L_p-Circular Functions

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Abstract

In this report we develop the basic properties of a set of functions analogous to the circular and hyperbolic functions, but based on L_p circles. The resulting identities may simplify analysis in L_p spaces in much the way that the circular functions do in Euclidean space. In any case, they are a pleasing example of mathematical generalization.

1 Introduction and Basic Definitions

Recall (Fig. 1) that the angle α is measured in circular radians by twice the area of the circular segment OAB, and that the basic circular functions are then defined

$$\sin\alpha = y/r, \quad \cos\alpha = x/r, \quad \tan\alpha = y/x.$$

Similarly (Fig. 2) the angle α is measured in hyperbolic radians by twice the

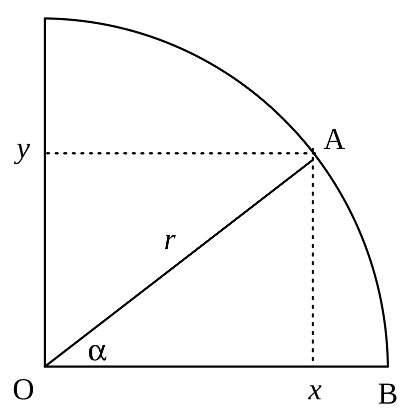

Figure 1: Definition of Circular Functions

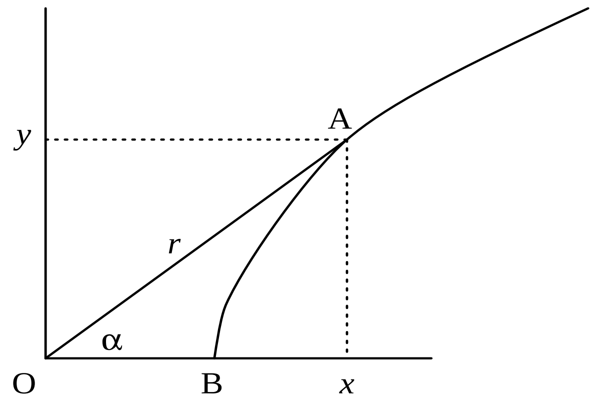

Figure 2: Definition of Hyperbolic Functions

area of OAB, and the basic hyperbolic functions are defined by

$$\sinh\alpha = y/r, \quad \cosh\alpha = x/r, \quad \tanh\alpha = y/x.$$

These definitions suggest several generalizations; in this report we develop the basic properties of functions defined with reference to L_p circles, that is, curves defined by

$$|x|^p + |y|^p = |r|^p, \ 0 < p \leq \infty.$$

By strict analogy (Fig. 3) with the circular and hyperbolic cases, we define the L_p-circular measure of α, in L_p-circular radians, by:

$$\alpha = xy + 2\int_x^r y\mathrm{d}x, \tag{1}$$

where $|x|^p + |y|^p = r^p$. Table 1 shows some common angles in L_p radians, for $p = 1, 2, 3, 4$.[1] Then we may define the L_p-circular functions:

$$\begin{aligned}\sin_p\alpha &= y/r\\ \cos_p\alpha &= x/r\\ \tan_p\alpha &= y/x\end{aligned}$$

Thus $\sin_2$, $\cos_2$ and $\tan_2$ are the familiar (i.e. trigonometric) circular functions. The other basic functions are defined in the usual way:

$$\begin{aligned}\csc_p\alpha &= r/y\\ \sec_p\alpha &= r/x\\ \cot_p\alpha &= x/y\end{aligned}$$

For simplicity we will generally take $r = 1$. These functions may be called the "L_p-sine," "L_p-cosine," etc., or more briefly, "p-sine," "p-cosine," etc. Thus the familiar functions are the 2-sine, 2-cosine, etc. Figures 4 and 5 show the 3-sine and 5-sine as functions of angles in 3-radians and 5-radians, respectively.

[1] This table was produced by numerical integration of $xy + 2\int_x^1 y\mathrm{d}x$ where $y = (1 - x^p)^{1/p}$ and $x = (1 + \tan^p\theta)^{-1/p}$. The $p = \infty$ case is discussed later.

Table 1: Common Angles in L_p Radians.

angle	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = \infty$
0°	0.	0.	0.	0.	0.
10°	0.149896	0.174533	0.176166	0.17631	0.173648
20°	0.266846	0.349066	0.361111	0.363337	0.34202
30°	0.366025	0.523599	0.56035	0.571215	0.5
40°	0.456256	0.698132	0.773647	0.804238	0.642788
45°	0.5	0.785398	0.883319	0.927037	1.
50°	0.543744	0.872665	0.992992	1.04984	1.35721
60°	0.633975	1.0472	1.20629	1.28286	1.5
70°	0.733154	1.22173	1.40553	1.49074	1.65798
80°	0.850104	1.39626	1.59047	1.67776	1.82635
90°	1.	1.5708	1.76664	1.85407	2.

