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Lorentz Invariance and the Retarded Bohm Model

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Abstract

We show how a recently introduced retarded version of the Bohm Model evades the Hardy proof that hidden-variable models must violate Lorentz-Invariance. We also discuss a possible test of such models.

1 Introduction

Since the work of Bell [1], it has been known that any hidden-variable model of quantum theory must be non-local. That it must also violate Lorentz-invariance was shown by Hardy [2] (see also [3]). Recently one of us has shown how the Bohm model can be modified so that it becomes Lorentz invariant [4]. Clearly this modified model, which we call the Retarded Bohm Model, will violate quantum theory. It is not clear, however, whether it violates the results of any actual experiments.

The purpose of the present note is to show how the Retarded Bohm Model evades the Hardy proof of the lack of Lorentz-invariance of hidden-variable models. The work reveals another possible test of the retarded model, which is briefly discussed in the final section.

2 The Mach-Zehnder interferometer

A particle enters at point A (see fig.1) and encounters a beam splitter, which separates the wave-packet into two parts of equal magnitude:

$$|\psi> \rightarrow \frac{1}{\sqrt{2}}(|u_0> + |v_0>). \qquad (1)$$

The wave-function then evolves with time according to

$$|\psi_t> = \frac{1}{\sqrt{2}}(|u_t> + |v_t>), \qquad (2)$$

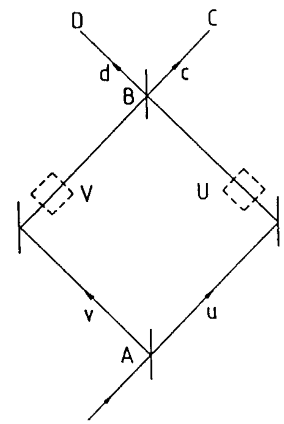

Figure 1: A Mach-Zehnder type interferometer

until either it is observed by the detectors, U or V, or the wave-packets reach region B. Here there is a second beam splitter. If T is the time when the wave-packet reaches B, then, for $t > T$, it can be arranged that

$$|u_t> \rightarrow \frac{1}{\sqrt{2}}(|c_t> + |d_t>) \qquad (3)$$

and

$$|v_t> \rightarrow \frac{1}{\sqrt{2}}(|c_t> - |d_t>), \qquad (4)$$

where path $|c>$ will cause the C detector to fire and path $|d>$ will cause the D detector to fire.

Hence, if detectors U and V are missing, we have for $t > T$

$$|\psi_t> = |c_t>. \qquad (5)$$

On the other hand, if the U(V) detector is present, it will either register the particle, in which case

$$|\psi_t> = 0, \qquad (6)$$

or it will fail to register, leading to

$$|\psi_t> = \frac{1}{\sqrt{2}}(|c_t> \pm |d_t>). \qquad (7)$$

It is possible to describe the interferometer in one space and one time dimension. Essentially this means taking time as the vertical axis in fig.1 and distance as the horizontal. The beamsplitter at $x = 0$ can be modelled by a potential $V(x) = \frac{\hbar^2}{m}\Omega\delta(x)$. A wave-packet travels towards the potential, and interacts with it at $t = 0$. This produces two wave-packets leaving A in opposite directions. Mirrors are placed at $kx = \pm 2\pi n$, where $n \in \mathbf{Z}$, and completely reflect the two parts of the wave-function back towards $x = 0$.

The solution of the scattering problem for plane waves is

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & x < 0 \\ Te^{ikx} & x > 0 \end{cases} \qquad (8)$$

where

$$R = -\frac{\Omega^2 + ik\Omega}{k^2 + \Omega^2} \quad \text{and} \quad T = \frac{k^2 - ik\Omega}{k^2 + \Omega^2}. \qquad (9)$$

A wave-packet whose momenta peaks sharply at $k = \Omega$ will be half transmited and half reflected as required. For $k = \Omega$ the above solution becomes

$$\psi(x) = \begin{cases} e^{ikx} - \frac{1+i}{2}e^{-ikx} & x < 0 \\ \frac{1-i}{2}e^{ikx} & x > 0. \end{cases} \qquad (10)$$

Here Schrödinger's Equation is invariant under $x \rightarrow -x$ and $\psi \rightarrow \psi^*$. Hence

$$\psi(x) = \begin{cases} \frac{1+i}{2}e^{ikx} & x < 0 \\ e^{ikx} - \frac{1-i}{2}e^{-ikx} & x > 0 \end{cases} \qquad (11)$$

is also a solution.

