ALGEBRAIC APPROACH TO QUANTUM GRAVITY I: RELATIVE REALISM

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ABSTRACT. In the first of three articles, we review the philosophical foundations of an approach to quantum gravity based on a principle of representation-theoretic duality and a vaguely Kantian-Buddist perspective on the nature of physical reality which I have called 'relative realism'. Central to this is a novel answer to the Plato's cave problem in which both the world outside the cave and the 'set of possible shadow patterns' in the cave have equal status. We explain the notion of constructions and 'co'constructions in this context and how quantum groups arise naturally as a microcosm for the unification of quantum theory and gravity. More generally, reality is 'created' by choices made and forgotten that constrain our thinking much as mathematical structures have a reality created by a choice of axioms, but the possible choices are not arbitary and are themselves elements of a higher-level of reality. In this way the factual 'hardness' of science is not lost while at the same time the observer is an equal partner in the process. We argue that the 'ultimate laws' of physics are then no more than the rules of looking at the world in a certain self-dual way, or conversely that going to deeper theories of physics is a matter of letting go of more and more assumptions. We show how this new philosophical foundation for quantum gravity leads to a self-dual and fractal like structure that informs and motivates the concrete research reviewed in parts II,III. Our position also provides a kind of explanation of why things are quantized and why there is gravity in the first place, and possibly why there is a cosmological constant.

1. Introduction

One of the two origins of quantum groups in mathematical physics in the 1980s was [14] precisely as toy models in an algebraic approach to quantum gravity. In this article we provide a fresh introduction to our philosophy of 'relative realism'[12] underlying this approach, which by now may be of more interest than it was when written twenty years ago at the time of my PhD thesis. The ideas go far beyond quantum groups and extend into category theory and the nature of logic, the notion of measurement, the nature of reality itself. It is this larger context which the present unashamedly philosophical 'Part I' article addresses, leaving to Parts II,III the subsequent applications [22, 23]. I shall also try to write at a less technical level for this reason, at least in the early sections. We begin with an explanation of why we do need a new philosophical foundation.

1.1. **Is modern theoretical physics missing a key philosophical insight?** Quantum theory and gravity have failed to be unified now for around 90 years.

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There have been ‘creative ideas’ that never really got off the ground, such as wonderful ideas of J.A. Wheeler, and there have been ‘serious’ but relatively less imaginative attempts (in the sense of extending more or less existing methodology for quantisation etc.) that also have not really panned out, and I include here much of string theory even though it can be quite technical. What then are we in theoretical physics doing wrong? I propose the following points:

(1) Nature does not necessarily use the maths already in maths books, hence theoretical physicists should be prepared to explore where needed *as* pure mathematicians (not ‘applied’) as well as the more obvious need to keep contact with experiment.
(2) Since the problem has been around for so long, it’s likely that there is a ‘missing ingredient’ in the form of a totally wrong mind set, overcoming which will need to give up one or more key assumptions.
(3) Since quantum gravity is expected to be the ‘last unification’ to achieve a theory of physics as we currently see it, the new ingredient should be of a fundamental significance on the border of what is currently physics and meta-physics, i.e. should in part transcend current physics.
(4) In this case the missing ‘input’ can be inspired from art, culture, life and we should be open to this possibility of wider cultural input.
(5) Such philosophy should complement and inform but not replace dramatic advances currently being made on the experimental/astro front (quantum gravity phenomenology) as a result of new technology and instrumentation.

On the first point, there are at the moment very different mind sets between theoretical physics, even if it involves quite fancy and advanced mathematics, and pure mathematics. I am proud to have been educated at the ‘Department of Applied Mathematics and Theoretical Physics’ in Cambridge, UK and yet for quantum gravity I propose that one really needs a ‘Department of *Pure* Mathematics and Theoretical Physics’. It’s a department that would have been more at home in earlier eras when mathematics and natural philosophy were as one (go back to medieval times for example).

It is true that theoretical physicists can eventually take on new mathematical structures, for example quantum groups[20, 21] and noncommutative geometry[3] which were arcane topics for physicists two decades ago but which due for example to the efforts of those of us who have been arguing for alternatives to a continuum assumption for spacetime, are now tools at the disposal of many working in quantum gravity. But even as this new machinery becomes ‘absorbed’ the mind set in theoretical physics is to seek to apply it and not to understand the conceptual issues and freedoms (result: a great many articles of dubious value in this particular area). By contrast a pure mathematician is sensitive to deeper structural issues and to the subtle interplay between definition and fact. It is precisely this subtle interplay which will be at the core of my proposal for quantum gravity.