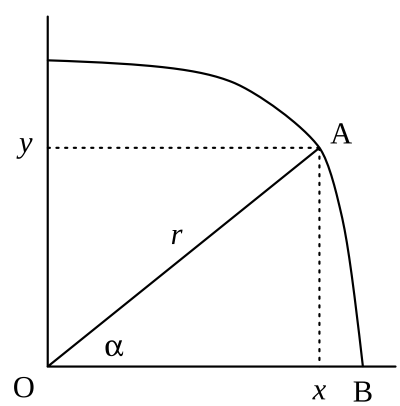

Figure 3: Definition of L_p-Circular Functions

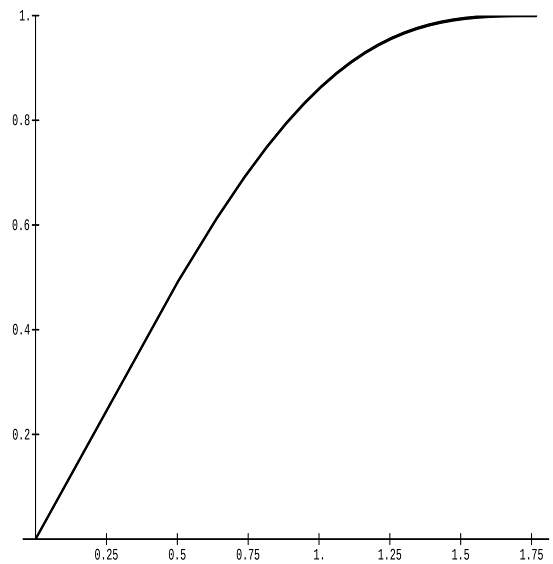

Figure 4: The L_3 Sine Function. A plot of $\sin_3 \alpha$ for $\alpha \in [0, \pi_3/2]$, α in units of L_3 radians.

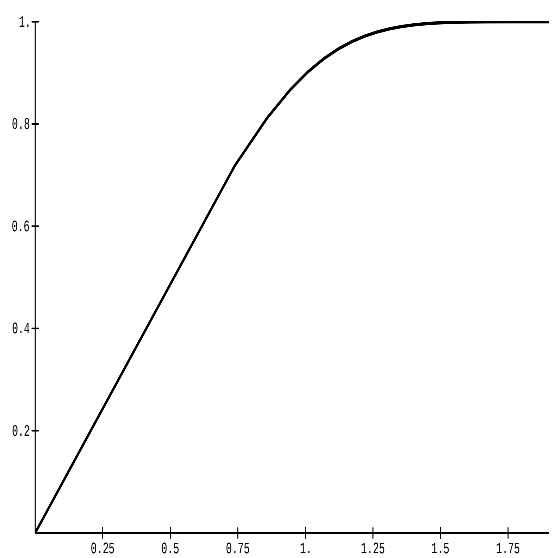

Figure 5: The L_5 Sine Function. A plot of $\sin_5 \alpha$ for $\alpha \in [0, \pi_5/2]$, α in units of L_5 radians.

2 Basic Identities

The most basic identities follow directly from the definition of an L_p circle:

$$\begin{aligned} |\sin_1 \alpha| + |\cos_1 \alpha| &= 1 \\ \sin_2^2 \alpha + \cos_2^2 \alpha &= 1 \\ &\vdots \\ |\sin_p \alpha|^p + |\cos_p \alpha|^p &= 1 \\ &\vdots \\ \max(|\sin_\infty \alpha|, |\cos_\infty \alpha|) &= 1 \end{aligned}$$

The latter equation is the usual $p \to \infty$ limit of the L_p circle.

Dividing the above equations by $|\cos_p \alpha|^p$ produces another group of identities:

$$\begin{aligned} 1 + |\tan_1 \alpha| &= |\sec_1 \alpha| \\ 1 + \tan_2^2 \alpha &= \sec_2^2 \alpha \\ &\vdots \\ 1 + |\tan_p \alpha|^p &= |\sec_p \alpha|^p \\ &\vdots \\ \max(1, |\tan_\infty \alpha|) &= |\sec_\infty \alpha| \end{aligned}$$

The last equation requires some verification, which we leave for later (Section 8). Dividing instead by $|\sin_p \alpha|^p$ yields:

$$\begin{aligned} 1 + |\cot_1 \alpha| &= |\csc_1 \alpha| \\ 1 + \cot_2^2 \alpha &= \csc_2^2 \alpha \\ &\vdots \\ 1 + |\cot_p \alpha|^p &= |\csc_p \alpha|^p \\ &\vdots \\ \max(1, |\cot_\infty \alpha|) &= |\csc_\infty \alpha| \end{aligned}$$

Notice that except for L_2 circles, L_p circles do not have rotational symmetry, which means, for example, that sum and difference of angle formulas cannot be derived geometrically.[2] L_p circles do, however, have other symmetries that are useful in deriving identities. In particular, they are symmetric through 90 degree rotations and through reflections across the axes. This leads to familiar

[2] In particular, angular measure is not rotationally invariant. For example, let α_p, β_p, γ_p be three angles measured in p-radians, and let α_q, β_q, γ_q be the same angles in q-radians. From $\alpha_p + \beta_p = \gamma_p$ we cannot conclude $\alpha_q + \beta_q = \gamma_q$.

