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How high can Baumgartner's $\mathcal{I}$-ultrafilters lie in the P-hierarchy?

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Abstract Under the continuum hypothesis we prove that for any tall P-ideal $\mathcal{I}$ on ω and for any ordinal $\gamma \leq \omega_1$ there is an $\mathcal{I}$-ultrafilter in the sense of Baumgartner, which belongs to the class $\mathcal{P}_\gamma$ of the P-hierarchy of ultrafilters. Since the class of $\mathcal{P}_2$ ultrafilters coincides with the class of P-points, our result generalizes the theorem of Flašková, which states that there are $\mathcal{I}$-ultrafilters which are not P-points.

Keywords I-ultrafilters · P-hierarchy · CH · P-points · Monotonic sequential contour

Mathematics Subject Classification 03E05 · 03E50

1 Introduction

Baumgartner in the article *Ultrafilters on ω* [1] introduced the notion of $\mathcal{I}$-ultrafilters:

Let $\mathcal{I}$ be an ideal on ω. An ultrafilter u on ω is an $\mathcal{I}$-ultrafilter, if and only if, for every function $f \in \omega^\omega$, there is a set $U \in u$ such that $f[U] \in \mathcal{I}$.

Ultrafilters of this kind have been the subject of research of a large group of mathematicians. Let us mention some of the most important papers in this subject from our point of view: Błaszczyk [2], Brendle [3], Laflamme [18], Shelah [19,20]. The theory of $\mathcal{I}$-ultrafilters on ω was developed by Flašková [8–11] in a series of articles, as well

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as in her Ph.D. thesis [12]. Flašková [12] proved, under the continuum hypothesis (CH), that for every tall P-ideal $\mathcal{I}$ that contains all singletons, there is an $\mathcal{I}$-ultrafilter that is not a P-point. Later she succeeded in replaceing the assumption of the CH by $\mathfrak{p} = \mathfrak{c}$ [8].

The main subject of this article is the P-hierarchy of ultrafilters on ω. The P-hierarchy can be viewed as one of the ways to classify ultrafilters with respect to their complicity. It is composed of ω_1 disjoint $\mathcal{P}_\alpha$ classes, where P-points correspond to class $\mathcal{P}_2$ as was proven by Starosolski in [23] Proposition 2.1:

Proposition 1.1 *An ultrafilter u is a P-point if and only if u belongs to class $\mathcal{P}_2$ in the P-hierarchy.*

Many important facts about the P-hierarchy are given in [23]. Additional information regarding sequential cascades and contours can be found in [5–7,21]. However, the most crucial definitions and conventions are presented here. Since P-points correspond to $\mathcal{P}_2$ ultrafilters in the P-hierarchy of ultrafilters the question arises: to which classes of the P-hierarchy can $\mathcal{I}$-ultrafilters belong? In this paper we shall show that it can be any class $\mathcal{P}_\alpha$. Let us introduce all the necessary definitions and tools.

The set of natural numbers (finite ordinal numbers) will be denoted by ω. The filters considered in this paper will be defined on a countable and infinite set (except for one case indicated later). This will usually be the set $\max V$ of maximal elements of a cascade V (see the definition of cascades below) and we will often identify it with ω without indication. The following conventions will be of constant use:

Conventions If $\mathcal{F}$ is a filter on A and $A \subset B$, then we identify $\mathcal{F}$ with the filter on B, for which $\mathcal{F}$ is a filter base. In particular, we identify the principal ultrafilter on $\{v\}$ with the principal ultrafilter generated on ω by v. If $\mathcal{F}$ is a filter base, then by $\langle \mathcal{F} \rangle$ we denote the filter generated by $\mathcal{F}$.

A *cascade* is a tree V without infinite branches and with the least element $\emptyset_V$. A cascade is *sequential* if, for each non-maximal element of V ($v \in V \setminus \max V$), the set v^{+V} of immediate successors of v (in V) is countable and infinite. We write v^+ instead of v^{+V} if it is known in which cascade the successors of v are considered. If $v \in V \setminus \max V$, then v^+ may be endowed with an order of type ω, and then by $(v_n)_{n\in\omega}$ we denote the sequence of elements of v^+.

The *rank* of $v \in V$, which will be denoted by $r_V(v)$ or simply by $r(v)$, is defined inductively as follows: if $v \in \max V$, then $r(v) = 0$; otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of v. The rank $r(V)$ of the cascade V is, by definition, the rank of $\emptyset_V$. If for $v \in V \setminus \max V$ the set v^+ can be ordered in type ω so that the sequence $(r(v_n)_{n<\omega})$ is non-decreasing, then the cascade V is *monotonic* and we fix such orders for V without indication.

For $v \in V$ we denote by $v^{\uparrow V}$ the subcascade of V consisting of v and all elements greater than v. We write $v^{\uparrow}$ instead of $v^{\uparrow V}$ if it is clear in which cascade the subcascade is included.

One may assume that the sequential cascade V is a family of subsets of an infinite and countable set ω and the order of V is the reverse inclusion. Indeed, cascade V is isomorphic to cascade $\bar{V}$ such that:

- $\emptyset_{\bar{V}} = \omega$;
- for every $\bar{v} \in \bar{V}$, $\bar{v}$ is the disjoint union of the elements of $\bar{v}^+$:
  - $\bar{v} = \bigcup\{\bar{w} : \bar{w} \in \bar{v}^+\}$,
  - for every $w, u \in v^+$, $\bar{w} \cap \bar{u} = \emptyset$,
- $\bar{v}$ is a singleton, for every $\bar{v} \in \max \bar{V}$.

If one identifies $\max V$ with ω, then the map $^- : V \to \bar{V}$ given by the formula $\bar{v} = \max v^{\uparrow}$ is such an isomorphism.

