THE JOURNAL OF SYMBOLIC LOGIC
Volume 62, Number 2, June 1997

INFINITARY LOGICS AND VERY SPARSE RANDOM GRAPHS

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Abstract. Let $L^{\omega}_{\infty\omega}$ be the infinitary language obtained from the first-order language of graphs by closure under conjunctions and disjunctions of arbitrary sets of formulas, provided only finitely many distinct variables occur among the formulas. Let $p(n)$ be the edge probability of the random graph on n vertices. It is shown that if $p(n) \ll n^{-1}$ satisfies certain simple conditions on its growth rate, then for every $\sigma \in L^{\omega}_{\infty\omega}$, the probability that σ holds for the random graph on n vertices converges. In fact, if $p(n) = n^{-\alpha}$, $\alpha > 1$, then the probability is either smaller than 2^{-n^d} for some $d > 0$, or it is asymptotic to cn^{-d} for some $c > 0$, $d \geq 0$. Results on the difficulty of computing the asymptotic probability are given.

§1. Introduction. Two of the main subjects of finite model theory are descriptive complexity and convergence laws. The objective of the first area is to characterize resource bounded computations by sentences in formal languages. The second area is concerned with the asymptotic probabilities of properties expressible by sentences in formal languages. By now, both of these areas are rather well-developed, although many open problems still remain.

One question that many computer scientists have about finite model theory, in particular the study of convergence laws, is its relevance to their discipline. It has been suggested that convergence laws have potential applications to computational complexity (via descriptive complexity), data base theory, and algorithm analysis. As yet, these connections are tenuous. On a more fundamental level, it can be argued that finite model theory has the same relationship to computer science that group theory has to physics. Even if the specific theorems of group theory rarely have applications to physics, the experience gained from the more abstract subject can be useful in studying physical phenomena. An eloquent exposition of these issues was written by Gurevich [10].

The limited applicability of convergence laws is due to two factors: the languages that they apply to are not powerful enough to express general computational processes, and the probability distributions on the classes of finite models are quite simple. Much effort in finite model theory has been involved with finding convergence laws for more expressive languages and broader classes of probability distributions. The results in this article are of that kind. They also include a new type of convergence law which may have applications to computer science.

The class of random structures we will investigate is the well-known class of random graphs introduced by Erdős and Rényi [6]. For any natural number n, we say $G = \langle V, E \rangle$ is a *graph on n* if $V = \{1, \dots, n\}$ and $E \subseteq \{\{x, y\} : x, y \in V$

Received December 21, 1993; revised November 1, 1994.
Research supported by NSF Grant CCR-9006303.

0022-4812/97/6202-0015/$2.50

and $x \neq y\}$. The function $p(n)$ will always represent a probability function, i.e., $0 \leq p(n) \leq 1$ for all n. The *random graph on n* is the graph on n whose edges are chosen independently with probability $p(n)$. That is, each pair $\{x, y\} \subseteq V$ is in E with probability $p(n)$. A property $\mathscr{P}$ is a set of graphs. We put $\mathrm{pr}(G \in \mathscr{P}, n)$ or simply $\mathrm{pr}(\mathscr{P}, n)$ for the probability that the random graph on n has property $\mathscr{P}$.

A widely studied class of edge probabilities is $p(n)$ of the form $\beta n^{-\alpha}$, where $\alpha, \beta \geq 0$. The probabilities of many interesting graph properties, including those defined by various logics, are strongly affected by α. On the other hand, the theorems of this article do not depend on β, as long as it is greater than 0. Thus we will take it to be 1.

The language we shall use to express properties is the infinitary language $L^{\omega}_{\infty\omega}$. This is an extension of the first-order language of graphs. It has the predicates $=$, interpreted as equality, and the binary relation symbol E, interpreted as the edge relation on a graph. All of the usual constructs of first-order logic are available, but in addition, conjunctions and disjunctions of arbitrary sets of formulas are allowed, provided only finitely many distinct variables occur among the formulas. That is, if Ψ is a set of formulas in $L^{\omega}_{\infty\omega}$, and there are only finitely many distinct variables among the formulas in Ψ, then $\vee\Psi$ and $\wedge\Psi$ are formulas in $L^{\omega}_{\infty\omega}$. For natural numbers k, $L^{k}_{\infty\omega}$ consists of those formulas in $L^{\omega}_{\infty\omega}$ with at most k distinct variables.

This language is of interest to descriptive complexity because it can capture certain computational problems that first-order logic cannot. For instance, it can express the property that a graph is connected. It is more expressive than partial-fixpoint logic (or, equivalently on finite structures, iterative logic, i.e., first-order logic augmented with **while** looping [3]).

The model theory of random finite structures was initiated independently by Fagin [7] and Glebskiĭ et. al. [9]. They proved a 0-1 law for first-order logic when the edge probability is constant. That is, if $p(n) = p$ for all n, then for every first-order sentence σ,

$$\lim_{n \to \infty} \mathrm{pr}(\sigma, n) = 0 \text{ or } 1.$$

In fact, their result holds for finite models of arbitrary relational type, where the tuples in each relation are chosen independently with constant probability. When variable edge probabilities are considered, the situation is not so simple. Shelah and Spencer [18] showed that a 0-1 law holds for first-order logic when $p(n) = n^{-\alpha}$ and α is irrational, $p(n) \ll n^{-2}$, or $n^{-k/(k-1)} \ll p(n) \ll n^{-(k+1)/k}$ for some integer $k \geq 2$; but for $p(n) = n^{-\alpha}$ where $\alpha \in (0, 1)$ is rational, there are first-order sentences σ for which $\mathrm{pr}(\sigma, n)$ has no limit as $n \to \infty$. Lynch [14] proved that convergence laws hold for the remaining cases $p(n) = n^{-k/(k-1)}$ and $p(n) = n^{-1}$. That is, $\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$ exists for every first-order σ; however the limit need not be 0 or 1.