Table 2: Fractions of L_p Circle in Various L_p-Circular Measures

p	$90°$	$180°$	$360°$
$1/2$	$1/3$	$\pi_{1/2} = 2/3$	$4/3$
1	1	$\pi_1 = 2$	4
2	$\pi/2$	$\pi_2 = \pi$	2π
3	$1.77\ldots$	$\pi_3 = 3.53\ldots$	$7.07\ldots$
4	$1.85\ldots$	$\pi_4 = 3.71\ldots$	$7.42\ldots$
∞	2	$\pi_\infty = 4$	8

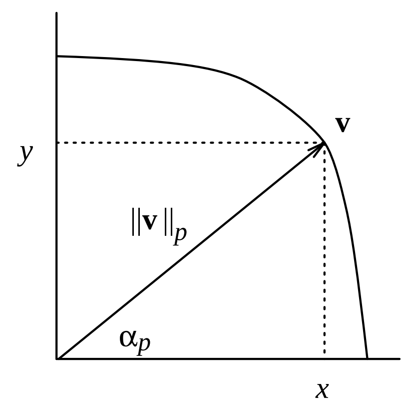

Figure 6: L_p-Circular Functions and Vectors

looking identities. Here we let π_p represent two right angles measured in L_p-circular radians (Table 2); thus:

$$\pi_p = 2\int_0^1 y\mathrm{d}x = 2\int_0^1 (1-x^p)^{1/p}\mathrm{d}x.$$

We immediately see:

$$\begin{aligned}
\sin_p \alpha &= \cos_p(\pi_p/2-\alpha) &&= \sin_p(\pi_p-\alpha)\\
\cos_p \alpha &= \sin_p(\pi_p/2-\alpha) &&= -\cos_p(\pi_p-\alpha)\\
\tan_p \alpha &= \cot_p(\pi_p/2-\alpha) &&= -\tan_p(\pi_p-\alpha)
\end{aligned}$$

$$\begin{aligned}
\sin_p(\pi_p/4\pm\alpha) &= \cos_p(\pi_p/4\mp\alpha)\\
\tan_p(\pi_p/4\pm\alpha) &= \cot_p(\pi_p/4\mp\alpha)
\end{aligned}$$

3 Relations Between Functions of Different p

In this section we derive some simple relations between the L_p and L_q functions for $p \neq q$. To do this, consider a vector $\mathbf{v}$, which lies on an L_p circle with radius $\|\mathbf{v}\|_p$, the L_p norm of $\mathbf{v}$ (Fig. 6). From the definitions of the L_p-circular functions we know

$$\begin{aligned} x &= \|\mathbf{v}\|_p \cos_p \alpha_p \\ y &= \|\mathbf{v}\|_p \sin_p \alpha_p \\ y/x &= \tan_p \alpha_p, \end{aligned}$$

where the subscript on α_p emphasizes the fact that the angle is measured in L_p radians. Since these equations hold for any p, $0 < p < \infty$, we have the following equations for $p \neq q$:

$$\|\mathbf{v}\|_p \cos_p \alpha_p = \|\mathbf{v}\|_q \cos_q \alpha_q \tag{2}$$

$$\|\mathbf{v}\|_p \sin_p \alpha_p = \|\mathbf{v}\|_q \sin_q \alpha_q \tag{3}$$

$$\tan_p \alpha_p = \tan_q \alpha_q. \tag{4}$$

The last equation reflects the fact that all the tangent functions measure the slope, which is independent of p. Of course it does not mean that $\tan_p$ and $\tan_q$ are the same function, since the equality holds only when the angles are measured in *natural units*, that is, radians consistent with the functions. This equation does, however, give us a formula for converting between different angular measures:

$$\alpha_p = \operatorname{arctan}_p(\tan_q \alpha_q).$$

4 Derivatives

From the definition of L_p-angular measure (Eq. 1) we have (for $r = 1$)

$$\alpha = xy - 2\int_1^x y \mathrm{d}x$$

and so the differential is

$$\mathrm{d}\alpha = x\mathrm{d}y - y\mathrm{d}x. \tag{5}$$

Dividing by $\mathrm{d}\alpha$ yields

$$1 = x\frac{\mathrm{d}y}{\mathrm{d}\alpha} - y\frac{\mathrm{d}x}{\mathrm{d}\alpha}.$$

Solving for $\mathrm{d}y/\mathrm{d}\alpha$:

$$\frac{\mathrm{d}y}{\mathrm{d}\alpha} = \frac{y\,\mathrm{d}x/\mathrm{d}\alpha + 1}{x}. \tag{6}$$

But we also know[3] $1 = x^p + y^p$. Taking derivatives:

$$0 = px^{p-1}\frac{\mathrm{d}x}{\mathrm{d}\alpha} + py^{p-1}\frac{\mathrm{d}y}{\mathrm{d}\alpha}.$$

[3]For convenience we restrict attention to the first quadrant; the results are easily extended by symmetry.

