Mathematical Structures of Simple Voting Games

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Abstract

We aim to systematize the quasi-algebraic operations involving simple voting games (SVGs), by constructing an appropriate category, consisting of a class of objects and mappings (morphisms) between these objects, in terms of which all the operations involving SVGs can be defined in a natural way. But what should we take as the objects of the desired category? After trying an obvious solution, which turns out to be a dead end, we present the right solution. All the operations on SVGs fall naturally into place. We discover the remarkable central role played by the operation of SVG composition.

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Motivation

- Systematize the theory of SVGs and clarify its struture.
- Bring it into line with other mathematical theories: category theory is the *structural* foundation of mathematics.
- Find connections with other branches of mathematics and obtain new results about SVGs

Terminology, notation

By "game" I mean simple voting game. A game is an ordered pair (V, G), where V is a finite set – the set of voters, aka the *assembly* – and G is the set of winning coalitions.

I say that (V, G) is a game *on* V.

I often use sloppy notation, omitting V and writing G instead of (V, G).

I denote by L_V the set of all games on V.

Operations involving games

- Application of a game as decision rule to a division of the voters into "yes" and "no" voters.
- Composition of games, including the special cases of forming the meet and join of SVGs.
- Formation of Boolean subgames, including the special cases of forming subgames and reduced games.
- Adding dummy voters to a game.
- Transforming an SVG by forming voter blocs, whereby coalitions of voters amalgamate to form new single voters.

An obvious attempt

Let $\varphi : V \to W$ be an arbitrary map from V to the finite set W. For any game G on V, define $\mathrm{L}\varphi\mathsf{G}$ as a game on W by putting

$$\mathrm{L}\varphi\mathsf{G} := \{Y \subseteq W : \varphi^{-1}[Y] \in \mathsf{G}\}.$$

This seems promising. We do get a category whose objects are the games, and with mappings of the form $\mathrm{L}\varphi$ as morphisms. (The notation '$\mathrm{L}\varphi$' anticipates an insight that will transpire later on.)

The mapping $\mathrm{L}\varphi$ is a sort of homomorphism.

$\mathrm{L}\varphi\mathsf{G}$ is the game on W resulting from G by formation of the blocs corresponding to the partition $\{\varphi^{-1}[\{w\}] : w \in W\}$ of V.

Moreover, if $w \in W - \varphi[V]$ (ie, $\varphi^{-1}[\{w\}] = \emptyset$) then w is a dummy in $\mathrm{L}\varphi\mathsf{G}$.

If φ is injective (one-to-one) but not surjective (onto) then $\mathrm{L}\varphi\mathsf{G}$ is essentially G with added dummies.

So this takes care of bloc formation and adding dummies.

But it doesn't take care of any of the other operations: application of a game to a division of the voters, composition, Boolean subgames.

An insight:

$\mathrm{L}\varphi$ is defined "in the same way" not just for one particular game G, but for all games in L_V and it maps L_V into L_W. This is conveyed by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \big\downarrow & & \big\downarrow \\ \mathrm{L}_V & \xrightarrow[\mathrm{L}\varphi]{} & \mathrm{L}_W \end{array}$$

The significance of the downward arrows will become clear later. Moreover, L_V and L_W are lattices, in fact distributive lattices; **and $\mathrm{L}\varphi$ respects the lattice structure.**

So the idea is to look at a category **whose objects are not individual games, but lattices of the form** L_V for all finite sets V, and whose morphisms are not just mappings of the form $\mathrm{L}\varphi$ but ***all*** **mappings between these objects that respect their structure as lattices. We denote this category by G.**

This is analogous to the insight of Peano who – following ideas of Grassmann – realized that to get a satisfactory vector algebra you must take as objets not individual vectors but *vector spaces*, and focus on the mappings between vector spaces that respect their structure, namely *linear mappings*.

Recall the definition of the lattice operations in L_V

$$(V, \mathsf{G}) \vee (V, \mathsf{H}) := (V, \mathsf{G} \cup \mathsf{H}), \quad (V, \mathsf{G}) \wedge (V, \mathsf{H}) := (V, \mathsf{G} \cap \mathsf{H}).$$

Liberalizing the definition of the L_V

For technical reasons that will become apparent later, we must liberalize the definition of the L_V, admitting games that are usually excluded because they are not useful as decision rules.