Kolaitis and Vardi [12] extended the first-order 0-1 law for constant edge probability to $L^{\omega}_{\infty\omega}$. They suggested investigating whether this logic has convergence laws for any variable edge probabilities.

This article shows that the convergence laws for very sparse random graphs, i.e., $p(n) \ll n^{-1}$, extend to $L^{\omega}_{\infty\omega}$. Specifically, if $n^{-k/(k-1)} \ll p(n) \ll n^{-(k+1)/k}$, then the 0-1 law still holds. This is an easy consequence of results by Kolaitis [11], Shelah

	First-Order Logic	$L^{\omega}_{\infty\omega}$
$p(n) \ll n^{-2}$	0-1 Law (trivial)	0-1 Law (trivial)
$n^{-k/(k-1)} \ll p(n) \ll n^{-(k+1)/k}$	0-1 Law (Shelah & Spencer [18])	0-1 Law (this article)
$p(n) = n^{-k/(k-1)}$	Convergence Law (Lynch [14])	Convergence Law (this article)
$p(n) = n^{-1}$	Convergence Law (Lynch [14])	Nonconvergence for TCL (Tyszkiewicz [19])
$p(n) = n^{-\alpha}$, $\alpha < 1$ irrational	0-1 Law (Shelah & Spencer [18])	0-1 Law for TCL, Nonconvergence for $L^{\omega}_{\infty\omega}$ (Lynch, McArthur, & Spencer, in preparation)
$p(n) = n^{-\alpha}$, $\alpha < 1$ rational	Nonconvergence (Shelah & Spencer [18])	Nonconvergence (a fortiori)
$p(n)$ constant	0-1 Law (Fagin [7], Glebskiĭ et. al. [9])	0-1 Law (Kolaitis & Vardi [12])

TABLE 1. Summary of convergence laws for first-order logic and $L^{\omega}_{\infty\omega}$.

and Spencer [18], and Ruciński and Vince [17]. We also show that the convergence law still holds for $L^{\omega}_{\infty\omega}$ when $p(n) = n^{-k/(k-1)}$. In fact, we prove a stronger result for all $p(n) = n^{-\alpha}$, $\alpha > 1$: either

$$\begin{aligned} \mathrm{pr}(\sigma, n) &< 2^{-n^d} \text{ for some } d > 0, \text{ or} \\ \mathrm{pr}(\sigma, n) &\sim cn^{-d} \text{ for some } c > 0, d \geq 0. \end{aligned}$$

If we take $f(n) \sim n^{-\infty}$ to mean $f(n) < n^{-d}$ for all $d > 0$, then the conclusion implies $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ for some $c \in (0, \infty)$, $d \in [0, \infty]$. This is known as a *power law* in physics and engineering. The 0-1 laws follow as a corollary because we show that if $d = 0$, then $c = 1$. The power law also applies to constant p: either

$$\begin{aligned} \mathrm{pr}(\sigma, n) &< 2^{-n^d} \text{ for some } d > 0, \text{ or} \\ \mathrm{pr}(\sigma, n) &\sim 1. \end{aligned}$$

Tyszkiewicz [19] has shown that there are sentences in the transitive closure logic (TCL) of graphs (and therefore in $L^{\omega}_{\infty\omega}$) whose probability does not converge when $p(n) = n^{-1}$. At present, the case when $p(n) = n^{-\alpha}$, $\alpha < 1$ irrational, is unpublished, but a manuscript in preparation by Lynch, McArthur, and Spencer shows that the 0-1 law holds for TCL but convergence fails for $L^{\omega}_{\infty\omega}$. Table 1 summarizes all these results. Note that the only case where the first-order convergence laws do not carry over to TCL is $p(n) = n^{-1}$.

A possible use of power laws is for estimating the frequency of critical events. Examples are the occurrence of faults in software systems, the satisfaction of database queries, termination of an algorithm, and deadlock in multiprocessing systems [5, 16]. In all these examples, the event can be described by some sentence σ. In many cases, the expected time until σ occurs is approximated by the inverse of $\mathrm{pr}(\sigma, n)$. However, critical events often have probability asymptotic to 0, and convergence laws do not yield any more information than that. But power laws can still provide estimates of the frequency of σ. The expected time until σ occurs is superpolynomial in n if $\mathrm{pr}(\sigma, n) \sim n^{-\infty}$, but it is approximated by a polynomial in n if $\mathrm{pr}(\sigma, n) \sim cn^{-d}$, $d < \infty$.

Related to these considerations is the computation of conditional probabilities. Let σ and τ be sentences in a language that has a convergence law. The conditional probability of σ given τ, $\mathrm{pr}(\sigma|\tau, n)$ is the ratio $\mathrm{pr}(\sigma \wedge \tau, n)/\mathrm{pr}(\tau, n)$. When $\lim_{n\to\infty} \mathrm{pr}(\tau, n) > 0$, the conditional probability of σ given τ also satisfies a convergence law. It is said that the class $\mathscr{C}$ of structures satisfying τ *inherits* the convergence law. When $\lim_{n\to\infty} \mathrm{pr}(\tau, n) = 0$, no conclusion can be drawn. But if the language has a power law and $\mathrm{pr}(\tau, n) \sim cn^{-d}$, $d < \infty$, then $\mathscr{C}$ inherits the power law.