Substituting Eq. 6 into this yields:

$$x^{p-1}\frac{\mathrm{d}x}{\mathrm{d}\alpha} + y^{p-1}\frac{y\,\mathrm{d}x/\mathrm{d}\alpha + 1}{x} = 0.$$

Multiply through by x:

$$x^p\frac{\mathrm{d}x}{\mathrm{d}\alpha} + y^p\frac{\mathrm{d}x}{\mathrm{d}\alpha} + y^{p-1} = 0.$$

Hence,

$$(x^p + y^p)\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -y^{p-1}$$

and so,

$$\mathrm{d}x/\mathrm{d}\alpha = -y^{p-1}. \tag{7}$$

The derivative of the L_p-cosine is thus

$$\mathrm{D}_\alpha \cos_p \alpha = -\sin_p^{p-1} \alpha, \tag{8}$$

and we have:

$$\begin{aligned} \mathrm{D}_\alpha \cos_1 \alpha &= -1 \\ \mathrm{D}_\alpha \cos_2 \alpha &= -\sin_2 \alpha \\ \mathrm{D}_\alpha \cos_3 \alpha &= -\sin_3^2 \alpha \\ &\vdots \end{aligned}$$

By a similar derivation we can get the formula for $\mathrm{D}_\alpha \sin_p \alpha$:

$$\mathrm{D}_\alpha \sin_p \alpha = \cos_p^{p-1} \alpha. \tag{9}$$

Thus we have:

$$\begin{aligned} \mathrm{D}_\alpha \sin_1 \alpha &= 1 \\ \mathrm{D}_\alpha \sin_2 \alpha &= \cos_2 \alpha \\ \mathrm{D}_\alpha \sin_3 \alpha &= \cos_3^2 \alpha \\ &\vdots \end{aligned}$$

Straight-forward differentiation yields

$$\mathrm{D}_\alpha \tan_p \alpha = \sec_p^2 \alpha \tag{10}$$

and

$$\mathrm{D}_\alpha \cot_p \alpha = -\csc_p^2 \alpha. \tag{11}$$

The derivatives allow us to derive closed forms for the inverse functions. First rewrite Eq. 7:

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -y^{p-1} = -[(1-x^p)^{1/p}]^{p-1} = -(1-x^p)^{(p-1)/p}.$$

Therefore,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = -(1 - x^p)^{p/(p-1)},$$

and so,

$$\alpha = x \mid_{\alpha=0} - \int_0^x (1 - x^p)^{p/(p-1)} \mathrm{d}x.$$

Thus the principal value of the L_p-arccosine is given by:

$$\operatorname{arccos}_p x = 1 - \int_0^x (1 - x^p)^{p/(p-1)} \mathrm{d}x. \tag{12}$$

Similarly, from Eq. 9 we get:

$$\operatorname{arcsin}_p y = \int_0^y (1 - y^p)^{(1-p)/p} \mathrm{d}y. \tag{13}$$

5 Power Series

The formulas for the derivatives of the L_p-circular functions allow the derivation of their MacLauren series expansions. For the cosines we have:

$$\begin{aligned}
\cos_1 \alpha &= 1 - \alpha \\
\cos_2 \alpha &= 1 - \frac{1}{2}\alpha^2 + \frac{1}{4!}\alpha^4 - \frac{1}{6!}\alpha^6 + \frac{1}{8!}\alpha^8 - \mathcal{O}(\alpha^{10}) \\
\cos_3 \alpha &= 1 - \frac{1}{3}\alpha^3 + \frac{1}{18}\alpha^6 - \frac{23}{2268}\alpha^9 + \frac{25}{13608}\alpha^{12} - \mathcal{O}(\alpha^{15}) \\
\cos_4 \alpha &= 1 - \frac{1}{4}\alpha^4 + \frac{9}{160}\alpha^8 - \frac{149}{9600}\alpha^{12} + \frac{15147}{3328000}\alpha^{16} - \mathcal{O}(\alpha^{20}) \\
\cos_5 \alpha &= 1 - \frac{1}{5}\alpha^5 + \frac{4}{75}\alpha^{10} - \frac{224}{12375}\alpha^{15} + \frac{4957}{742500}\alpha^{20} - \mathcal{O}(\alpha^{25}) \\
\cos_6 \alpha &= 1 - \frac{1}{6}\alpha^6 + \frac{25}{504}\alpha^{12} - \frac{15775}{825552}\alpha^{18} + \frac{21301825}{2635161984}\alpha^{24} - \mathcal{O}(\alpha^{30})
\end{aligned}$$

In general, one can show the MacLauren series for the $\cos_p$ is:

$$\cos_p \alpha = 1 - \frac{1}{p}\alpha^p + \frac{U_{2p}^p}{(2p)!}\alpha^{2p} - \mathcal{O}(\alpha^{3p}),$$

where the first coefficient derives from $(p-1)!/p! = 1/p$, and the numbers U_{2p}^p are defined recursively:

$$\begin{aligned}
U_2^p &= 0, \\
U_{k+1}^p &= (2p - k)U_k^p + (k-1)(p-1)(p-1)!/(p-k)!, \quad k \geq 2.
\end{aligned}$$

See Table 3 for examples.

Table 3: The Numbers U_k^p for $k = 2, \ldots, 20$ and $p = 1, \ldots, 10$

$k \setminus p$	1	2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
3		1	4	9	16	25	36	49	64	81
4		1	20	81	208	425	756	1225	1856	2673
5			40	378	1536	4300	9720	19110	34048	56376
6			40	1134	8064	32500	96120	233730	496384	954504
7				2268	32256	198000	790560	2425500	6225408	14043456
8				2268	96768	990000	5559840	22041180	69447168	185830848
9					193536	3960000	33359040	176576400	696729600	2241400896
10					193536	11880000	166795200	1236034800	6273146880	24681537216
11						23760000	667180800	7416208800	50185175040	246844765440
12						23760000	2001542400	37081044000	351296225280	2221602888960
13							4003084800	148324176000	2107777351680	17772823111680
14							4003084800	444972528000	10538886758400	124409761781760
15								889945056000	42155547033600	746458570690560
16								889945056000	126466641100800	3732292853452800
17									252933282201600	14929171413811200
18									252933282201600	44787514241433600
19										89575028482867200
20										89575028482867200