First, like Taylor and Zwicker in *Simple Games*, we admit into each L_V a *bottom* and a *top* game which are, respectively, a game in which no coalition is winning, and a game in which every coalition (including the empty one!) is winning:

$$\bot_V := (V, \emptyset), \quad \top_V := (V, \wp V).$$

And we insist that morphisms of our category $\mathbf{G}$ respect these trivial games; so if $f : \mathrm{L}_V \to \mathrm{L}_W$ is a morphism of $\mathbf{G}$, it must not only respect the lattice operations $\vee$ and $\wedge$,

$$f(\mathsf{G} \vee \mathsf{H}) = f\mathsf{G} \vee f\mathsf{H}, \quad f(\mathsf{G} \wedge \mathsf{H}) = f\mathsf{G} \wedge f\mathsf{H},$$

but also obey

$$f\bot_V = \bot_W, \quad f\top_V = \top_W.$$

In addition, unlike anyone else, **we admit the degenerate object $\mathrm{L}_\emptyset$, the lattice of games without any voters.** There are exactly two such 'rubberstamp' games, $\bot_\emptyset$ and $\top_\emptyset$. They play the role of truth values, *false* and *true*.

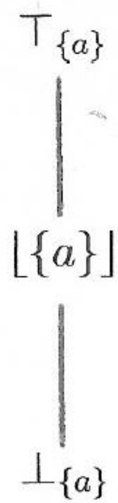
For $A \subseteq V$ we denote by $\lfloor A \rfloor$ the game that has A as its sole minimal winning coalition (MWC). In this game a bill is passed iff all members of A vote for it. The voters in $V - A$ are dummies. In lattice-algebraic terms, $\lfloor A \rfloor$ is a *principal* member of L_V.

In particular, if $a \in V$, $\lfloor \{a\} \rfloor$ is the *dictatorial* game with a as dictator.

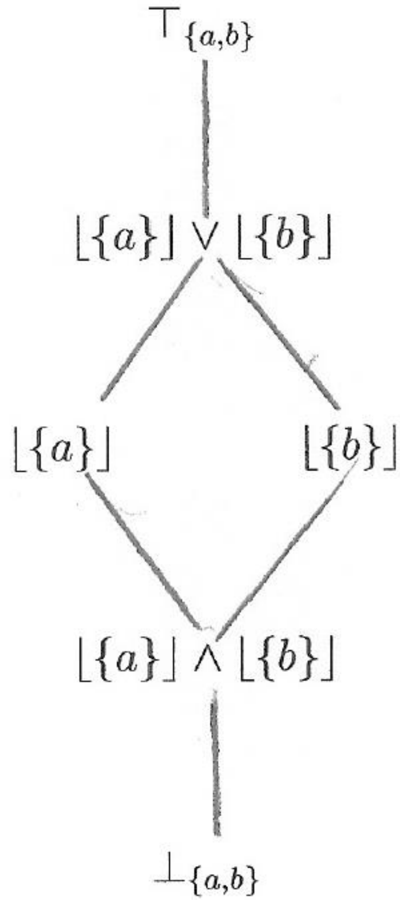
Here is what the 3 simplest objects of $\mathbf{G}$ look like:

$L_{\emptyset}$

$L_{\{a\}}$



$L_{\{a,b\}}$



$\top_{\{a,b\}}$
$\lfloor\{a\}\rfloor \vee \lfloor\{b\}\rfloor$
$\lfloor\{a\}\rfloor$
$\lfloor\{b\}\rfloor$
$\lfloor\{a\}\rfloor \wedge \lfloor\{b\}\rfloor$
$\bot_{\{a,b\}}$

Characterization of the L_V

Theorem *Any game* G *on* V *can be presented as a join of a set of pairwise incomparable principal games:*

$$\mathsf{G} = \bigvee_{i=1}^{k} \lfloor A_i \rfloor, \textit{ where } k \geq 0 \textit{ and } i \neq j \Rightarrow A_j \not\subseteq A_i.$$