If $\mathbb{F} = \{\mathcal{F}_s : s \in S\}$ is a family of filters on X and if $\mathcal{G}$ is a filter on S, then the *contour of* $\{\mathcal{F}_s\}$ *along* $\mathcal{G}$ is defined by

$$\int_{\mathcal{G}} \mathbb{F} = \int_{\mathcal{G}} \{\mathcal{F}_s : s \in S\} = \bigcup_{G \in \mathcal{G}} \bigcap_{s \in G} \mathcal{F}_s.$$

This construction has been used by many authors ([13–15]), and is also known as a sum and limit of filters. We apply the operation of contour along filter to define *the contour of a cascade*: Fix a cascade V. Let $\mathcal{G}(v)$ be a filter on v^+, for every $v \in V \setminus \max V$. For $v \in \max V$, let $\mathcal{G}(v)$ be a trivial ultrafilter on the singleton $\{v\}$, which we can treat as a principal ultrafilter on $\max V$, according to the convention we assumed. In this way we obtain the function $v \longmapsto \mathcal{G}(v)$. We define the contour of every subcascade $v^{\uparrow}$ inductively with respect to the rank of v:

$$\int^{\mathcal{G}} v^{\uparrow} = \{\{v\}\},$$

for $v \in \max V$ (i.e. $\int^{\mathcal{G}} v^{\uparrow}$ is just a trivial ultrafilter on the singleton $\{v\}$);

$$\int^{\mathcal{G}} v^{\uparrow} = \int_{\mathcal{G}(v)} \left\{ \int^{\mathcal{G}} w^{\uparrow} : w \in v^+ \right\}$$

for $v \in V \setminus \max v$. Eventually

$$\int^{\mathcal{G}} V = \int^{\mathcal{G}} (\emptyset_V)^{\uparrow}.$$

Usually we shall assume that all the filters $\mathcal{G}(v)$ are Fréchet filters (for $v \in V \setminus \max V$). In this case, we shall write $\int V$ instead of $\int^{\mathcal{G}} V$.

Filters defined in a similar way were also considered in [4,16,17].

Let V be a monotonic sequential cascade and let $u = \int V$. Then the *rank* $r(u)$ of u is, by definition, the rank of V.

It was shown in [7] that, if $\int V = \int W$, then $r(V) = r(W)$.

We shall say that the set F *meshes* the contour $\mathcal{V}$ ($F \# \mathcal{V}$) if and only if $\mathcal{V} \cup \{F\}$ has the finite intersection property, i.e., can be extended to a filter. If $\omega \setminus F \in \mathcal{V}$, then we say that F is *small* with respect to $\mathcal{V}$.

For a countable ordinal number $\alpha \geq 1$ we define the class $\mathcal{P}_\alpha$ of the P-hierarchy (see [23]) as follows: $u \in \mathcal{P}_\alpha$ iff

1. there is no monotonic sequential contour C_α of rank α such that $C_\alpha \subset u$, and
2. for each $\beta < \alpha$, $\beta \neq 0$, there exists a monotonic sequential contour C_β of rank β such that $C_\beta \subset u$.

If for each $\alpha < \omega_1$ there exists a monotonic sequential contour C_α of rank α such that $C_\alpha \subset u$, then we say that u belongs to the class $\mathcal{P}_{\omega_1}$.

Let us consider a monotonic cascade V and a monotonic sequential cascade W. We will say that W is a *sequential extension* of V if:

1. V is a subcascade of the cascade W,
2. if v^{+V} is infinite, then $v^{+V} = v^{+W}$,
3. $r_V(v) = r_W(v)$ for each $v \in V$.

It is clear that sequential extensions are not uniquely defined.

Note that, if W is a sequential extension of V and $U \subset \max V$, then U is small for V if and only if U is small for W.

It cannot be proven in ZFC that all classes $\mathcal{P}_\alpha$ are nonempty. The following theorem is Theorem 2.8 of [23].

Theorem 1.2 *The following statements are equivalent:*

1. *P-points exist,*
2. *$\mathcal{P}_\alpha$ classes are non-empty for all countable successor α,*
3. *There exists a countable successor $\alpha > 1$ such that the class $\mathcal{P}_\alpha$ is non-empty.*

Starosolski showed in [25] Theorem 6.7 that:

Theorem 1.3 *Assuming CH every class $\mathcal{P}_\alpha$ is non-empty*

The main theorem presented in this paper can be viewed as an extension of Starosolski's result.

Let us consider another technical notion which can be called a *"restriction of a cascade"*. Let V be a monotonic sequential cascade and let H be a set such that $H \# \int V$. By $V^{\downarrow H}$ we denote the biggest monotonic sequential cascade such that $V^{\downarrow H} \subset V$ and $\max V^{\downarrow H} \subset H$. It is easy to see that $H \in \int V^{\downarrow H}$.

2 Lemmata

The following lemmata will be used in the proof of the main theorem.

The first lemma is given in [24] (see: Lemma 6.3):

Lemma 2.1 *Let $\alpha < \omega_1$ be a limit ordinal and let $(\mathcal{V}_n : n < \omega)$ be a sequence of monotonic sequential contours such that $r(\mathcal{V}_n) < r(\mathcal{V}_{n+1}) < \alpha$ for every n, and $\bigcup_{n<\omega} \mathcal{V}_n$ has the finite intersection property. Then there is no monotonic sequential contour $\mathcal{W}$ of rank α such that $\mathcal{W} \subset \langle \bigcup_{n<\omega} \mathcal{V}_n \rangle$.*

As a corollary we get:

Lemma 2.2 *Let $\alpha < \omega_1$ be a limit ordinal, let $(\mathcal{V}_n)_{n<\omega}$ be an increasing (in the sense of inclusion) sequence of monotonic sequential contours, such that $r(\mathcal{V}_n) < \alpha$ and let $\mathcal{F}$ be a countable family of sets such that $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}$ has the finite intersection property. Then $\langle \bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F} \rangle$ does not contain a monotonic sequential contour of rank α.*

Proof If $\mathcal{F}$ is finite then set

$$\mathcal{W}_n = \left\{U \cap \bigcap \mathcal{F} : U \in \mathcal{V}_n\right\}.$$

It is easy to see that $\mathcal{W}_n$ is a monotonic sequential contour of the same rank as $\mathcal{V}_n$. Consider the sequence $(\mathcal{W}_n : n < \omega)$. By Lemma 2.1 the union $\bigcup_{n<\omega} \mathcal{W}_n$ does not contain a contour of rank α.

If $\mathcal{F}$ is infinite then we enumerate elements of $\mathcal{F}$ by natural numbers obtaining a sequence $(F_n)_{n<\omega}$. Next, set

$$\mathcal{W}_n = \left\{U \cap \bigcap_{i \leq n} F_i : U \in \mathcal{V}_n\right\}.$$

Consider the sequence $(\mathcal{W}_n : n < \omega)$ and use Lemma 2.1 to show that the union $(\mathcal{W}_n : n < \omega)$ does not contain a contour of rank α.

The following lemma is a straightforward extension of the claim contained in the proof of Theorem 3.2 [8]. We leave a proof to the reader.