Our results lend theoretical support to the growing awareness of fundamental difficulties in software assessment. For example Butler and Finelli [2] claim that there is no practical way to test software for high reliability. Even in our simple model of computation, there are events with nonzero probability that are difficult to detect by the usual methods of software testing. Furthermore, there is no practical means for determining which kind of event σ characterizes: one whose frequency is superpolynomial in n, or one whose frequency is polynomial. We will show that there is no effective procedure for determining which case holds, given arbitrary $\sigma \in L^{\omega}_{\infty\omega}$. On the other hand, the techniques used in our proofs may be useful in estimating $\mathrm{pr}(\sigma, n)$ in special cases, or when additional information is known. For example, we will show that it is decidable whether $\mathrm{pr}(\sigma, n) = o(n^{-d})$ or $\mathrm{pr}(\sigma, n) = \Omega(n^{-d})$, given σ and d.

This is certainly an oversimplification of the difficulties involved in software assessment, just as the complexity class P is a crude characterization of feasible computation. In particular, it is usually unclear what the probability distribution of the inputs is, and it is not likely to be precisely of the forms studied here. We do not expect that the results of finite model theory apply directly to software assessment, but we hope that the ideas and techniques will lead to further results that will be useful.

Another area that should be explored is general relational structures, i.e., finite structures that have several random relations of arbitrary degree. Characterizations of those types of structures and probability distributions that have convergence or power laws would be interesting.

§2. Convergence laws. We first describe the logical and combinatorial notions that will be used in the proofs. Let σ have k distinct variables. A *k-class* is a set of all graphs that satisfy the same sentences in $L^{k}_{\infty\omega}$. We put $G \equiv_k H$ for two graphs G and H if they belong to the same k-class. As shown by Kolaitis [11], the truth of σ on any graph depends only on how many (up to k) components G has in each k-class. Specifically, letting $C \sqsubseteq G$ mean that C is a component of G,

THEOREM 2.1 (Kolaitis). *Let G_0 and G_1 be two graphs such that for every connected graph C, either*

$$|\{C_0 \sqsubseteq G_0 : C_0 \equiv_k C\}|, |\{C_1 \sqsubseteq G_1 : C_1 \equiv_k C\}| \geq k$$

or

$$|\{C_0 \sqsubseteq G_0 : C_0 \equiv_k C\}| = |\{C_1 \sqsubseteq G_1 : C_1 \equiv_k C\}|.$$

Then $G_0 \equiv_k G_1$.

We will actually use a weaker version of this result. Let $G \cong H$ mean that the graphs G and H are isomorphic.

COROLLARY 2.2. *Let G_0 and G_1 be two graphs such that for every connected graph C, either*

$$|\{C_0 \sqsubseteq G_0 : C_0 \cong C\}|, |\{C_1 \sqsubseteq G_1 : C_1 \cong C\}| \geq k$$

or

$$|\{C_0 \sqsubseteq G_0 : C_0 \cong C\}| = |\{C_1 \sqsubseteq G_1 : C_1 \cong C\}|.$$

Then $G_0 \equiv_k G_1$.

We will use $G_0 \cong_k G_1$ to mean the condition in the Corollary is satisfied by G_0 and G_1. Thus $\cong_k$ is a refinement of $\equiv_k$.

We will categorize connected graphs in three ways:

(I) Trees with v vertices, $v < \alpha/(\alpha - 1)$.
(II) Trees with v vertices, $v = \alpha/(\alpha - 1)$.
(III) Connected graphs with v vertices and e edges, $v - \alpha e < 0$.

To see that these three classes partition the set of connected graphs, consider any connected graph that is not of type (I) or (II). If it is not a tree then $v \leq e$, which implies $v - \alpha e < 0$ since $\alpha > 1$. If it is a tree, then $v > \alpha/(\alpha - 1)$ and $e = v - 1$, again implying $v - \alpha e < 0$.

These three types are special cases of notions introduced by Shelah and Spencer [18] in their proof of the 0-1 law for first-order logic and $p(n) = n^{-\alpha}$, α irrational. A rooted graph is a pair (R, G), where $G = \langle V, E\rangle$ is a graph, and $R \subseteq V$. Let $\alpha > 0$ be fixed. (R, G) is sparse if $|V - R| > \alpha|E - E \restriction R|$, and it is dense if $|V - R| < \alpha|E - E \restriction R|$. If α is irrational, then every rooted graph is either sparse or dense. The rooted graph is safe if $(R, \langle S, E \restriction S\rangle)$ is sparse for every set S such that $R \subset S \subseteq V$. Thus the type (I) graphs are the only possible connected graphs G such that $(\emptyset, G)$ is safe for $\alpha > 1$, and the type (III) graphs are all the connected graphs G such that $(\emptyset, G)$ is dense. The type (II) graphs are the connected graphs G such that $(\emptyset, G)$ is neither sparse nor dense, and when $\alpha > 1$, they exist only for $\alpha = k/(k - 1)$.

Our first result is a straightforward application of these notions. It extends a 0-1 law for first-order logic due to Shelah and Spencer [18], to $L^{\omega}_{\infty\omega}$.

THEOREM 2.3. *Let $p(n) \ll n^{-2}$ or $n^{-k/(k-1)} \ll p(n) \ll n^{-(k+1)/k}$ for some integer $k \geq 2$. Then for every sentence $\sigma \in L^{\omega}_{\infty\omega}$,*

$$\lim_{n\to\infty} \mathrm{pr}(\sigma, n) = 0 \text{ or } 1.$$

PROOF. In the first case, almost all graphs have no edges. In the second case, almost all graphs have at least k copies of each tree with at most k vertices, and no other components. In particular, for $p(n) = n^{-\alpha}$ where $(k+1)/k < \alpha < k/(k-1)$, almost all graphs will have k components isomorphic to each connected graph of type (I), and no other components. These facts are direct consequences of more general theorems due to Ruciński and Vince [17]. The theorem follows immediately from Corollary 2.2. ⊣

THEOREM 2.4. *Let $p(n) = n^{-\alpha}$ where $\alpha > 1$. Then for every sentence $\sigma \in L^{\omega}_{\infty\omega}$, either*

$$\text{(1)} \qquad \mathrm{pr}(\sigma, n) < 2^{-n^d} \text{ for some } d > 0, \text{ or}$$

$$\text{(2)} \qquad \mathrm{pr}(\sigma, n) \sim cn^{-d} \text{ for some } c > 0,\ d \geq 0.$$

There are three cases to consider:

Case 1. For every graph G such that $G \models \sigma$, there is some tree T of type (I) such that G has less than k components isomorphic to T.

