed (or uniform) proof* ([16, 17]) essentially is a sequent calculus proof obtained by applying (at each step) the $(\odot R)$ rule, where $\odot$ is the top-level logical symbol of the consequent (or goal). When the goal is an atom A, the legal step (so-called "backchaining") essentially consists on of applying $(\to L)$ to some formula $\varphi \to A$ in the antecedent. Goal-directed proofs have become an important proof-theoretical foundation for LP languages and goal-directed proof systems have been developed for fragments of first order logic ([18]), intuitionistic logic ([7, 16]), intermediate logics lying in between intuitionistic and classical logic ([6]), and many other fragments of (higher-order, modal, etc.) logics. A more detailed discussion of this topic is outside the scope of this paper. [1] deals with a fragment of $\mathcal{FO}^{\supset}$ which is a logic programming language satisfying the existence of a goal-directed proof for every provable sequent. Let us illustrate the goal-oriented ability of $\mathcal{FO}^{\supset}$ by means of a simple example. Consider the sequent $p,\, p \to q \mapsto p \supset q$. It has a very easy goal-directed proof, whereas its translation to $S4$ modal logic, $\Box p, \Box p \to \Box q \mapsto \Box(\Box p \to \Box q)$, does not have a goal-directed proof. For the former, by $(R \supset)$, we obtain $p,\, p \to q\ ;\, p \mapsto q$. Now, by applying $(\to L)$ ("backchaining") to $p \to q$, two initial sequents $p \mapsto p$ and $p, q; p \mapsto q$ are obtained. For the latter, a goal-directed proof must firstly apply $(R\Box)$, hence it obtains the non-provable sequent $\Box p \mapsto \Box p \to \Box q$. There is a (non goal-directed) $S4$-proof which begins with the $(\to L)$ to obtain the initial sequent $\Box p \mapsto \Box p$ and also the sequent $\Box p, \Box q \mapsto \Box(\Box p \to \Box q)$ which now can be proved by $(R\Box)$.
Fariñas and Herzig ([2]) have investigated the combination of classical and intuitionistic logic, in the propositional case. They provide a Hilbert-style axiomatization and also a tableaux method. Actually, a completeness proof for the *propositional subset* of $\mathcal{FO}^{\supset}$ could be obtained by deriving the axiomatization of the propositional logic introduced in [2]. A similar work is [12] where a natural deduction system is given. A common characteristic of both systems, [2] and [12], is that some deduction steps depend on the *persistence* of formulas. From the semantical point of view, persistent formulas are the "every greater world" ones in the sense mentioned above. In $S4$ only $\Box$-formulas are persistent, whereas the combination of classical and intuitionistic connectives leads to an inductive characterization of persistence. In [2, 12] persistent goals require the elimination of non-persistent premises, like the $(R\Box)$-rule does in $S4$. Hence their deductive styles are similar and quite far from the (structured sequents based) $\mathcal{FO}^{\supset}$ style, especially with regard to goal-directed proofs. Let us consider again the above example and the tableaux method introduced in [2], in order to obtain a closed tableau for the sentence $\neg((p \wedge (p \to q)) \to (p \supset q))$. After some simple steps, we have a linear tableau containing the three nodes: p, $p \to q$, and $\neg(p \supset q)$. From the goal-directed point of view, the first two play the role of premises and the last one is the goal. Therefore, to build a "goal-directed tableau" we must enlarge the unique branch with p and $\neg q$ and, simultaneously, we must eliminate all premises, since they are not persistent. Hence, this tableau will not close. A closed tableau could be obtained and to do this it suffices to use the premise $p \to q$ before the goal.

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