Lemma 2.3 *Let $\mathcal{I}$ be a tall P-ideal that contains all singletons, let $\{U_n : n < \omega\}$ be a pairwise disjoint sequence of subsets of ω, let $\{u_n : n < \omega\}$ be a sequence of $\mathcal{I}$-ultrafilters such that $U_n \in u_n$, and let v be an $\mathcal{I}$-ultrafilter. Then $\int_v \{u_n : n < \omega\}$ is an $\mathcal{I}$-ultrafilter.*

As an immediate consequence we get

Lemma 2.4 *If V is a monotonic sequential cascade, $\mathcal{G}(v)$ is a P-point and an $\mathcal{I}$-ultrafilter for each $v \in V \setminus \max V$ and $\mathcal{G}(v)$ is a trivial ultrafilter on a singleton $\{v\}$ for $v \in \max V$, then $\int^{\mathcal{G}} V$ is an $\mathcal{I}$-ultrafilter.*

Lemmata similar to the one above can be formulated for certain classes of the P-hierarchy instead of $\mathcal{I}$-ultrafilters, see [23] Theorem 2.5:

Theorem 2.5 *Let γ be an ordinal. Let V be a monotonic sequential cascade of rank γ, let $\mathcal{G}(v)$ be a principal ultrafilter on $\{v\}$ for $v \in \max V$, and let $\mathcal{G}(v)$ be a P-point on v^+ for $v \in V \setminus \max V$. Then $\int^{\mathcal{G}} V \in P_{\gamma+1}$.*

Let W be a cascade, and let $\{V^w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V^w \cap W = \emptyset$ for each $w \in \max W$. The *confluence* of cascades V^w with respect to the cascade W (we write $W \hookleftarrow V^w$) is defined as a cascade constructed by the identification of $w \in \max W$ with $\emptyset_{V^w}$ and according to the following rules: (1) $\emptyset_W = \emptyset_{W \hookleftarrow V^w}$; (2) if $w \in W \setminus \max W$, then $w^{+W \hookleftarrow V^w} = w^{+W}$; (3) if $w \in V^{w_0}$, for a certain $w_0 \in \max W$, then $w^{+W \hookleftarrow V^w} = w^{+V^{w_0}}$; (4) in each case we also assume that the order on the set of successors remains unchanged. By $(n) \hookleftarrow V^n$ we denote $W \hookleftarrow V^w$ where W is a sequential cascade of rank 1.

At the end of this section we shall make a remark concerning bases of countours.

Remark 2.6 *Each filter base of the contour of any cascade of rank 2 is uncountable.*

Proof Let V be a cascade of rank 2. We may assume that V is obtained by the confluence of cascades V_n of rank 1 i.e. $V = (n) \hookleftarrow V_n$. Assume that $\int V$ has a countable base. Thus there is a set A such that $A \backslash F$ is finite for every $F \in \int V$. It is evident that A meshes $\int V$. There are two cases:

Case I: there is $n < \omega$ such that $A \cap \max V_n$ is infinite. This cannot happen: it is sufficient to take $F = \max V \backslash V_n$ and observe that $F \in \int V$ but $A \backslash F$ is not finite.

Case II: for each $n < \omega$ the intersection $A \cap \max V_n$ is finite. This also cannot happen: it is sufficient to take $F = \bigcup_{n<\omega} \max V_n \backslash A$ and observe that $F \in \int V$ (because we remove only finite many elements from each V_m) but A is disjoint with F.

In both cases we get a contradiction.

In fact one can prove a stronger result than the above: bases of contours of cascades of rank greater than 1 must have the cardinality greater or equal to the dominating number $\mathfrak{d}$.

3 Main result

In this section we shall present the main result of the paper.

Theorem 3.1 *Assume that CH holds. Let $\mathcal{I}$ be a tall P-ideal that contains all singletons and let $\gamma \leq \omega_1$ be an ordinal. Then there exists an $\mathcal{I}$-ultrafilter u which belongs to $\mathcal{P}_\gamma$.*

Proof We shall split the proof into five steps: $\gamma = 1$, $\gamma = 2$, $\gamma > 2$ is a countable successor ordinal, $\gamma < \omega_1$ is a limit ordinal, and $\gamma = \omega_1$.

Step 0: $\gamma = 1$. This is clear, since an image of singleton ($\mathcal{P}_1$ is a class of principal ultrafilters) is a singleton, and thus belongs to $\mathcal{I}$.

Step 1: $\gamma = 2$.

Since CH is assumed, we can fix enumerations of length ω_1, of contours of rank 2 and of functions $\omega \to \omega$, say $(\mathcal{W}_\alpha)_{\alpha<\omega_1}$, $(f_\alpha)_{\alpha<\omega_1}$ respectively. Applying transfinite induction we build countably generated filters $\mathcal{F}_\alpha$ together with their decreasing bases $(F_\alpha^n)_{n<\omega}$, such that:

(W1) $\mathcal{F}_0$ is a Fréchet filter;
(W2) for each $\alpha < \omega_1$, $(F_\alpha^n)_{n<\omega}$ is a strictly decreasing base of $\mathcal{F}_\alpha$;
(W3) $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, for $\alpha < \beta$;
(W4) $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, if α is a limit ordinal;
(W5) for each $\alpha < \omega_1$ there is $F \in \mathcal{F}_{\alpha+1}$ such that $f_\alpha[F] \in \mathcal{I}$;
(W6) for each $\alpha < \omega_1$ there is $F \in \mathcal{F}_{\alpha+1}$ such that $F^c \in \mathcal{W}_\alpha$.

Assume that $\mathcal{F}_\alpha$ is already defined; we shall show how to build $\mathcal{F}_{\alpha+1}$. Since F_α^n is strictly decreasing, one can pick any $x_n \in F_\alpha^n \backslash F_\alpha^{n+1}$ for every $n < \omega$. Set $T = \{x_n : n < \omega\}$. The are two possibilities:

If $f_\alpha[T]$ is finite, then set $G_\alpha = T$.

If $f_\alpha[T]$ is infinite, then since $\mathcal{I}$ is tall, there is an infinite $I \in \mathcal{I}$ such that $I \subset f_\alpha[T]$. In this case set $G_\alpha = f_\alpha^{-1}[I]$.

Note that $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\}$ has the finite intersection property and is countable. Subbases of any sequential contour of rank 2 have to be uncountable. Thus none of them is contained in $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\}$. This means that there is a set A_α such that its complement belongs to $\mathcal{W}_\alpha$ and the family $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha, A_\alpha\}$ has the finite intersection property. We can order $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha, A_\alpha\}$ to obtain a sequence $(\tilde{F}_{\alpha+1}^n : n < \omega)$. Set $F_{\alpha+1}^n = \bigcap_{m \leq n} \tilde{F}_{\alpha+1}^m$ to get a decreasing sequence and let

$$\mathcal{F}_{\alpha+1} = \left\langle \{F_{\alpha+1}^n : n < \omega\} \right\rangle.$$

Take an ultrafilter u that extends $\bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$. By (W5) u is an $\mathcal{I}$-ultrafilter and by (W6) u does not contain a monotonic sequential contour of rank 2. Since by (W1) u contains a Fréchet filter u cannot be a principal ultrafilter. Thus u is a P-point. Note that in this step we did not use the assumption that $\mathcal{I}$ is a P-ideal.