The reflection of $-\frac{1+i}{2}e^{-ikx}$ in the left mirror leads to $\frac{1+i}{2}e^{ikx}$ and the reflection of $\frac{1-i}{2}e^{ikx}$ in the right to $-\frac{1-i}{2}e^{-ikx}$. Hence, when the two wave-packets meet again at $x = 0$, they will combine constructively for $x > 0$ and destructively for $x < 0$. Thus a single wave-packet will leave $x = 0$ heading in the direction $x \rightarrow +\infty$. In other words, the *reflected* direction corresponds to the d-path, and with the single interferometer no particles will be observed on it.

3 The Hardy Experiment

Here we have two overlapping interferometers, one for electrons and one for positrons, labeled (1) and (2) respectively. This means the state will exist in 3-dimensional configuration space, with $\psi = \psi(x_1, x_2, t)$. The interferometers are made to overlap as in fig.2 so that, if the electron and positron are in the

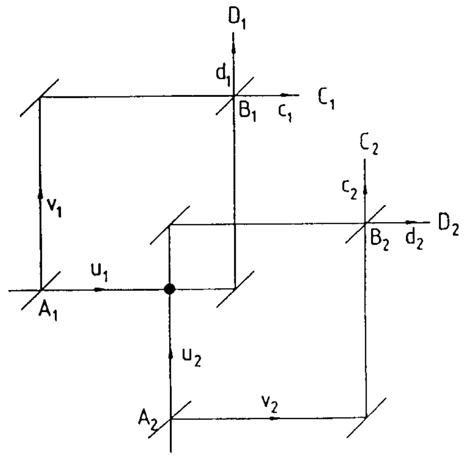

Figure 2: The Hardy Experiment with two overlapping interferometers

states $|u_1>$ and $|u_2>$, they will annihilate each other with probability 1. The quantum evolution then depends on the sign of $(T_1 - T_2)$ where T_1, T_2 are the times when the particles reach the respective interference regions B_1, B_2. We have, after the annihilation region has been passed,

$$\begin{array}{lll} \psi \rightarrow & u_1v_2 + v_1u_2 + v_1v_2 & t < T_1 \text{ and } t < T_2 \quad \text{(a)} \\ \rightarrow & \begin{cases} u_1c_2 - u_1d_2 + 2v_1c_2 & T_2 < t < T_1 \quad \text{(b)} \\ c_1u_2 - d_1u_2 + 2c_1v_2 & T_1 < t < T_2 \quad \text{(c)} \end{cases} \\ \rightarrow & 3c_1c_2 - c_1d_2 - c_2d_1 - d_1d_2 & t > T_1 \text{ and } t > T_2. \quad \text{(d)} \end{array} \qquad (12)$$

If the regions where the two $|u_i>$ and $|v_i>$ meet, B_1 and B_2, are arranged to have a space-like separation, then the description of the experiment depends on the choice of a particular time coordinate, i.e., on a particular foliation of space-time by a series of space-like hypersurfaces, each labelled by a time coordinate. Clearly it is possible for $(T_1 - T_2)$ to have either sign, according to the choice of foliation, and hence for either (b) or (c) of eq.12 to be appropriate.

To see how this leads to the Hardy contradiction with Lorentz-invariance we note first that, according to eq.12(d), there is a probability of $\frac{1}{12}$ that both particles will end up at the dark detectors, i.e., along the paths d_1, d_2. The standard Bohm model has trajectories given by

$$\dot{x}_i(x_1, x_2, S) = \mathrm{Re}\frac{p_i\psi(x_1, x_2, S)}{m\psi(x_1, x_2, S)}, \qquad (13)$$

where S is the hypersurface $t = constant$, t being the time variable in some Lorentz frame. The model is designed to give exactly the statistical predictions of orthodox quantum theory and so must contain trajectories going along these paths. Let us consider such trajectories from the point of view of a Lorentz frame in which $T_1 < T_2$. Then, from eq.12(c), it is clear that a path along d_1 requires that particle 2 is on the u_2 path. Hence, in this frame we have a unique description of the event: particle 2 went along u_2 and particle 1 went along v_1. It could not have gone along u_1, otherwise it would have been annihilated.

However, it is clear that from a different Lorentz frame in which $T_2 < T_1$, we have exactly the opposite description: particle 1 went along u_1 and particle 2 along v_2. In the Bohm model the paths actually exist, so only one of the descriptions can correspond to what actually happened. Hence there is a preferred frame of reference, in clear violation of Lorentz-invariance.