For the sines we have the MacLauren series:

$$\begin{aligned}
\sin_1 \alpha &= \alpha \\
\sin_2 \alpha &= \alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 - \frac{1}{7!}\alpha^7 + \frac{1}{9!}\alpha^9 - \mathcal{O}(\alpha^{11}) \\
\sin_3 \alpha &= \alpha - \frac{1}{6}\alpha^4 + \frac{2}{63}\alpha^7 - \frac{13}{2268}\alpha^{10} + \frac{23}{22113}\alpha^{13} - \mathcal{O}(\alpha^{16}) \\
\sin_4 \alpha &= \alpha - \frac{3}{20}\alpha^5 + \frac{19}{480}\alpha^9 - \frac{469}{41600}\alpha^{13} + \frac{189611}{56576000}\alpha^{17} - \mathcal{O}(\alpha^{21}) \\
\sin_5 \alpha &= \alpha - \frac{2}{15}\alpha^6 + \frac{34}{825}\alpha^{11} - \frac{719}{49500}\alpha^{16} + \frac{3034}{556875}\alpha^{21} - \mathcal{O}(\alpha^{26}) \\
\sin_6 \alpha &= \alpha - \frac{5}{42}\alpha^7 + \frac{265}{6552}\alpha^{13} - \frac{253595}{15685488}\alpha^{19} + \frac{91657229}{13175809920}\alpha^{25} - \mathcal{O}(\alpha^{31})
\end{aligned}$$

The general MacLauren series takes the form:

$$\sin_p \alpha = \alpha - \frac{p-1}{p(p+1)}\alpha^{p+1} + \frac{V_{2p}^p}{(2p+1)!}\alpha^{2p+1} - \mathcal{O}(\alpha^{3p+1}).$$

The coefficient $(p-1)/(p+1)p$ derives from $(p-1)(p-1)!/(p+1)!$, and the numbers V_{2p}^p are defined recursively:

$$\begin{aligned}
V_1^p &= 0, \\
V_{k+1}^p &= (2p-k)V_k^p + (p-1)(kp-k-1)(p-1)!/(p-k)!, \quad k \geq 1.
\end{aligned}$$

See Table 4 for examples.

The equation of the tangents (Eq. 4) allows us to derive useful power series for the L_p-circular functions. We have seen that (in the first quadrant) $1 + \tan_p^p \alpha_p = \sec_p^p \alpha_p$; therefore, $\cos_p \alpha_p = (1 + \tan_p^p \alpha_p)^{-1/p}$. Since it doesn't

Table 4: The Numbers V_k^p for $k = 1, \ldots, 20$ and $p = 1, \ldots, 10$

$k \setminus p$	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0	0	2	6	12	20	30	42	56	72
3		1	20	81	208	425	756	1225	1856	2673
4		1	80	549	1984	5225	11376	21805	38144	62289
5			160	2394	13344	47500	130320	301350	617344	1155384
6			160	7182	68544	346900	1235520	3514770	8549632	18528264
7				14364	274176	2098800	10035360	35870940	105122304	266607936
8				14364	822528	10494000	70424640	324531900	1165215744	3499651008
9					1645056	41976000	422547840	2598195600	11672478720	42111752256
10					1645056	125928000	2112739200	18187369200	105075210240	463490548416
11						251856000	8450956800	109124215200	840601681920	4635196151040
12						251856000	25352870400	545621076000	5884211773440	41716765359360
13							50705740800	2182484304000	35305270640640	333734122874880
14							50705740800	6547452912000	176526353203200	2336138860124160
15								13094905824000	706105412812800	14016833160744960
16								13094905824000	2118316238438400	70084165803724800
17									4236632476876800	280336663214899200
18									4236632476876800	841009989644697600
19										1682019979289395200
20										1682019979289395200

matter which tangent we use, let $t = \tan_p \alpha_p = y/x = \tan_q \alpha_q$, and expand $\cos_p \alpha_p = (1 + t^p)^{-1/p}$ by the binomial theorem:

$$\cos_p \alpha_p = 1 - \frac{1}{p} t^p + \frac{(1-p)}{2! p^2} t^{2p} - \frac{(1-p)(1-2p)}{3! p^3} t^{3p} + \cdots,$$

which converges in the first quadrant for $t < 1$, that is, for $\alpha_p < \pi_p/4$. Similarly,

$$\sin_p \alpha_p = 1 - \frac{1}{p} t^{-p} + \frac{(1-p)}{2! p^2} t^{-2p} - \frac{(1-p)(1-2p)}{3! p^3} t^{-3p} + \cdots,$$

which converges in the first quadrant for $t > 1$, that is, for $\alpha_p > \pi_p/4$.