Step 2: $\gamma > 2$ is a countable successor ordinal.

Let V be an arbitrary monotonic sequential cascade of rank $\gamma - 1$. Let $V \ni v \longmapsto \mathcal{G}(v)$ be a function such that:

1. $\mathcal{G}(v)$ is a P-point and is an $\mathcal{I}$-ultrafilter, for each $v \in V \setminus \max V$ (such ultrafilters exist by step 1);
2. $\mathcal{G}(v)$ is a trivial ultrafilter on the singleton $\{v\}$, for $v \in \max V$.

Theorem 2.5 guarantees that $\int^{\mathcal{G}} V \in \mathcal{P}_\gamma$, whilst Lemma 2.4 guarantees that $\int^{\mathcal{G}} V$ is an $\mathcal{I}$-ultrafilter.

Step 3: $\gamma < \omega_1$ is a limit ordinal. The proof is based on the same idea as in step 1 but it is more complicated.

Let $(\mathcal{V}_n)_{n<\omega}$ be an increasing sequence of monotonic sequential contours such that their ranks $r(\mathcal{V}_n)$ are smaller than γ and converging to γ. For each $n < \omega$, denote by V_n a (fixed) monotonic sequential cascade such that $\int V_n = \mathcal{V}_n$. Let $\{\mathcal{W}_\alpha : \alpha < \omega_1\}$ be an enumeration of all monotonic sequential contours of rank γ. Let $\omega^\omega = \{f_\alpha : \alpha < \omega_1\}$.

By transfinite induction we build filters $\mathcal{F}_\alpha$ together with their decreasing bases $(F_\alpha^n)_{n<\omega}$ such that conditions (W1)-(W6) hold together with the additional condition (W7) i.e.

(W7) $\bigcup_{i<\omega} \mathcal{V}_i \cup \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ has the finite intersection property.

Assume that $\mathcal{F}_\alpha$ is already defined; we shall show how to build $\mathcal{F}_{\alpha+1}$. This will be done in five substeps. Firstly, for each $\mathcal{V}_n$ and for each F_α^i we shall find $H_{n,i}$ such that $\mathcal{V}_n \cup \{F_\alpha^i, H_{n,i}\}$ has the finite intersection property and $f_\alpha[H_{n,i}] \in \mathcal{I}$. Next, we will replace all the sets $H_{n,i}$ by one set H_n such that $\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}$ has the finite intersection property and $f_\alpha[H_n] \in \mathcal{I}$. In the third step we shall replace all the sets H_n by one set G_α such that $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha\}$ has the finite intersection property and $f_\alpha[G_\alpha] \in \mathcal{I}$. The set G_α will take care of all the contours $\mathcal{V}_n$. Adding it as a generator to $\mathcal{F}_{\alpha+1}$ will guarantee that the conditions (W5) and (W7) will hold true. The fourth step will deal with the condition (W6) by adding a set A_α to the list of generators of $F_{\alpha+1}$. Finaly we will define a decreasing base of a filter $\mathcal{F}_{\alpha+1}$ and a filter itself.

Substep (i) Fix n and i. Let us introduce an auxiliary definition.

Definition Fix a monotonic sequential cascade V, a set F that meshes V and a function $f \in \omega^\omega$. For each $v \in V$, we write $U \in \mathbf{S}(v)$ if

1. $U \subset \max v^{\uparrow}$;
2. $(U \cap F)\# \int v^{\uparrow}$;
3. $\mathrm{card}\,(f[U \cap F]) = 1$.

The following claim is crucial in our argument:

Proposition 3.2 *One of the following conditions holds:*

(A) $\boldsymbol{S}(\emptyset_V) \neq \emptyset$;
(B) there is an antichain (with respect to the order of the cascade) $\mathbb{A} \subset V$ *such that:*
  1. $\boldsymbol{S}(v) = \emptyset$, *for all* $v \in \mathbb{A}$,
  2. $\left(\bigcup\{\max w^{\uparrow} : w \in v^{+}, \boldsymbol{S}(w) \neq \emptyset\}\right) \# \int v^{\uparrow}$, *for all* $v \in \mathbb{A}$,
  3. $\left(\bigcup\{\max v^{\uparrow} : v \in \mathbb{A}\}\right) \# \int V$.

Proof (of the proposition) Firstly note that in the definition of **S** condition $\mathrm{card}\,(f[U \cap F]) = 1$ can by replaced by $\mathrm{card}\,(f[U \cap F]) < \aleph_0$, and that this change has no influence on non-emptiness of $\mathbf{S}(v)$.

The proof is by induction on the rank of cascade V.

First step $r(V) = 1$. If case A holds, then we are done. Thus without loss of generality we may assume that the image $f(U \cap F)$ is infinite, for each $U \cap F \in \max V$, such that $U\# \int V$. However, since $r(V) = 1$, $\mathrm{card}\,(f(\max w \cap F)) \leq 1$, for each $w \in \emptyset_V^{+}$. Moreover, since $F\# \int V$,

$$\left(\bigcup\left\{(\max w \cap F) : w \in v^{+}, \mathrm{card}\,(f(\max w \cap F)) = 1\right\}\right) \# \int V.$$

We set $\mathbb{A} = \{\emptyset_V\}$ and see that case (B) holds.

Inductive step: Assume that the proposition holds for each $\beta < \alpha$. Let V be a monotonic sequential cascade of rank $r(V) = \alpha$. Again, if case A holds, then we are done; thus without loss of generality we assume that the image $f(U \cap F)$ is infinite for each $U \cap F \subset \max V$ such that $U\# \int V$. By the inductive assumption, for each successor w of $\emptyset_V$, either case A holds for cascade $w^{\uparrow}$, or case B holds for the cascade $w^{\uparrow}$.

Split the set $\emptyset_V^{+}$ of immediate successors of $\emptyset_V$ into two subsets:

$$V^{A} = \{w \in \emptyset_V : \text{ case A holds}\}, \quad V^{B} = \{w \in \emptyset_V : \text{case B holds}\}.$$

Since $F\# \int V$, there are two possibilities:

$$\left(\bigcup_{w \in V^{A}} (\max w^{\uparrow} \cap F)\right) \# \int V \text{ or } \left(\bigcup_{w \in V^{B}} (\max w^{\uparrow} \cap F)\right) \# \int V.$$

In the first case, $\mathbb{A} = \{\emptyset_V\}$ is what we are looking for.