Case 2. Not Case 1, and $\alpha/(\alpha - 1)$ is not an integer.

Case 3. Not Case 1, and $\alpha/(\alpha - 1)$ is an integer.

The proof depends on estimates of the rates at which the existence of safe graphs and nonexistence of dense graphs approach probability 1. Lemma 2.7 shows that there will be arbitrarily many components of every type (I) graph, with probability approaching 1 exponentially fast. Lemma 2.9 shows that there will be a dense component with probability approaching 0 polynomially fast. Further, as shown in Lemma 2.11, the existence of components of type (II) has a Poisson distribution. Thus Formula (1) of Theorem 2.4 holds in Case 1, and Formula (2) holds in Cases 2 and 3. Also, in Case 2, if $d = 0$ it will be seen that $c = 1$, i.e., the 0-1 law holds.

Some general combinatorial results will be used. Let F and I be finite sets where a probability measure pr is defined on F. For every $i \in I$, let Q_i be a collection of properties of members of F, say the elements of Q_i are $P_{i0}, P_{i1}, \ldots$ where each $P_{ia} \subseteq F$.

Take any family of sets $\vec{S} = \{S_i : i \in I\}$ such that each $S_i \subseteq Q_i$, i.e., it is a set of properties. Let

$$E^{\geq}(\vec{S}) = \bigcap_{i \in I} \left(\bigcap_{P_{ia} \in S_i} P_{ia} \right)$$

$$E^{=}(\vec{S}) = E^{\geq}(\vec{S}) - \bigcup_{i \in I} \left(\bigcup_{P_{ia} \in Q_i - S_i} P_{ia} \right).$$

That is, $E^{\geq}(\vec{S})$ is the set of elements in F that have all the properties in each S_i, and $E^{=}(\vec{S})$ is the set of elements in F that have exactly those properties in each S_i. Let $\vec{s} = \langle s_i : i \in I \rangle$ be a sequence of nonnegative integers. Let $L(\vec{s}) = \sum_{\vec{S}} \mathrm{pr}(E^{\geq}(\vec{S}))$ where the sum is taken over all $\vec{S}$ such that $|S_i| = s_i$ for all $i \in I$. For $J \subseteq I$ let $M(J, \vec{s}) = \bigcup_{\vec{S}} E^{=}(\vec{S})$ where the union is taken over all $\vec{S}$ such that $|S_i| = s_i$ for $i \in J$ and $|S_i| \geq s_i$ for $i \in I - J$. Thus $M(J, \vec{s})$ is the set of elements in F with exactly s_i properties in Q_i for $i \in J$ and at least s_i properties in Q_i for $i \in I - J$.

The following two lemmas are generalizations of the inclusion-exclusion principle and Bonferroni's inequalities. Proofs for a specific distribution pr may be found in Lynch [13], but they generalize easily to arbitrary distributions. We put $\sum(\vec{s})$ for $\sum_{i \in I} s_i$, $\vec{t} \geq \vec{s}$ if $t_i \geq s_i$ for all $i \in I$, and we use the convention that $\binom{a}{b} = 0$ for natural numbers $a < b$.

LEMMA 2.5. *If $s_i > 0$ for all $i \in I - J$, then*

$$\mathrm{pr}(M(J,\vec{s})) = \sum_{\vec{t}} (-1)^{\Sigma(\vec{t})-\Sigma(\vec{s})} \prod_{i \in J} \binom{t_i}{s_i} \times \prod_{i \in I-J} \binom{t_i - 1}{s_i - 1} \times L(\vec{t}).$$

LEMMA 2.6. *If $s_i > 0$ for all $i \in I - J$, then for every integer r,*

$$\sum_{\Sigma(\vec{t}) \geq r} (-1)^{\Sigma(\vec{t})-r} \prod_{i \in J} \binom{t_i}{s_i} \times \prod_{i \in I-J} \binom{t_i - 1}{s_i - 1} \times L(\vec{t}) \geq 0.$$

Only Case 3 needs the full strength of these lemmas, but for the sake of uniformity we will use them for all three cases. Also, understanding how they are applied in the simpler cases should help in following the argument in the third case.

The proof for Case 1 follows from the next lemma.

LEMMA 2.7. *Let T be a tree of type* (I) *and h be any integer. Then for some $d > 0$*

$$\mathrm{pr}(G \text{ has at most } h \text{ components isomorphic to } T, n) < 2^{-n^d}.$$

PROOF. Let $\beta = \alpha(v-1)/v$. Then $\beta < 1$ since $v < \alpha/(\alpha-1)$, and $\beta v - \alpha(v-1) = 0$.

Consider a fixed natural number n. Take a maximal collection $\{V_a : 1 \leq a \leq A\}$ of disjoint subsets of $\{1, \dots, n\}$, each of size $m = \lceil n^\beta \rceil$. Then $A \sim n^{1-\beta}$. We will show that the probability that only h of these subsets contain vertices that induce a component isomorphic to T is less than 2^{-n^d}, for some suitable $d > 0$.