As to what key assumption in physics one should give up, a usual suspect here is the nature of measurement. In quantum theory we learned that somehow the observer has to ‘get in the act’ in the form of a postulate about collapsing wave functions, and this led its early pioneers including Bohr and Schrödinger to argue that a new (more Eastern) world view was therefore needed for quantum theory. Heisenberg wrote already in his 1927 paper[9]:

> "The existing scientific concepts cover always only a very limited part of reality, and the other part that has not yet been understood is infinite. Whenever we proceed from the known into the unknown we may hope to understand, but we may have to learn at the same time a new meaning of the word 'understanding'."

and later for example countered Einstein's famous criticism of the Copenhagen interpretation with an explicit attack on scientific realism or, as he put it, the 'ontology of materialism' implicit in Einstein's argument. At root here is the conviction that somehow the observer should be a partner in the very concept of reality. Yet this vision has not really come to pass; the measurement postulate is typically viewed as an approximation of a less mystical theory of measurement the details of which could be found in principle. Most scientists do still assume today that there is a fundamental observer-independent physical reality 'out there' and that we do experiments and make theories to come closer to it, a view going back to Francis Bacon, Hooke and to the historical concept in the West of what is science. The indications are that for quantum gravity, however, the measurement problem is more serious and can't be brushed off. This is because the division into atomic system and observer is also a division into micro and macro, which quantum gravity has to unify. If so, then perhaps we can't get further because the usual realist-reductionist assumption is just plain wrong whereas we refuse to give up something so seemingly essential to science as this.

The way I like to explain this is with reference to Figure 1 where I indicate the mass-energy (understood broadly) and size of everything in the universe on a log scale. Everything to the left is forbidden by quantum theory in the sense that smaller masses or lower energy photons have larger size in the sense of wavelength. Everything to the right is forbidden by gravity in the sense that too much mass in a given space forms a black hole and adding mass only makes the black hole bigger. Everything else lies in the triangular region while the intersection of the two contraint lines is the Planck scale, approaching which we would need a theory of quantum gravity to understand. What is interesting is that life, our macroscopic scales, lie somewhat in the middle. The usual 'explanation' is that things become simpler on the above two boundaries as phenomena are close to being forbidden hence tightly controlled there; hence conversely they should get more complicated as one approaches the centre of the triangle and hence this is most likely where life would develop. However, another point of view which we shall promote is that *we are in the middle because we in some sense built the edifice of physics around ourselves starting with things we could easily perceive and our assumptions in doing this have now 'boxed us in' and not without some inconsistencies at the corners.*

If so then this is really a limitation of the larger cultural mind set in which we operate as scientists. Maybe there is an alternative, no less rigourous and no less equation-based concept of science which if coming from a different philosophical background we might be able to accept, and within which our current inconsistencies can be understood and resolved. My proposal for this mind set, which I called '*relative realism*' [12], is not inspired by Eastern philosophy but rather by the nature of pure mathematics. But for better or for worse it would be fair to say that it has starting points in common with eastern philosophies such as Buddism, while also having critical differences. Indeed, the main difference is that one does not have to go off the deep end and assume anything about consciousness any more

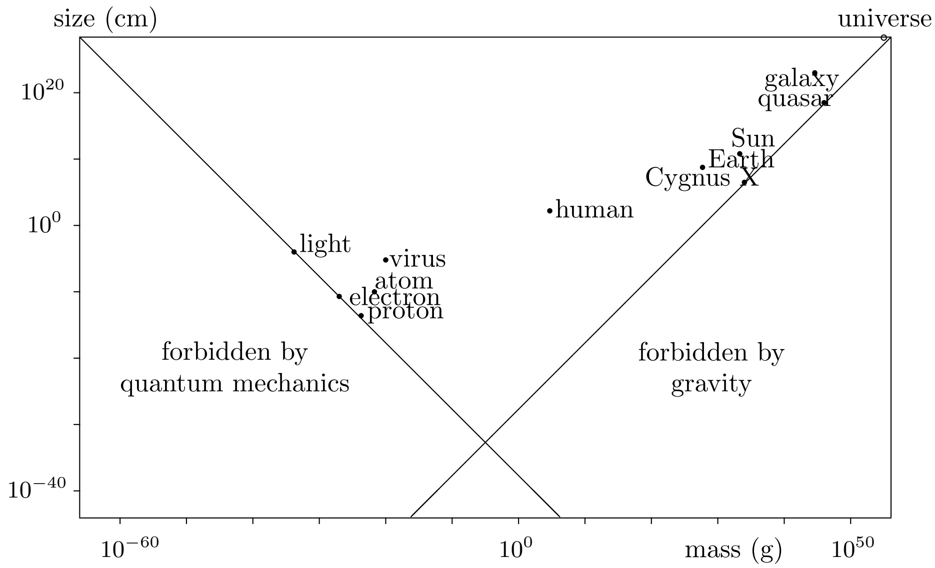

FIGURE 1. Everything in the universe on a log plot of size against mass-energy. Did we 'box ourselves in'?