In the second case, for each $w \in V^{B}$, there is an antichain $\mathbb{A}_w$ in $w^{\uparrow}$ as in case (B). Set $\mathbb{A} = \bigcup_{w \in V^{B}} \mathbb{A}_w$. This completes the proof of the proposition.

We can go back to the main proof.

We apply the proposition to cascade V_n, set F^i_α and function f_α. If case (A) holds, then we take any $U \in \mathbf{S}(\emptyset_{V_n})$ and set $H_{n,i} = U$.

If case (B) holds, then for each $v \in \mathbb{A}$ and each $w \in v^+$, such that $\mathbf{S}(w) \neq \emptyset$, we fix some $U_w \in \mathbf{S}(w)$; for the remaining $w \in v^+$ let $U_w = \emptyset$. For $v \in \mathbb{A}$, consider $T_v = \bigcup_{w \in v^+} U_w$, and note that $f_\alpha[T_v]$ is infinite. Since $\mathcal{I}$ is tall, there is an infinite $I_v \in \mathcal{I}$ such that $I_v \subset f_\alpha[T_v]$. Since $\mathcal{I}$ is a P-ideal, there is infinite $I_{n,i} \in \mathcal{I}$ such that for all $v \in \mathbb{A}$ the difference $I_v \backslash I_{n,i}$ is finite. Set $H_{n,i} = f^{-1}[I_{n,i}]$.

Substep (ii) Now we will show how to replace sets $H_{n,i}$ by one set H_n. Consider two possibilities:

1. there is an infinite $K \subset \omega$ such that $f_\alpha[H_{n,i}]$ is infinite for each $i \in K$;
2. there is an infinite $K \subset \omega$ such that $f_\alpha[H_{n,i}]$ is a singleton for each $i \in K$.

In both cases, since $(F^i_\alpha)_{i<\omega}$ is decreasing without loss of generality we may assume that $K = \omega$.

Assume that the case (1) holds. Since $\mathcal{I}$ is an P-ideal, there is an infinite $I_n \in \mathcal{I}$ such that for each $i < \omega$ the difference $I_{n,i} \backslash I_n$ is finite. Set $H_n = f_\alpha^{-1}[I_n]$.

In case (2), we have two sub-cases:

If $f_\alpha[\bigcup_{i<\omega} H_{n,i}]$ is infinite, then since $\mathcal{I}$ is tall, there is an infinite $I_n \in \mathcal{I}$ such that $I_n \subset f_\alpha[\bigcup_{i<\omega} H_{n,i}]$ and we set $H_n = f_\alpha^{-1}[I_n]$.

Otherwise $f_\alpha[\bigcup_{i<\omega} H_{n,i}]$ is finite and there is $j \in f_\alpha[\bigcup_{i<\omega} H_{n,i}]$ such that $f_\alpha^{-1}[\{j\}] = H_{n,i}$ for infinitely many i. We set $H_n = f_\alpha^{-1}[\{j\}]$.

Clearly, in both cases $\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}$ has the finite intersection property and $f_\alpha[H_n] \in \mathcal{I}$.

Substep (iii) In this step we have to find a set G_α which can replace each H_n. We have shown that, for each n, there is a set H_n such that $f_\alpha[H_n] \in \mathcal{I}$. In fact, we have got a little bit more: either $f_\alpha[H_n]$ is infinite but it belongs to $\mathcal{I}$ or $f_\alpha[H_n]$ is a singleton. We set

$$S = \{n < \omega : (\exists R_n) \ : \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{R_n\} \text{ has the f.i.p. and } f_\alpha[R_n] \text{ is a singleton}\}.$$

It could happen that $f_\alpha[H_n]$ is infinite, but $n \in S$ i.e. that for some $R_n \neq H_n$ the image $f_\alpha[R_n]$ is a singleton. In this case we replace H_n by R_n. For $n \in \omega \backslash S$ we leave H_n unchanged. Once again the proof splits into two cases: either S is infinite, or it is finite.

S is infinite Without loss of generality, since $(\mathcal{V}_n)_{n<\omega}$ is increasing, we may assume that $S = \omega$ i.e. $f_\alpha[H_n]$ is a singleton for each $n < \omega$.

If $f_\alpha[\bigcup_{n<\omega} H_n]$ is finite, then there is $j \in f_\alpha[\bigcup_{n<\omega} H_n]$ such that $f_\alpha[H_n] = \{j\}$, for infinitely many n. Since $\mathcal{V}_n$ is increasing and (F^n_α) is decreasing, the family $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup f_\alpha^{-1}[\{j\}]$ has the finite intersection property. Set $G_\alpha = f_\alpha^{-1}[\{j\}]$.

If $f_\alpha[\bigcup_{n<\omega} H_n]$ is infinite, then, since $\mathcal{I}$ is tall, there is an infinite $I_\alpha \in \mathcal{I}$ such that $I_\alpha \subset f_\alpha[\bigcup_{n<\omega} H_n]$. Since $\mathcal{V}_n$ is increasing and (F^n_α) is decreasing, the family $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup f_\alpha^{-1}[I_\alpha]$ has the finite intersection property. Set $G_\alpha = f_\alpha^{-1}[I_\alpha]$.

S is finite: Without loss of generality, since $(\mathcal{V}_n)_{n<\omega}$ is increasing, we may assume that $S = \emptyset$ i.e. that $f_\alpha[H_n]$ is infinite for each $n < \omega$.

Since $\mathcal{I}$ is a P-ideal and $f_\alpha[H_n] \in \mathcal{I}$, there exists $I_\alpha \in \mathcal{I}$ such that for each $n < \omega$ the difference $f_\alpha[H_n] \backslash I_\alpha$ is finite and there is no R_n such that $f_\alpha[R_n]$ is finite and that $\{\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{R_n\}\}$ has the finite intersection property. Fix $n < \omega$. Since $\{\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}\}$ has the finite intersection property and since $H_n \subset f_\alpha^{-1}[f_\alpha[H_n] \backslash I_\alpha] \cup f^{-1}[I_\alpha]$ thus $\{\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{f_\alpha^{-1}[f_\alpha[H_n] \backslash I_\alpha]\}\}$ has the finite intersection property or $\{\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{f^{-1}[I_\alpha]\}\}$ has the finite intersection property. The first case cannot occur since $f_\alpha[f_\alpha^{-1}[f_\alpha[H_n] \backslash I_\alpha]]$ is finite and thus the second case holds true. Set $G_\alpha = f_\alpha^{-1}[I_\alpha]$.

Substep (iv) Since the family $\mathcal{F}_\alpha \cup \{G_\alpha\}$ is countable, by Lemma 2.2 there exists a set A_α which does not mesh the contour $\mathcal{W}_\alpha$ and such that the family $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha, A_\alpha\}$ has the finite intersection property.