Consider any $a = 1, \dots, A$. We now apply Lemmas 2.5 and 2.6. There is only one collection of properties, i.e., $I = \{1\}$, so we will drop all the subscripts i. Let $Q = \{P_X\}$ where X ranges over all sets of v vertices in V_a and P_X is the set of graphs on n such that X induces a subgraph isomorphic to T, i.e., $T \cong \langle X, E \restriction X \rangle$. Then $E^=(\emptyset)$ is the set of graphs on n that have no subgraph in V_a isomorphic to T, and $M(I,0) = E^=(\emptyset)$.

Taking $r = 3$ in Lemma 2.6,

$$\mathrm{pr}(M(I,0),n) \leq L(0) - L(1) + L(2).$$

Let γ_X be the probability that $X \subseteq V_a$ induces a subgraph isomorphic to T, and for distinct $X, Y \subseteq V_a$, let δ_{XY} be the probability that both X and Y induce such a subgraph. Then, letting X and Y range over all distinct sets of v vertices in V_a,

$$\begin{aligned} L(0) &= 1, \\ L(1) &= \textstyle\sum_X \gamma_X, \text{ and} \\ L(2) &= \textstyle\sum_{X,Y} \delta_{XY}. \end{aligned}$$

It is easily seen that γ_X is the same for all X. Therefore

$$L(1) = \tfrac{m^{\underline{v}}}{v!}\gamma_X$$

for any fixed X. (We are using the falling factorial power notation $r^{\underline{i}} = r(r-1)\dots(r-i+1)$.) Also, by independence

$$\gamma_X = \zeta v! \times n^{-\alpha(v-1)} \times (1 - n^{-\alpha})^{\binom{v}{2}-v+1}$$

where $\zeta = 1/|\mathrm{Aut}(T)|$, the reciprocal of the number of automorphisms on T. Since $m \sim n^\beta$ and $\beta v - \alpha(v-1) = 0$, $\lim_{n\to\infty} m^{\underline{v}} n^{-\alpha(v-1)} = 1$, and $\lim_{n\to\infty} L(1) = \zeta$.

To estimate $L(2)$, first consider the case when $X \cap Y = \emptyset$. The contribution to $L(2)$ from all such X and Y is

$$\frac{m^{\underline{2v}}}{2(v!)^2}\delta_{XY}$$

for any fixed disjoint X and Y. Also, $\delta_{XY} = (\gamma_X)^2$, so the contribution of disjoint X and Y is asymptotic to $\zeta^2/2$. Now consider the contribution of those X and Y whose intersection is nonempty. If both X and Y induce subgraphs isomorphic to T, then $X \cup Y$ induces some connected graph U on $u > v$ vertices. Thus the contribution to $L(2)$ of all X and Y such that $X \cup Y$ induces a subgraph isomorphic to U is bounded above by

$$\begin{aligned} m^{\underline{u}} \times n^{-\alpha(u-1)} &\sim n^{(\beta-\alpha)u+\alpha} \\ &= n^{(-\alpha/v)u+\alpha} \\ &= o(1). \end{aligned}$$

There are only finitely many possibilities for U, so $L(2) = \zeta^2/2 + o(1)$. Therefore

$$\mathrm{pr}(M(I,0),n) \leq 1 - \zeta + \zeta^2/2 + \varepsilon$$

for any $\varepsilon > 0$ and sufficiently large n, and since $0 < \zeta \leq 1$,

$$1 - \zeta + \zeta^2/2 < 1.$$

That is, there exists $c < 1$ such that $\mathrm{pr}(M(I,0),n) \leq c$ for all $a = 1,\dots,A$ when n is sufficiently large.

Using a simple version of Chernoff bound [4], the probability that fewer than $(1-c)n^{1-\beta}/2$ of the V_a's have a subgraph isomorphic to T is bounded above by $\mathrm{e}^{-(1-c)n^{1-\beta}/8}$, where $\mathrm{e} = \lim_{n\to\infty}(1+1/n)^n$. Thus, assuming there is a set W of size $(1-c)n^{1-\beta}/2$ consisting of disjoint sets $X \subseteq \{1,\dots,n\}$ such that X induces a subgraph isomorphic to T, let K be the graph whose vertex set is W, and such that there is an edge between $X, Y \in W$ if and only if there exist $x \in X$ and $y \in Y$ such that $\{x,y\}$ is an edge of G. Then, using Lemmas 2.5 and 2.6 in the same way as before, K is a random graph with edge probability $q = \Theta(n^{-\alpha})$.

Again by a Chernoff bound, the probability that K has more than

$$2\binom{|W|}{2}q = \Theta(n^{2-2\beta-\alpha}) \ll |W|$$

edges is bounded above by $(\mathrm{e}/4)^{\binom{|W|}{2}q} \leq 2^{-n^d}$ for some $d > 0$. Therefore with probability $\geq 1-2^{-n^d}$, K has $(1-c)n^{1-\beta}/4$ isolated vertices, i.e., G has $(1-c)n^{1-\beta}/4$ disjoint sets that induce copies of T and have no edges between them. Let Z be this collection of sets.

For every $X \in Z$, the probability r that X is a component is $(1-n^{-\alpha})^{v(n-|Z|v)}$, and these are independent events. Since $\alpha > 1$, $r \geq 1/2$ for sufficiently large n. One more application of a Chernoff bound shows that the probability that there are fewer than $|Z|/4$ sets in Z that are components is $\mathrm{e}^{-|Z|/16} \leq 2^{-n^d}$ for some $d > 0$, and this completes the proof. ⊣

Case 1 of Theorem 2.4 follows by noting that there are only finitely many $v < \alpha/(\alpha - 1)$ and only finitely many trees with a given number of vertices.