6 Exponential Forms

We consider complex numbers $x + iy$ that lie on the L_p unit circle in the complex plane (Fig. 3). These points have the form $\cos_p \alpha_p + i \sin_p \alpha_p$, for an angle α_p measured in L_p radians. By analogy with the L_2-circular functions, we define the L_p-exponential function on imaginary values by:

$$\exp_p(i\alpha_p) = \cos_p \alpha_p + i \sin_p \alpha_p. \tag{14}$$

Clearly, then $\exp_p(\pi_p i) = -1$ and $\exp_p(2\pi_p i) = 1$. As α_p increases, the complex number $\exp_p(i\alpha_p)$ rotates around an L_p circle in the complex plane. Since $\sin_p(-\alpha_p) = -\sin_p \alpha_p$ and $\cos_p(-\alpha_p) = \cos_p \alpha_p$, it immediately follows that

$$\begin{aligned} \sin_p \alpha_p &= \frac{\exp_p(i\alpha_p) - \exp_p(-i\alpha_p)}{2i}, \\ \cos_p \alpha_p &= \frac{\exp_p(i\alpha_p) + \exp_p(-i\alpha_p)}{2}. \end{aligned}$$

By using the Eq. 14 and the MacLauren series for $\sin_p$ and $\cos_p$ we can derive the following power series for $\exp_p$:

$$\begin{aligned}
\exp_1 z &= 1 + z + iz \\
\exp_2 z &= 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \frac{1}{6!}z^6 + \mathcal{O}(z^7) \\
\exp_3 z &= 1 + z - \frac{i}{3}z^3 - \frac{i}{6}z^4 - \frac{1}{18}z^6 - \frac{2}{63}z^7 + \frac{23i}{2268}z^9 + \frac{13i}{2268}z^{10} + \\
&\quad \frac{25}{13608}z^{12} + \frac{23}{22113}z^{13} - \mathcal{O}(z^{15}) \\
\exp_4 z &= 1 + z - \frac{1}{4}z^4 - \frac{3}{20}z^5 + \frac{9}{160}z^8 + \frac{19}{480}z^9 - \frac{149}{9600}z^{12} - \frac{469}{41600}z^{13} + \\
&\quad \frac{15147}{3328000}z^{16} + \frac{189611}{56576000}z^{17} - \frac{4679969}{3394560000}z^{20} - \frac{1157629}{1131520000}z^{21} + \\
&\quad \mathcal{O}(z^{24}) \\
\exp_5 z &= 1 + z + \frac{i}{5}z^5 + \frac{2i}{15}z^6 - \frac{4}{75}z^{10} - \frac{34}{825}z^{11} - \frac{224i}{12375}z^{15} - \frac{719i}{49500}z^{16} + \\
&\quad \frac{4957}{742500}z^{20} + \frac{3034}{556875}z^{21} + \mathcal{O}(z^{25}) \\
\exp_6 z &= 1 + z + \frac{1}{6}z^6 + \frac{5}{42}z^7 + \frac{25}{504}z^{12} + \frac{265}{6552}z^{13} + \frac{15775}{825552}z^{18} + \\
&\quad \frac{253595}{15685488}z^{19} + \frac{21301825}{2635161984}z^{24} + \frac{91657229}{13175809920}z^{25} + \mathcal{O}(z^{30})
\end{aligned}$$

In general,

$$\exp_p z = 1 + z - \frac{i^{-p}}{p}z^p - \frac{i^{-p}(p-1)}{p(p+1)}z^{p+1} + \frac{i^{-2p}U_{2p}^p}{(2p)!}z^{2p} + \frac{i^{-2p}V_{2p}^p}{(2p+1)!}z^{2p+1} - i^{-3p}\mathcal{O}(z^{3p}),$$

where the numbers U_{2p}^p, V_{2p}^p are defined in Section 5.

7 The Case $p = 1$

The $p = 1$ case is especially interesting, first, because it is so simple that it permits direct analysis, second, because of the importance of L_1 spaces. For example, because a probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ must satisfy $\sum p_k = 1$, it is a point on the positive orthant of an n-dimensional L_1 unit circle. Further, the probabilities p_k are just the L_1 direction cosines of the vector $\mathbf{p}$:

$$\cos_1 \alpha_k = p_k,$$

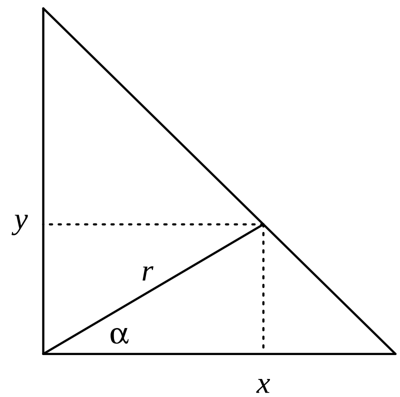

Figure 7: Definition of L_1-Circular Functions

where α_k is the angle (measured in L_1 radians) between $\mathbf{p}$ and the k-th axis. Thus we may apply the L_1-circular functions to the analysis of probability distributions.