4 The Retarded Model

Here we use a similar expression to that in eq.13 except that we replace the space-like surfaces S, by the backward light-cone from the i^{th} particle. Since B_1 and B_2 have a space-like separation each particle will react as though it reached the area of interference first. Thus, using eq.12(c), when B_1 is reached by particle 1 it will go along c_1 or d_1 on the basis of whether particle 2 *was* along u_2 or v_2. In particular, it can go along d_1 only if particle 2 is on the path u_2. (Note that the time when the wave-packets reach the annihilation region is inside the backward light-cones from the times when the particles reach the respective regions B). Likewise, using now eq.12(b), particle 2 can go along d_2 only if particle 1 is on path u_1. Hence, for both particles to end up in the *dark* detectors, D_1 and D_2, they have to have followed the u_1 and u_2 paths, respectively. Since this is not possible, for they would then have annihilated, we conclude that in the retarded model there are no such events.

It follows that the Hardy proof of a lack of Lorentz-invariance cannot be used. Of course the reason why we have been able to evade the Hardy Theorem is that we have violated one of the assumptions, namely that the hidden variable model should in all cases give the results of orthodox quantum theory. The retarded model fails in this respect, so it can in principle be distinguished from orthodox quantum theory by experiment. One possible such experiment is discussed in the next section.

5 Experimental Tests

The previous section suggests a very clear experimental test of the retarded Bohm model (see [4] for other possible tests). If the Hardy Experiment could actually be performed then any event in which D_1 and D_2 record particles would immediately rule out the retarded model.

Unfortunately such experiments are not 100% efficient and it is necessary to see what happens if the annihilation mechanism is not perfect. Perhaps surprisingly it turns out that an imperfect annihilation mechanism (which of course reduces the d_1, d_2 probability in the quantum theory case) increases it from zero in the retarded model. To see this suppose the probability of a $|u_1u_2\rangle$ state not being annihilated is α^2. Then, according to orthodox quantum mechanics, we have, after the annihilation region has been passed,

$$\begin{aligned} \psi &\rightarrow \alpha u_1u_2 + u_1v_2 + v_1u_2 + v_1v_2 && t < T_1 \text{ and } t < T_2 && \text{(a)} \\ &\rightarrow \begin{cases} (1+\alpha)u_1c_2 - (1-\alpha)u_1d_2 + 2v_1c_2 & T_2 < t < T_1 \quad \text{(b)} \\ (1+\alpha)c_1u_2 - (1-\alpha)d_1u_2 + 2c_1v_2 & T_1 < t < T_2. \quad \text{(c)} \end{cases} \end{aligned} \tag{14}$$

These equations replace the previous eqs.12(a,b,c). The important point to note now is that in the retarded Bohm model the path taken by particle 2 at the second beam splitter is determined by eq.14(b). Thus it will go along d_2 only if particle 1 is on u_1 and then with probability

$$\frac{(1-\alpha)^2}{(1+\alpha)^2 + (1-\alpha)^2} = \frac{(1-\alpha)^2}{2(1+\alpha^2)}. \tag{15}$$

A similar argument holds for particle 1. Thus the probability of obtaining d_1 and d_2 in the retarded model is given by

$$\begin{aligned} P_R(d_1d_2) &= \frac{1-\alpha)^4}{4(1+\alpha^2)^2} P(u_1u_2) \\ &= \frac{\alpha^2(1-\alpha)^4}{4(3+\alpha^2)(1+\alpha^2)^2}, \end{aligned} \tag{16}$$

where the $P(u_1u_2)$ is the probability of u_1u_2 paths calculated from 14(a).

We want to compare this with the result of orthodox quantum theory. This comes from the result

$$\psi \rightarrow (3+\alpha)c_1c_2 - (1-\alpha)c_1d_2 - (1-\alpha)d_1c_2 - (1-\alpha)d_1d_2 \quad t > T_1 \text{ and } t > T_2, \tag{17}$$

which gives

$$P_S(d_1d_2) = \frac{(1-\alpha)^2}{4(3+\alpha^2)}. \tag{18}$$

It is clear that $P_S(d_1d_2) \gg P_R(d_1d_2)$. In fact:

$$\frac{P_R}{P_S} = \frac{2\alpha^3(\alpha-1)^3}{(1+\alpha^2)(\alpha^2+2\alpha-1)} \tag{19}$$

which reaches a maximum value of $(12+8\sqrt{2})^{-1} \approx 0.0429$ at $\alpha = \sqrt{2}-1 \approx 0.414$ and tends to zero as $\alpha \rightarrow 0$ and $\alpha \rightarrow 1$. It must be remembered that both

$P_R(d_1 d_2) \rightarrow 0$ and $P_S(d_1 d_2) \rightarrow 0$ as $\alpha \rightarrow 1$, but as long as the experiment has a moderate degree of efficiency the two results can be distinguished.

We are happy to have this article included in a volume honouring Jean-Pierre Vigier, who has long recognised the significance of the hidden-variable model following from the work of L.de Broglie and D.Bohm.

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