Moreover, this presentation is unique (up to the order of the A_i*).*

But a principal game $\lfloor A \rfloor$ can be presented as a meet of dictatorial games:

$$\lfloor A \rfloor = \bigwedge_{x \in A} \lfloor \{x\} \rfloor.$$

Hence we have:

Characterization of the L_V (continued)

Join normal form theorem *Any game* G *on* V *can be presented as*

$$\mathsf{G} = \bigvee_{i=1}^{k} \bigwedge \mathrm{R}_i, \text{ where } k \geq 0 \text{ and each } \mathrm{R}_i \text{ is a set of dictatorial games}$$

such that $i \neq j \Rightarrow \mathrm{R}_j \not\subseteq \mathrm{R}_i$.

Moreover, this presentation is unique (up to the order of the R_i *and the order of the dictatorial games in each* R_i*).*

This provides a characterization of the L_V: Let L be a bounded lattice. Suppose there are n elements in L – call them 'atoms' – such that any element g of L has a unique JNF presentation as a join of meets of atoms similar to the above, then L is isomorphic (in the category of all bounded lattices) to L_V with $|V| = n$.

The category $\mathbf{G}$; Main Lemma

Recall that $\mathbf{G}$ is the category whose objects are the L_V for all finite sets V and whose morphisms are the mappings between these objects that respect their structure as bounded lattices.

Main Lemma *A morphism $f : \mathrm{L}_V \to \mathrm{L}_W$ is uniquely determined by the images under f of the dictatorial games $\{\lfloor\{v\}\rfloor : v \in V\}$. Moreover, these images, namely $\{f\lfloor\{v\}\rfloor : v \in V\}$, can be chosen freely as arbitrary games in the codomain L_W.*

So in $\mathbf{G}$ **the dictatorial games play a role of *free generators*, analogous to a basis of a vector space in the category of vector spaces:** to determine a linear transformation, you can choose freely the images of the basis vectors, and this determines the transformation uniquely. But a vector space has infinitely many bases, whereas in L_V the dictatorial games are the only 'basis'.

We have an explicit formula for $f\mathsf{G}$, where $\mathsf{G} \in \mathrm{L}_V$, in terms of the $f\lfloor\{v\}\rfloor$:

$$f\mathsf{G} = \{Y \subseteq W : \{v \in V : Y \in f\lfloor\{v\}\rfloor\} \in \mathsf{G}\}.$$

Another form of this is

$$\forall Y \subseteq W : Y \in f\mathsf{G} \Leftrightarrow \{v \in V : Y \in f\lfloor\{v\}\rfloor\} \in \mathsf{G}.$$

The category G; Another way of writing $f\mathsf{G}$

Without loss of generality, we take $V = \widehat{n} := \{1, 2, \ldots, n\}$. (This is the *canonical* assembly of cardinality n).

Let W be any finite set and let $f : \mathrm{L}_{\widehat{n}} \to \mathrm{L}_W$ be a morphism in our category.

Let us put $\mathsf{H}_i := f\lfloor\{i\}\rfloor$ for all $i \in \widehat{n}$. Then using our formula for $f\mathsf{G}$ we get, for all $\mathsf{G} \in \mathrm{L}_{\widehat{n}}$:

$$f\mathsf{G} = \mathsf{G}[\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_n].$$

Here we use the notation for game composition defined (for a special case) by Shapley (1962) and in complete generality by Felsenthal and Machover (1998).

What this means is that ***the most general morphism*** in our category **G** produces as image of any game G in its domain the composition of G with the images (in its codomain) of the dictatorial games in its domain.

This result surprised us. We knew that composition is important; but we had not realized *how* important. It is the most general operation on games!

I shall now show how the other operations listed in the beginning are obtained as special cases, by special choice of the $f\lfloor\{v\}\rfloor$.

Bloc formation revisited

To define a morphism $f : \mathrm{L}_V \to \mathrm{L}_W$, we may choose the images $f\lfloor\{v\}\rfloor$ of the dictatorial games in L_V to be *completely arbitrary* games in L_W. Let us now see what happens when we choose the latter to be *arbitrary dictatorial* games (in L_W).