Figure 7 illustrates the definition of the L_1-circular functions. In the first quadrant (where probability distributions lie) these functions take a very simple form:

$$\sin_1 \alpha = \alpha \tag{15}$$

$$\cos_1 \alpha = 1 - \alpha \tag{16}$$

$$\tan_1 \alpha = \frac{\alpha}{1-\alpha} \tag{17}$$

Thus, $\sin_1 \alpha$ is the probability of a thing happening, $\cos_1 \alpha$ is its probability of not happening, and $\tan_1 \alpha$ is its *odds* of happening.[4]

From Eq. 17 it's easy to show that

$$\alpha_1 = \frac{\tan_1 \alpha_1}{1 + \tan_1 \alpha_1}.$$

Therefore we have an explicit formula for the arctangent in the first quadrant:

$$\arctan_1 t = \frac{t}{1+t}.$$

Since all tangents are the same (in natural units), we then have formulas for converting between L_1 and L_p angular measures:

$$\alpha_1 = \frac{\tan_p \alpha_p}{1 + \tan_p \alpha_p}, \qquad \alpha_p = \arctan_p \left(\frac{\alpha_1}{1 - \alpha_1} \right).$$

When one considers all quadrants, it is apparent that $\sin_1$ is a triangular wave and that $\cos_1$ is the same, but delayed $\pi_1/2$ in phase.

[4]I.e., what Peirce called the *chance* of an event (Buchler, Justus (ed.). *Philosophical Writings of Peirce.* New York, NY: Dover, 1955. Previously published as *The Philosophy of Peirce: Selected Writings.* London, UK: Routledge and Kegan Paul, 1940, p. 177).

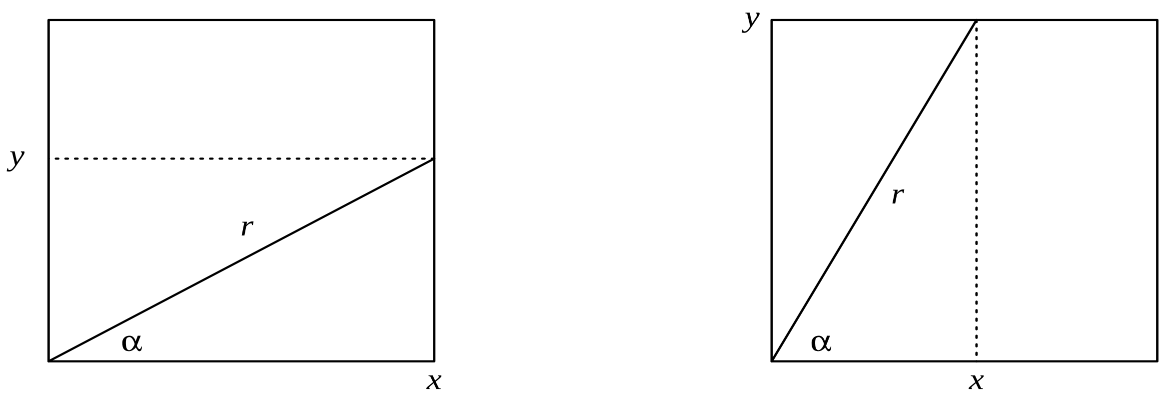

Figure 8: Definition of L_∞-Circular Functions

8 The Case $p = \infty$

Although the case $p = \infty$ is special, it is consistent with the usual interpretation

$$\sqrt[\infty]{|x|^\infty + |y|^\infty} = \lim_{p\to\infty} (|x|^p + |y|^p)^{1/p} = \max(|x|, |y|).$$

Again, for simplicity we limit our attention to the first quadrant. Then direct inspection of Fig. 8 shows that in the first quadrant:

$$\begin{aligned} \sin_\infty \alpha &= \min(\alpha, 1) \\ \cos_\infty \alpha &= \min(1, 2-\alpha) \\ \tan_\infty \alpha &= \begin{cases} \alpha & \text{if } 0 \le \alpha \le 1 \\ \frac{1}{2-\alpha} & \text{if } 1 \le \alpha < 2 \end{cases} \end{aligned}$$

With these formulas it's easy to verify the equations $\max(1, \tan_\infty \alpha) = \sec_\infty \alpha$ and $\max(1, \cot_\infty \alpha) = \csc_\infty \alpha$ given in Section 2. For example, for $0 \le \alpha \le 1$, $\max(1, \tan_\infty \alpha) = 1 = 1/\cos_\infty \alpha$. Conversely, for $1 \le \alpha \le 2$, $\max(1, \tan_\infty \alpha) = 1/(1-\alpha) = 1/\cos_\infty \alpha$.

When extended to the entire real axes, the $\sin_\infty$ and $\cos_\infty$ functions become "trapezoid waves" (truncated triangular waves).

The analysis of the L_∞ functions is simplified by the use of the *Heaviside* or *unit step* function:

$$u(s) = \begin{cases} 1 & \text{if } s \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

We will also need the integral of the Heaviside function:

$$U(s) = \int_{-\infty}^{s} u(s)\mathrm{d}s = \begin{cases} s & \text{if } s \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Also note $U(s) = su(s)$.

The L_∞ functions in the first quadrant can now be expressed in these terms:

$$\begin{aligned}
\sin_\infty \alpha &= 1 - U(1-\alpha) \\
\cos_\infty \alpha &= 1 - U(\alpha - 1) \\
\tan_\infty \alpha &= \frac{u(\alpha-1)}{2-\alpha} - \alpha u(1-\alpha) \\
\alpha &= 1 - U(1-y) + U(x-1)
\end{aligned}$$

From the relation $u(s) = U'(s)$ we can then derive the derivatives of the L_∞ functions:

$$\begin{aligned}
\mathrm{D}_\alpha \sin_\infty \alpha &= u(1-\alpha) \\
\mathrm{D}_\alpha \cos_\infty \alpha &= -u(\alpha-1)
\end{aligned}$$

A Table of Formulas

A.1 Definitions

In the following x, y and r indicate the sides of a right triangle, and α indicates the angle at the origin in L_p radians, as shown in Fig. 3. In this appendix we assume $0 < p < \infty$.