We now prove Case 2. Although there are infinitely many isomorphism types among the connected graphs, the next lemma will enable us to ignore all but finitely many of them in the asymptotic analysis of $\mathrm{pr}(\sigma, n)$.

LEMMA 2.8. *For every d there is a V such that*

$$\mathrm{pr}(G \textit{ has a component with more than } V \textit{ vertices}, n) = o(n^{-d}).$$

PROOF. Take $V > (\alpha + d)/(\alpha - 1)$. Then, letting W be the number of trees on V vertices,

$$\begin{aligned} \mathrm{pr}(G \text{ has a component with more than } V \text{ vertices}, n) &\leq Wn^{V-\alpha(V-1)} \\ &= o(n^{-d}). \end{aligned} \qquad \dashv$$

Let $\{C_i : i \in I\}$ be a list of connected graphs of type (III). Say each C_i has v_i vertices and e_i edges. For $i \in I$ let $Q_i = \{P_{iX}\}$ where X ranges over all subsets of $\{1, \dots, n\}$ with v_i vertices, and P_{iX} is the collection of graphs on $\{1, \dots, n\}$ where X induces a component isomorphic to C_i. Let $\vec{s} = \langle s_i : i \in I \rangle$ be a sequence of nonnegative integers and $J \subseteq I$. Then $M(J, \vec{s})$ is the set of graphs with exactly s_i components isomorphic to C_i for $i \in J$ and at least s_i components isomorphic to C_i for $i \in I - J$.

LEMMA 2.9. *We have*

$$\mathrm{pr}(M(J, \vec{s}), n) \sim c \prod_{i \in I} n^{(v_i - \alpha e_i)s_i}$$

for some $c > 0$.

PROOF. For $i \in I$ let $\vec{\xi_i}$ be the vector of $|I|$ zeros except $\xi_i = 1$. Then

$$L(\vec{s}) - \sum_{i \in J}(s_i + 1)L(\vec{s} + \vec{\xi_i}) - \sum_{i \in I - J} s_i L(\vec{s} + \vec{\xi_i}) \leq \mathrm{pr}(M(J, \vec{s}), n) \leq L(\vec{s}).$$

The left side follows from Lemma 2.6 when $r = 2$ and the right side when $r = 1$. Let $a = \sum(\vec{s})$ and $b = \sum_{i \in I} s_i v_i$. Partition $\{1, \dots, b\}$ into a sets $V_1, \dots, V_a$, where for each $i \in I$, exactly s_i of the sets have cardinality v_i. Then

$$L(\vec{s}) = n^{\underline{b}} \times \varepsilon \times \prod_{i \in I} \frac{(\zeta_i n^{-\alpha e_i})^{s_i}}{s_i!}$$

where ε is the probability that there are no edges between $\{1, \dots, n\} - V_j$ and V_j, for any $j = 1, \dots, a$, and each $\zeta_i = 1/|\mathrm{Aut}(C_i)|$. Now $\varepsilon = (1 - n^{-\alpha})^{(n-b)b+d}$, where $d = \sum_{1 \leq i < j \leq a} |V_i||V_j|$. Since $\alpha > 1$, $(n - b)b + d = o(n^\alpha)$ and $\lim_{n \to \infty} \varepsilon = 1$. Therefore

$$L(\vec{s}) \sim c \prod_{i \in I} n^{(v_i - \alpha e_i)s_i}$$

for some $c > 0$. Further, for each $i \in I$,

$$\begin{aligned} L(\vec{s}+\vec{\xi_i}) &\sim L(\vec{s}) \times \frac{\zeta_i n^{v_i - \alpha e_i}}{(s_i+1)!} \\ &= o(L(\vec{s})) \text{ since } v_i - \alpha e_i < 0. \end{aligned}$$

The Lemma then follows. ⊣

Lemma 2.10. *Assume $\alpha/(\alpha-1)$ is not an integer. Take any graph H such that H has at least k components isomorphic to T, for every T of type* (I). *Then*

$$\mathrm{pr}(G \cong_k H, n) \sim cn^{-d}$$

for some $c > 0$ and $d \geq 0$. In fact, if $d = 0$, then $c = 1$.

Proof. Let $\{C_i : i \in I\}$ be the list of components of H of type (III). If H has no such component, this is just $\emptyset$. For $i \in I$ let $s_i = \min(k, |\{D \sqsubseteq H : D \cong C_i\}|)$, and put $J = \{i \in I : s_i < k\}$. By Lemma 2.9, $\mathrm{pr}(M(J, \vec{s}), n) \sim cn^{-d}$ where $-d = \sum_{i \in I}(v_i - \alpha e_i)s_i$.

Now the condition that $G \in M(J, \vec{s})$ is not the same as $G \cong_k H$ since G could have less than k components isomorphic to T, for some T of type (I), or some type (III) component $D \notin \{C_i : i \in I\}$. However, the first possibility has probability $o(n^{-d})$ by Lemma 2.7, and we will show that the probability that G has such a component D is $o(n^{-d})$. The Lemma will then follow from Corollary 2.2. First, by Lemma 2.8, we may assume that D has $v \leq V$ vertices and e edges where $v - \alpha e < 0$. By Lemma 2.9, the probability that $G \in M(J, \vec{s})$ and also has a component isomorphic to D is $O(n^{-d+v-\alpha e}) = o(n^{-d})$. Since there are only finitely many graphs with at most V vertices,

$$\mathrm{pr}(G \cong_k H, n) \sim \mathrm{pr}(M(J, \vec{s}), n)$$

and we are done. Also, it can be seen that $d = 0$ if and only if $I = \emptyset$, and in that case $\mathrm{pr}(M(\emptyset, \emptyset), n) \sim 1$. ⊣

We can now complete the proof of Case 2. Since Case 1 does not hold, there exists some graph H with at least k components isomorphic to T, for every T of type (I), such that $H \models \sigma$. By Lemma 2.10, $\mathrm{pr}(G \cong_k H, n) \sim cn^{-d}$ for some $c > 0$ and $d \geq 0$, and therefore $\mathrm{pr}(\sigma, n) = \Omega(n^{-d})$.