So – as in our first obvious attempt (which led nowhere) – let us take any map $\varphi : V \to W$, and consider the morphism f such that

$$\forall v \in V : f\lfloor\{v\}\rfloor = \lfloor\{\varphi v\}\rfloor \textit{ in } \mathrm{L}_W.$$

Putting this in our formula for $f\mathsf{G}$, we obtain

$$f\mathsf{G} = \{Y \subseteq W : \varphi^{-1}[Y] \in \mathsf{G}\},$$

which is exactly the same as what we had for our old $\mathrm{L}\varphi\mathsf{G}$. So this f is our old $\mathrm{L}\varphi$. As we know, it yields the operation of bloc formation, with optional added dummies.

The reason our first attempt failed is that game composition cannot be obtained as a special case of bloc formation, because the exact opposite is true.

The old diagram revisited

We draw the old diagram with some added decoration:

$$
\begin{array}{ccc}
V & \xrightarrow{\varphi \text{ (in } \mathbf{FinSet}\text{)}} & W \\
{\scriptstyle \mathrm{L}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{L}} \\
\mathrm{L}_V & \xrightarrow[\mathrm{L}\varphi \text{ (in } \mathbf{G}\text{)}]{} & \mathrm{L}_W
\end{array}
$$

FinSet is the category of finite sets, with set mappings (such as φ) as morphisms. Those familiar with category theory will see at once that L is a *functor* from **FinSet** to **G**.

In fact, L is the left part of an adjointness relation; the corresponding right adjoint is the forgetful functor

$$F : \mathbf{G} \to \mathbf{FinSet}.$$

Boolean subgames

Let A and N be disjoint subsets of V and let $W = V - (A \cup N)$. In their book, Taylor and Zwicker define, for any game G on V, *the Boolean subgame of* G *determined by* N *and* A, which we (but not they) denote by $\sqsubset_N^A \mathsf{G}$ as the game on W given by

$$\sqsubset_N^A \mathsf{G} := \{Y \subseteq W : Y \cup A \in \mathsf{G}\}.$$

Explanation Consider G is a decision rule with V as its set of voters. Suppose that voters belonging to subsets A and N of V are committed in advance to voting "aye" and "nay" respectively, come what may. When a bill is put to the vote, the outcome will then depend only on the votes of the remaining voters, members of $W = V - (A \cup N)$. We are left with a decision rule with W as the *de facto* set of voters. This rule is precisely $\sqsubset_N^A \mathsf{G}$.

Special cases are:

- $A = \emptyset$. Then $\sqsubset_N^{\emptyset} \mathsf{G}$ is *the subgame of* G *determined by* W.
- $N = \emptyset$. Then $\sqsubset_{\emptyset}^A \mathsf{G}$ is *the reduced game of* G *determined by* W.

Boolean subgames (continued)

It turns out that $\sqsubset_N^A$ is a morphism of **G**. We obtain the morphism

$$\sqsubset_N^A : \mathrm{L}_V \to \mathrm{L}_W$$

by choosing:

$$\sqsubset_N^A \lfloor\{v\}\rfloor := \begin{cases} \top_W & \text{if } v \in A, \\ \bot_W & \text{if } v \in N, \\ \lfloor\{v\}\rfloor \text{ on } W & \text{if } v \in W. \end{cases}$$

A very special case

With V, A and N as above, suppose $W = \emptyset$, so $V = A \cup N$. Then

$$\sqsubset_N^A : \mathrm{L}_V \to \mathrm{L}_\emptyset.$$

In fact we obtain,

$$\sqsubset_N^A \mathsf{G} = \begin{cases} \top_\emptyset & \text{if } A \in \mathsf{G}, \\ \bot_\emptyset & \text{if } A \notin \mathsf{G}. \end{cases}$$

So $\sqsubset_N^A$ is the operator that, when applied to the game G, yields the output (truth value) under G of the division of V in which A is the coalition of “aye” voters and N is the coalition of “nay” voters.