1. $|x|^p + |y|^p = r^p$
2. $\mathrm{d}\alpha = x\mathrm{d}y - y\mathrm{d}x$
3. $\sin_p \alpha = y/r$
4. $\cos_p \alpha = x/r$
5. $\tan_p \alpha = y/x$
6. $\csc_p \alpha = r/y$
7. $\sec_p \alpha = r/x$
8. $\cot_p \alpha = x/y$
9. $\pi_p \ = 2\int_0^1 y\mathrm{d}x = 2\int_0^1 (1 - x^p)^{1/p}\mathrm{d}x$

A.2 Identities

1. $|\sin_p \alpha|^p + |\cos_p \alpha|^p = 1$
2. $1 + |\tan_p \alpha|^p = |\sec_p \alpha|^p$
3. $1 + |\cot_p \alpha|^p = |\csc_p \alpha|^p$
4. $\sin_p \alpha = \pm\sqrt[p]{1 - \cos_p^p \alpha}$
5. $\cos_p \alpha = \pm\sqrt[p]{1 - \sin_p^p \alpha}$
6. $\tan_p \alpha = \pm\sqrt[p]{\sec_p^p \alpha - 1}$
7. $\sec_p \alpha = \pm\sqrt[p]{1 + \tan_p^p \alpha}$
8. $\cot_p \alpha = \pm\sqrt[p]{\csc_p^p \alpha - 1}$
9. $\csc_p \alpha = \pm\sqrt[p]{1 + \cot_p^p \alpha}$
10. $\sin_p \alpha = 1/\csc_p \alpha$
11. $\cos_p \alpha = 1/\sec_p \alpha$
12. $\tan_p \alpha = 1/\cot_p \alpha = \sin_p \alpha/\cos_p \alpha$
13. $\sec_p \alpha = 1/\cos_p \alpha$
14. $\csc_p \alpha = 1/\sin_p \alpha$
15. $\cot_p \alpha = 1/\tan_p \alpha = \cos_p \alpha/\sin_p \alpha$

16. $\sin_p(-\alpha) = -\sin_p \alpha$
17. $\cos_p(-\alpha) = \cos_p \alpha$
18. $\tan_p(-\alpha) = -\tan_p \alpha$
19. $\sin_p \alpha = \cos_p(\pi_p/2 - \alpha) = \sin_p(\pi_p - \alpha)$
20. $\cos_p \alpha = \sin_p(\pi_p/2 - \alpha) = -\cos_p(\pi_p - \alpha)$
21. $\tan_p \alpha = \cot_p(\pi_p/2 - \alpha) = -\tan_p(\pi_p - \alpha)$
22. $\sin_p(\pi_p/4 \pm \alpha) = \cos_p(\pi_p/4 \mp \alpha)$
23. $\tan_p(\pi_p/4 \pm \alpha) = \cot_p(\pi_p/4 \mp \alpha)$
24. $\|\mathbf{v}\|_p \cos_p \alpha_p = \|\mathbf{v}\|_q \cos_q \alpha_q$
25. $\|\mathbf{v}\|_p \sin_p \alpha_p = \|\mathbf{v}\|_q \sin_q \alpha_q$
26. $\tan_p \alpha_p = \tan_q \alpha_q$
27. $\alpha_p = \arctan_p(\tan_q \alpha_q)$
28. $\exp_p(i\alpha_p) = \cos_p \alpha_p + i \sin_p \alpha_p$
29. $\sin_p \alpha_p = \dfrac{\exp_p(i\alpha_p) - \exp_p(-i\alpha_p)}{2i}$
30. $\cos_p \alpha_p = \dfrac{\exp_p(i\alpha_p) + \exp_p(-i\alpha_p)}{2}$
31. $\mathrm{arccos}_p\, x = 1 - \displaystyle\int_0^x (1 - x^p)^{p/(p-1)} \mathrm{d}x$
32. $\mathrm{arcsin}_p\, y = \displaystyle\int_0^y (1 - y^p)^{(1/p)-1} \mathrm{d}y$

A.3 Special Values

1. $\sin_p 0 = \cos_p(\pi_p/2) = \sin_p \pi_p = \cos_p(3\pi_p/2) = 0$
2. $\tan_p 0 = \cot(\pi_p/2) = \tan_p \pi_p = \cot_p(3\pi_p/2) = 0$
3. $\cos_p 0 = \sin_p(\pi_p/2) = 1$
4. $\cos_p \pi_p = \sin_p(3\pi_p/2) = -1$
5. $\exp_p[(\pi_p/2)i] = i$
6. $\exp_p(\pi_p i) = -1$
7. $\exp_p(2\pi_p i) = 1$

A.4 Derivatives

1. $\mathrm{D}_\alpha \sin_p \alpha = \cos_p^{p-1} \alpha$
2. $\mathrm{D}_\alpha \cos_p \alpha = -\sin_p^{p-1} \alpha$
3. $\mathrm{D}_\alpha \tan_p \alpha = \sec_p^2 \alpha$
4. $\mathrm{D}_\alpha \cot_p \alpha = -\csc_p^2 \alpha$