Consider any other $\cong_k$ class whose members have at least k components isomorphic to T, for every T of type (I). Again by Lemma 2.10, each such class has probability asymptotic to $c'n^{-d'}$ for some $c' > 0$ and $d' \geq 0$. By Lemma 2.8, there are only finitely many classes whose probability is not $o(n^{-d})$, i.e., $d' \leq d$. Thus we may as well assume that d is minimal among all such d'. Let $H_1, \ldots, H_m$ be representatives of those $\cong_k$ classes for which $d' = d$. Then by Corollary 2.2,

$$\begin{aligned} \mathrm{pr}(\sigma, n) &\sim \textstyle\sum_{i=1}^m \mathrm{pr}(G \cong_k H_i, n) \\ &\sim cn^{-d} \end{aligned}$$

for some $c > 0$.

Finally, the third case is based on a lemma whose statement is similar to that of Lemma 2.10, but whose proof is considerably more complicated. Let $\{C_i : i \in I\}$ enumerate all connected graphs of type (II) (there are only finitely many of them).

For $i \in I$ let Q_i be as defined in Case 2, but now for the new set $\{C_i : i \in I\}$. Let $\vec{s} = \langle s_i : i \in I\rangle$ such that $0 \leq s_i \leq k$ for $i \in I$, and $J = \{i \in I : s_i < k\}$.

LEMMA 2.11. *We have*

$$\operatorname{pr}(M(J, \vec{s}), n) \sim c$$

for some $c > 0$.

PROOF. Using the notation of Lemma 2.9,

$$L(\vec{t}) = n^{\underline{b}} \times \varepsilon \times \prod_{i \in I} \frac{(\zeta_i n^{-\alpha(v_i - 1)})^{t_i}}{t_i!}.$$

For $m, n \in \omega$, let

$$u(n, m) = \sum_{\sum(\vec{t})=m} \prod_{i \in J} \binom{t_i}{s_i} \prod_{i \in I-J} \binom{t_i - 1}{k - 1} \times L(\vec{t}).$$

By Lemma 2.5,

$$\operatorname{pr}(M(J, \vec{s}), n) = \sum_{m \geq \sum(\vec{s})} (-1)^{m - \sum(\vec{s})} u(n, m) \tag{3}$$

and by Lemma 2.6,

$$\sum_{m \geq r} (-1)^{m-r} u(n, m) \geq 0 \text{ for all } n, r \in \omega. \tag{4}$$

Further, let

$$u(m) = \sum_{\substack{\vec{t} \geq \vec{s} \\ \sum(\vec{t})=m}} \prod_{i \in J} \frac{\zeta_i^{t_i}}{s_i!(t_i - s_i)!} \times \prod_{i \in I-J} \frac{\zeta_i^{t_i}}{(k-1)!(t_i - k)!t_i}.$$

Then for every m,

$$\lim_{n \to \infty} u(n, m) = u(m). \tag{5}$$

For $i \in J$,

$$\begin{aligned} \sum_{t \geq s_i} (-1)^{t - s_i} \frac{\zeta_i^t}{s_i!(t - s_i)!} &= \frac{\zeta_i^{s_i}}{s_i!} \sum_{t \geq s_i} \frac{(-\zeta_i)^{t - s_i}}{(t - s_i)!} \\ &= \frac{\zeta_i^{s_i}}{s_i!} \mathrm{e}^{-\zeta_i}. \end{aligned}$$

For $i \in I - J$,

$$\begin{aligned}\sum_{t\geq k}(-1)^{t-k}\frac{\zeta_i^t}{(k-1)!(t-k)!t} &= \sum_{t\geq k}(-1)^{t-k}\frac{\zeta_i^t}{t!}\left[\sum_{j<k}(-1)^{k-1-j}\binom{t}{j}\right] \\ &= 1-\sum_{t<k}\frac{\zeta_i^t}{t!}\left[\sum_{j\leq t}(-1)^{t-j}\binom{t}{j}\right]- \\ &\quad \sum_{t\geq k}\frac{\zeta_i^t}{t!}\left[\sum_{j<k}(-1)^{t-j}\binom{t}{j}\right] \\ &= 1-\sum_{j<k}\left[\sum_{t\geq j}\frac{\zeta_i^t}{t!}(-1)^{t-j}\binom{t}{j}\right] \\ &= 1-\sum_{j<k}\frac{\zeta_i^j}{j!}\mathrm{e}^{-\zeta_i}.\end{aligned}$$

Therefore

$$\begin{aligned}(6)\quad \sum_{m\in\omega}(-1)^{m-\Sigma(\vec{s})}u_m &= \left[\prod_{i\in J}\frac{\zeta_i^{s_i}}{s_i!}\mathrm{e}^{-\zeta_i}\right]\times\left[\prod_{i\in I-J}\left(1-\sum_{j<k}\frac{\zeta_i^j}{j!}\mathrm{e}^{-\zeta_i}\right)\right] \\ &= c\end{aligned}$$

for some constant $c > 0$.

Now by Lemma 7.4 in [13], Equations (3), (4), (5), and (6) together imply

$$\lim_{n\to\infty}\mathrm{pr}(M(J,\vec{s}),n) = c. \qquad \dashv$$

LEMMA 2.12. *Assume $\alpha/(\alpha-1)$ is an integer. Take any graph H such that H has at least k components isomorphic to T, for every T of type* (I). *Then*

$$\mathrm{pr}(G \cong_k H, n) \sim cn^{-d}$$

for some $c > 0$ and $d \geq 0$.