Substep (v) We complete the proof as for $\gamma = 2$.

Step 4: $\gamma = \omega_1$.

Again, we list ${}^\omega\omega = \{f_\alpha : \alpha < \omega_1\}$, as well as all pairs (set and its complement) in the ω_1-sequence of pairs $(A_\alpha, \omega \backslash A_\alpha)_{\alpha<\omega_1}$, so that each set appears in the sequence only once: either as set A_α, or as its complement $\omega \backslash A_\alpha$.

We will build an ω_1-sequence $(V_\alpha : \alpha < \omega_1)$ of monotonic sequential cascades such that

(Z1) $\int V_\beta \subset \int V_\alpha$, for each $\beta < \alpha < \omega_1$.
(Z2) $r(V_\alpha) = \alpha$, for every $\alpha < \omega_1$;
(Z3) $\max V_\alpha = \omega$, for every $\alpha < \omega_1$;
(Z4) there exists $U \in \int V_{\alpha+1}$ such that $f_\alpha[U] \in \mathcal{I}$;
(Z5) $A_\alpha \in \int V_{\alpha+1}$ or $\omega \backslash A_\alpha \in \int V_{\alpha+1}$.

Let V_1 be any monotonic sequential cascade of rank 1. Suppose that we have already defined cascades V_β, for all $\beta < \alpha$.

Case 1: $\alpha = \beta + 1$ is a successor. Let us consider V_β. By step 3, there is a set H_α such that $H_\alpha \# \int V_\beta$ and $f_\alpha[H_\alpha] \in \mathcal{I}$. Consider the cascade $V_\beta^{\downarrow H_\alpha}$. This is a monotonic sequential cascade of rank β. By the proof of Theorem 4.6 from [7], there is a monotonic sequential cascade $\tilde{V}_\alpha$ of rank α such that $\int V_\beta^{\downarrow H_\alpha} \subset \int \tilde{V}_\alpha$. At least one of the elements of the pair $(A_\alpha, \omega \backslash A_\alpha)$ meshes $\int \tilde{V}_\alpha$. Denote it by B_α and let $V_\alpha = \tilde{V}_\alpha^{\downarrow B_\alpha}$.

Case 2: α is a limit ordinal. Let V_α be any monotonic sequential cascade of rank α such that $\int V_\beta \subset \int V_\alpha$ for each $\beta < \alpha$. A construction of such a cascade one can find in the proof of Theorem 4.6 in [7].

Now it suffices to take $u = \bigcup_{\alpha<\omega_1} \int V_\alpha$.

The assumption that the ideal $\mathcal{I}$ is tall is essential: Flašková proved in Proposition 2.2 [8], that if $\mathcal{I}$ is not tall, then there is no $\mathcal{I}$-ultrafilter. One can easily see that an ideal $\mathcal{I}$ has to contain all singletons.

4 Rudin–Keisler ordering

Let $\mathcal{F}$ be a filter on X, and let $\mathcal{G}$ be a filter on Y; we say that $\mathcal{F}$ is *Rudin–Keisler greater* than $\mathcal{G}$ (we write $\mathcal{F} \geq_{RK} \mathcal{G}$) if there is a map $f : X \to Y$ such that $f(\mathcal{F}) \supset \mathcal{G}$. We say that $\mathcal{F}$ and $\mathcal{G}$ are Rudin–Keisler equivalent (denoted by $\mathcal{F} \approx_{RK} \mathcal{G}$) iff $\mathcal{F} \geq_{RK} \mathcal{G}$ and $\mathcal{G} \geq_{RK} \mathcal{F}$. Rudin–Keisler order is called *Katětov order* by some authors.

The main result of the paper can be improved as follows:

Theorem 4.1 (CH) *Let $\mathcal{I}$ be a tall P-ideal that contains all singletons and let $1 < \gamma < \omega_1$ be an ordinal. Then there exists a pair u, o of Rudin–Keisler incomparable $\mathcal{I}$-ultrafilters that belong to $\mathcal{P}_\gamma$.*

To prove the theorem, we need an additional lemma which states that the contour operation preserves the Rudin–Kiesler ordering:

Lemma 4.2 *Assume that α is a countable ordinal number and $u, v \in \mathcal{P}_\alpha$ are, such that $u \not\geq_{RK} v$. Fix a P-point p and an arbitrary free filter s. Next let $(U_n)_{n<\omega}$ and $(V_i)_{i<\omega}$ be partitions of ω into infinite sets and let (u_n) and (v_i) be sequences of ultrafilters such that $u_n \approx_{RK} u$, $U_n \in u_n$, $v_i \approx_{RK} v$, $V_i \in v_i$. Then $\int_p u_n \not\geq_{RK} \int_s v_i$.*

Proof Suppose on the contrary that there is a function $f : \omega \to \omega$ such that $f(\int_p u_n) = \int_s v_i$, without loss of generality f is a surjection. Take a non-decreasing sequence of ordinals $(\alpha_n)_{n<\omega}$ such that $\alpha_n < \alpha$, $\lim_{n\to\infty}(\alpha_n+1) = \alpha$. For each V_i, fix a monotonic sequential cascade C_i of rank α_i such that $\int C_i \subset v_i$ and $\max C_i = V_i$.

In addition define cascades D_i from C_i as follows: leave all non-maximal elements unchanged and for every v of rank 1 replace v^+ by $f^{-1}(v^+)$. More formally, this means that $v^{+D_i} = f^{-1}(v^{+C_i})$. Consider a cascade $K = (i) \hookleftarrow D_i$. Clearly K is a monotone sequential cascade of rank α.

Since $u_n \in \mathcal{P}_\alpha$ thus there are sets $U^n \in u_n$ small with respect to K. These sets mesh $\int D_i$ only for finitely many i. Since u_n is an ultrafilter there is $T_n \in u_n$ such that either:

1. T_n meshes $\int D_i$ for exactly one i that will be denoted by $i(n)$, and $f[T_n] \subset V_i$; or
2. T_n never meshes $\int D_i$.

Let A be a set of those n for which the first case holds. There are two possibilities:

1. $A \in p$;
2. $\omega \setminus A \in p$.

Assume that $A \in p$. Without loss of generality $A = \omega$. Consider the partition of ω into sets $A_i = \{n < \omega : i(n) = i\}$. Since p is a P-point, there is a $P \in p$ such that $P \cap A_i$ is finite for each i. The possibility that there is a $P \in p$ which is contained in some A_i can be excluded since $f(\bigcup_{n\in P'} T_n) \in \int_s v_n$ for each co-finite $P' \subset P$.

Define the sets

$$N(j) = \{n : i(n) = j \text{ and } n \in P\}.$$

These sets are finite. For each $n \in N(j)$, since $u \not\geq_{RK} v$, there are pairs of sets $\tilde{U}_n \in u_n$, $\tilde{V}_{j,n} \in v_j$ such that $f[\tilde{U}_n] \cap \tilde{V}_{j,n} = \emptyset$. Put

$$U'_n = \tilde{U}_n \cap T_n \quad \text{and}$$

$$V'_j = \begin{cases} \bigcap_{n\in N(j)} \tilde{V}_{j,n}, & j = i(n) \text{ for any } n < \omega \\ V'_j = V_j, & \text{otherwise} \end{cases}.$$

Clearly

$$U'_n \in u_n,\ V'_i \in v_i \quad \text{and} \quad f[U'_n] \cap V'_i = \emptyset$$

for each pair (n, i) such that $n \in P$. Therefore

$$\bigcup_{n\in P} U'_n \in \int_p u_n \text{ and } \bigcup_{i<\omega} V'_i \in \int_p v_i \text{ and } f(\bigcup_{n\in P} U'_n) \cap \bigcup_{i<\omega} V'_i = \emptyset.$$

Indeed, it is so because $f[U_n \cap T_n] \subset f[T_n] \subset V_{i(n)}$.

Assume that $\omega\setminus A \in p$. Without loss of generality, $A = \emptyset$.

If the set $\{n < \omega :$ there is $i < \omega$ and a set $R_n \subset U_n$, $R_n \in u_n$, such that $F[R_n] \subset V_i$, $f[R_n] \in v_i\}$ belongs to p then we proceed as in the case when $A \in p$.

Otherwise without loss of generality we may assume that for each $n, i < \omega$ the set $T_n\setminus f^{-1}[\bigcup_{j\leq i} V_j]$ belongs to u_i.

Since $u \not\geq_{RK} v$ for each n there are sets $\tilde{U}_n$ and $\tilde{V}_n$ such that

$$\tilde{U}_n \in u_n, \tilde{V}_n \in v_n \quad \text{and} \quad f[\tilde{U}_n] \cap \tilde{V}_n = \emptyset.$$

Define

$$U'_n = (\tilde{U}_n \cup T_n)\setminus f^{-1}\left[\bigcup_{j\leq n} V_j\right] \quad \text{and} \quad V'_n = \tilde{V}_n\setminus f\left[\bigcup_{j<n} T_j\right].$$

Notice that $U'_n \in u_n$ and $V'_n \in v_n$.

Clearly

$$\bigcup_{n<\omega} U'_n \in \int_p u_n,\ \bigcup_{n<\omega} V'_n \in \int_s v_i \quad \text{and} \quad f\left[\bigcup_{n<\omega} U'_n\right] \cap \bigcup_{n<\omega} V'_n = \emptyset.$$

Proof (of the theorem) Generally speaking, the proof is similar to the proof of the main result.

Step 1: for $\gamma = 2$. The proof is similar to the one in Step 3 below but easier. Hence, we skip it.

Step 2: $\gamma > 2$ is a countable successor ordinal number. Let u' and o' be a pair of RK incomparable $\mathcal{I}$-ultrafilters that belong to $\mathcal{P}_{\gamma-1}$. Fix a partition $(A_n : n < \omega)$ of ω into infinitely many infinite sets. Let $(u'_n : n < \omega)$ be a sequence of ultrafilters such that $u'_n \approx u'$, $A_n \in u'_n$. Also let $(o'_n : n < \omega)$ be a sequence of ultrafilters such that $o'_n \approx o'$, $A_n \in o'_n$.

Take the ultrafilters $u = \int_p u'_n$ and $o = \int_p o'_n$, where p is any P-point that is $\mathcal{I}$-ultrafilter. Theorem 2.5 guarantees that u and o belong to $\mathcal{P}_\gamma$ whilst Lemma 2.3 guarantees that u and o are $\mathcal{I}$-ultrafilters. Finally, Lemma 4.2 ensures that u and o are RK-incomparable.

Step 3: $\gamma < \omega_1$ is a countable limit ordinal number. We construct $\mathcal{F}_\alpha$ and its decreasing base $(F^n_\alpha)_{n<\omega}$ as in Theorem 3.1. Simultaneously, we construct $\mathcal{E}_\alpha$ and $(E^n_\alpha)_{n<\omega}$ with exactly the same properties as $\mathcal{F}_\alpha$ and $(F^n_\alpha)_{n<\omega}$. Additionally we have to make sure that $\mathcal{F}_\alpha$ and $\mathcal{E}_\alpha$ are related as follows:

(W8) for each $\alpha < \omega_1$, there is $F \in \mathcal{F}_{\alpha+1}$ such that $(f_\alpha[F])^c \in \mathcal{E}_{\alpha+1}$ and there is $E \in \mathcal{E}_{\alpha+1}$ such that $(f_\alpha[E])^c \in \mathcal{F}_{\alpha+1}$.

Firstly we define F^n_α, A_α and G_α as in Threorem 3.1 and set $\mathcal{F}^*_{\alpha+1} = \{F^n_\alpha : n < \omega\} \cup \{A_\alpha, G_\alpha\}$. We define $\mathcal{E}^*_{\alpha+1}$ in exactly the same way.

Since $\langle \mathcal{F}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \rangle$ is not an ultrafilter, there is a set Y such that both $\mathcal{F}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \{Y\}$ and $\mathcal{F}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \{Y^c\}$ have the finite intersection property.

Now $\{f_\alpha^{-1}[Y]\} \cup \mathcal{E}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n$ has the finite intersection property (case 1) or $\{(f_\alpha^{-1}[Y])^c\} \cup \mathcal{E}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n$ has the finite intersection property (case 2).

In the first case we set $Z^1_{\alpha+1} = f_\alpha^{-1}[Y]$ and $X^1_{\alpha+1} = Y^c$ whilst in the second case we set $Z^1_{\alpha+1} = (f_\alpha^{-1}[Y])^c$ and $X^1_{\alpha+1} = Y$.

Similarly, since $\langle \mathcal{E}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \{Z^1_{\alpha+1}\} \rangle$ is not an ultrafilter, there is a set Y such that both $\mathcal{E}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \{Z^1_{\alpha+1}, Y\}$ and $\mathcal{E}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \{Z^1_{\alpha+1}, Y^c\}$ have the finite intersection property.

Now $\{f_\alpha^{-1}[Y], X^1_{\alpha+1}\} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}^*_{\alpha+1}$ has the finite intersection property (case 1) or $\{(f_\alpha^{-1}[Y])^c, X^1_{\alpha+1}\} \cup \bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}^*_{\alpha+1}$ has the finite intersection property (case 2).