PROOF. Let $\vec{s} = \langle s_i : i \in I\rangle$ be the sequence that characterizes the type (II) components of H. That is, for each $i \in I$, $s_i = \min(k, |\{D \sqsubseteq H : D \cong C_i\}|)$.

Let τ be the sentence that holds for a graph G if and only if it has the same number of components (up to k) as H, for each type (II) connected graph. In other words, $G \in M(J,\vec{s})$.

We will now repeat the same arguments used in Case 2, but conditioned on τ. Let F' be the set of graphs on $\{1,\dots,n\}$ that satisfy τ and $\{C_i' : i \in I'\}$ be the list of components of H of type (III). (We assume $I \cap I' = \emptyset$.) For $i \in I'$ let $Q_i' = \{P_{iX}'\}$ where X ranges over all subsets of $\{1,\dots,n\}$ with v_i' vertices, and P_{iX}' is the collection of graphs on $\{1,\dots,n\}$ where X induces a component isomorphic to C_i'. Let $\vec{s}' = \langle s_i' : i \in I'\rangle$ where $s_i' = \min(k, |\{D \sqsubseteq H : D \cong C_i'\}|)$.

By Lemma 2.11, $\mathrm{pr}(\tau, n) \sim c$ for some $c > 0$. Using the same methods as in the proofs of Lemmas 2.9 and 2.10,

$$\mathrm{pr}(G \cong_k H | \tau, n) \sim c' n^{-d}$$

for some $c' > 0$ and $d \geq 0$. Then $\mathrm{pr}(G \cong_k H, n) = \mathrm{pr}(G \cong_k H | \tau, n) \times \mathrm{pr}(\tau, n) \sim c' n^{-d} \times c = cc' n^{-d}$. ⊣

The rest of the proof is the same as in Case 2.

§3. **Related results.** Theorems 2.3 and 2.4 give a fairly complete picture of very sparse random graphs from the viewpoint of $L^{\omega}_{\infty\omega}$. However, for edge probabilities $p(n) \gg n^{-1}$, the main known results (see Table 1) are for $p(n) = n^{-\alpha}$, $\alpha < 1$. Investigating other examples of edge probabilities in this region is an open problem. Interestingly, when $p(n)$ and $1 - p(n)$ are much larger than n^{-1}, for example when it is constant, a power law holds. This was shown for first-order sentences in Lynch [15], and essentially the same proof carries over to $L^{\omega}_{\infty\omega}$.

THEOREM 3.1. *If* $\sigma \in L^{k}_{\infty\omega}$ *and* $n^{-\alpha} \leq p(n) \leq 1 - n^{-\alpha}$ *where* $\alpha < 1/(k-1)$, *then there is* $d > 0$ *such that*

$$\begin{aligned} \mathrm{pr}(\sigma, n) &\leq 2^{-n^d} \text{ or} \\ \mathrm{pr}(\sigma, n) &\geq 1 - 2^{-n^d}. \end{aligned}$$

Our final results pertain to the computability of $\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$. In order for this to make sense, we must consider how sentences in $L^{\omega}_{\infty\omega}$ can be encoded as strings over a finite alphabet. Clearly no encoding can capture every sentence since $L^{\omega}_{\infty\omega}$ is uncountable. Thus let us assume our sentences are from some fragment F of $L^{\omega}_{\infty\omega}$ that is encoded by a recursive language. Further, given any such encoding, there is an effective procedure for determining how many distinct variables the sentence has, and given any graph, there is an effective procedure for determining if the graph satisfies the sentence. We also assume $p(n) = n^{-\alpha}$ where α is a recursive real greater than 1. The next results show that with rather mild conditions, there is a recursive procedure for computing the limit, but if the limit is 0, there is no decision procedure for separating the two cases of Theorem 2.4.

THEOREM 3.2. *There is a recursive procedure such that given any* $\delta \geq 0$ *and sentence* $\sigma \in F$, *it decides if* $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ *for some* $c > 0$ *and* $d \leq \delta$. *Further, it generates* c *and* d *if they exist.*

PROOF. By Lemma 2.8, there exists V such that

$$\mathrm{pr}(G \text{ has a component with more than } V \text{ vertices}, n) = o(n^{-\delta}).$$

Thus, $\mathrm{pr}(\sigma, n) = \Omega(n^{-\delta})$ if and only if there is some graph $G \models \sigma$ such that every component of G is of size $\leq V$. By Lemma 2.7, we may assume G has at least k components isomorphic to T, for every T of type (I). By Corollary 2.2, we may assume G has at most k components of any isomorphism type. There are only finitely many such graphs, and they can be effectively listed, given δ. If none of them satisfy σ, then $\mathrm{pr}(\sigma, n) \not\sim cn^{-d}$ for any $c > 0$ and $d \leq \delta$.

If there are such graphs, then the proofs of Cases 2 and 3 of Theorem 2.4 show that $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ for some c and d. Furthermore, the proofs give an algorithm for finding c and d. ⊣

COROLLARY 3.3. *There is a recursive procedure such that given $\sigma \in F$, it computes $\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$.*

PROOF. Take $\delta = 0$ in the Theorem. ⊣

COROLLARY 3.4. *There is a procedure such that if $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ for some c, d, then it recursively generates c and d.*

COROLLARY 3.5. *The set of σ such that $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ for some c, d is recursively enumerable.*

THEOREM 3.6. *Assume F contains the first-order language of graphs. There is no recursive procedure such that given $\sigma \in F$, it decides if $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ for some $c > 0$ and $d \geq 0$.*

PROOF. This was already shown for the first-order language of graphs in [15]. ⊣

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