In the first case we set $X^2_{\alpha+1} = f_\alpha^{-1}[Y]$ and $Z^2_{\alpha+1} = Y^c$ whilst in the second we set $X^2_{\alpha+1} = (f_\alpha^{-1}[Y])^c$ and $Z^2_{\alpha+1} = Y$.

Order $\mathcal{F}^*_{\alpha+1} \cup \{X^1_{\alpha+1}, X^2_{\alpha+1}\}$ in a sequence $(\tilde{F}^n_{\alpha+1})_{n<\omega}$. Set $F^n_{\alpha+1} = \bigcap_{m\leq n} \tilde{F}^n_{\alpha+1}$ to get a decreasing sequence, and let $\mathcal{F}_{\alpha+1} = \langle \{F^n_{\alpha+1} : n < \omega\} \rangle$.

We proceed similarly with $\mathcal{E}_{\alpha+1}$.

Finally we take any ultrafilter u that extends $\bigcup_{n<\omega} \mathcal{V}_n \cup \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ and any ultrafilter o that extends $\bigcup_{n<\omega} \mathcal{V}_n \cup \bigcup_{\alpha<\omega_1} \mathcal{E}_\alpha$. Ultrafilters o and u have the same properties. It is easy to see that u and o are RK-incomparable.

Step 4: $\gamma = \omega_1$.

The proof is as in Theorem 3.1, but we build two sequences $(V_\alpha : \alpha < \omega_1)$, $(W_\alpha : \alpha < \omega_1)$ of monotonic sequential cascades instead of one. We demand that they are related as follows:

(Z6) for each $\alpha < \omega_1$ there is $V \in \int V_{\alpha+1}$, such that $(f_\alpha[V])^c \in \int W_{\alpha+1}$, and there is $W \in \int W_{\alpha+1}$, such that $(f_\alpha[W])^c \in \int V_{\alpha+1}$.

On the inductive step we have two cases. For a limit ordinal α there is no change in comparision to the proof of Theorem 3.1. Assume that $\alpha = \beta + 1$ is a successor ordinal. Exactly as in the original proof we find $\tilde{V}_\alpha^{\downarrow B_\alpha}$. Next, there is a change in the proof. Set $V^*_\alpha = \tilde{V}_\alpha^{\downarrow B_\alpha}$ and define its counterpart W^*_α in the same way.

We define sets Y, Z^1_α, X^1_α for $\int V^*_\alpha$ and $\int W^*_\alpha$ exactly as we defined sets Y, $Z^1_{\alpha+1}$, $X^1_{\alpha+1}$ for $\langle \mathcal{F}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n \rangle$ and $\mathcal{E}^*_{\alpha+1} \cup \bigcup_{n<\omega} \mathcal{V}_n$ on step 3. This argument we repeat once again to get another sets Z^2_α, X^2_α for $\int W^*_\alpha \cup \{Z^1_\alpha\}$ and $\int V^*_\alpha \cup \{X^1_\alpha\}$.

Let

$$V_\alpha = V_\alpha^{*\downarrow(X^1_\alpha \cap X^2_\alpha)} \quad \text{and} \quad W_\alpha = W_\alpha^{*\downarrow(Z^1_\alpha \cap Z^2_\alpha)}$$

Now it suffices to take $u = \bigcup_{\alpha<\omega_1} \int V_\alpha$ and $o = \bigcup_{\alpha<\omega_1} \int W_\alpha$. Condition (Z6) guarantees that u and o are RK-incomparable.

5 Ordinal ultrafilters and relatively minimal ultrafilters

Baumgartner in his paper [1] defined the notion of ordinal ultrafilters for an indecomposable ordinal α as J_α-ultrafilters where are ideals defined on ω_1 as follows:

$$J_\alpha = \{A \subset \omega_1 : A \text{ has order type } < \alpha\}.$$

Let J^*_α-ultrafilters be ultrafilters which are J_α-ultrafilters but are not J_β-ultrafilters for any $\beta < \alpha$.

Baumgartner ([1, Theorem 4.1] and [1, Theorem 4.2]) proved that

Theorem 5.1 *$J^*_{\omega^2}$-ultrafilters are P-points.*

Theorem 5.2 *Let $(\alpha_n)_{n<\omega}$ be a non-decreasing sequence of countable ordinal numbers. Let $\alpha = \lim_{n<\omega} \alpha_n$ and let $(X_n)_{n<\omega}$ be a partition of ω. If $(p_n)_{n<\omega}$ is a sequence of ultrafilters such that $X_n \in p_n \in J^*_{\omega^{\alpha_n}}$ and p is a P-point, then $\int_p p_n \in J^*_{\omega^{\alpha+1}}$.*

Applying the above theorems in our proof we obtain the following:

Theorem 5.3 (CH) *For each successor ordinal $1 < \alpha < \omega_1$ and for each tall P-ideal $\mathcal{I}$ there are two RK-incomparable $\mathcal{I}$-ultrafilters that belongs to $J^*_{\omega^\alpha}$.*

It is still an open problem whether classes $J^*_{\omega^\alpha}$ are nonempty for any limit infinite α, in any model of ZFC; an unpublished (as yet) Starosolski's result [22] state that the class $J^*_{\omega^\omega}$ is empty (ZFC).

Let $\mathbb{C}$ be any set of filters on a fixed set X. We say that an ultrafilter u is relatively RK-$\mathbb{C}$-minimal, whenever $u \in \mathbb{C}$ and $f(u) \approx_{RK} u$ or $f(u) \notin \mathbb{C}$ for each function $f : X \to X$.

For the P-hierarchy and for ordinal ultrafilters we have the following:

Theorem 5.4 ([24], reformulation of Theorems 4.4 and 4.7) *Let $\alpha < \omega$. If (p_n) is a discrete sequence of relatively RK-P_α ($J^*_{\omega^\alpha}$)-minimal ultrafilters on ω and p is a RK-minimal ultrafilter, then $\int_p p_n$ is relatively RK-$P_{\alpha+1}$ ($J^*_{\omega^{\alpha+1}}$)-minimal.*

A standard modification of the proof of Theorem 4.1 proof in virtue of Theorems 5.1 and 5.2 gives us the following:

Theorem 5.5 (CH) *For each natural number n and for each tall P-ideal $\mathcal{I}$ there are two RK-incomparable relatively RK-P_n ($J^*_{\omega^n}$)-minimal ultrafilters.*

Relatively RK-P_α-minimal ultrafilters for successor infinite α's do not exist (see [24]). For limit α and for classes of ordinal ultrafilters of infinite rank the question remains open.

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