than one needs to do this to do quantum theory. We also don't regard concrete reality as empty of value, indeed for us as 'hard scientists' it remains the very object of interest in this article, which is entirely different from Buddism. I stress this both because I am not an advocate of religion in science, not even a nontheistic one such as Buddism, and because any mention of it will alarm many readers as a dangerous precipice over which others have fallen. With regard to previous attempts inspired by the quantum measurement problem, the difference now is that we do not address that directly but see it as being part of quantum gravity, which we have addressed with this new philisophical outlook and with some limited success as I see it [12, 13, 14]. In particular, I propose a position for physics within relative realism, which could be called '*self-dualism*' (but in a certain specific sense) and which provides, for example, insight into why there is quantum theory and why there is gravity (see Section 4.4) and why there should be a cosmological constant (see Section 5.4). The first two are usually taken for granted, but in my view a true philosophy of physics should help explain even these most basic of discoveries. The third by contrast is a major theoretical problem of our times.

The last of my points above is about timeliness. The received dogma 20 years ago was that Planck scale effects were well out of reach of experimental measurability. This is certainly true from the point of view of naively smashing particles in accelerators and seeing what particles come out, but by now we know of several more subtle situations where these smallest of effects can sufficiently accumulate or 'amplify'. This greater flexibility combined with giant leaps in technology and the quality and content of cosmological data, not least (but not only) due to the Hubble telescope, means that theoretical predictions for quantum gravity effects are today beginning to be accessed through phenomena such as γ-ray bursts, neutrino oscillations, active galactic nucleii, and earthbound experiments (such as VIRGO and LISA inteferometers intended for gravitational wave detection). The latter could also detect quantum gravity effects at least in principle although not without retooling. See [22] for more details.

Also changed in the last two decades, as mentioned above, is much greater familiarity among theoretical physicists with algebraic and categorical methods. That is to say, there has also been a giant leap in the available *mathematical technology.* I recall being told at a 1988 reception in MIT (by a very famous physicist) that my just finished PhD thesis must be nonsense because without a continuum spacetime and a Lagrangian it would be impossible to maintain conservation of energy between an atom and, say, a baseball bat. By now it's more accepted that most likely spacetime after all is not a continuum and there is no actual Lagrangian, that things are more algebraic. To be fair many physicists over the decades have questioned that assumption but in the 1980s the received wisdom was 'we do not have a coherent alternative'. We now have at least one fairly well-developed alternative technology in the form of noncommutative geometry, the quantum groups approach to which we will cover in [23]. Quantum groups also led directly into braided categories, 3-manifold invariants and ultimately into combinatorial methods including spin-networks on the one hand and linear logic on the other. Both noncommutative geometry and generalised logic tie up with other preexisting aspects of quantum gravity such as causal sets and quantum topology. So we start to have a kind of convergence to a new set of available tools.

What is relevant to the present article is that we therefore have just at the moment two of the three key ingredients for a revolution in physics: rich streams of relevant data coming on line and new mathematical tools. If one asks what made Einstein's general relativity possible, it was perhaps not only cutting-edge experimental possibilities and the new mathematics of Riemann geometry whether properly acknowledged or not, but also their conjunction with deep philosophical insight which he attributed in its origins to Mach, but which was probably also tied up with wider developments of the era. As should be clear from the above, if relative realism provides some of the correct philosophy then it may ultimately be part of a perhaps more general cultural shift or 'new renaissance' involving pure mathematics, philosophy, art, humanity, but as for the Italian Renaissance, driven in fact by advances in technology and scientific fact. I do not know if this larger vision will be realised, our main concern in this article will be how to get actual equations and ideas for quantum gravity out of this philosophy.

2. First steps: Models with observer-observed duality

Rather than justifying the above in general terms, I think it more informative to illustrate my ideas by outlining my concrete alternative to the realist-reductionist assumption. Even if one does not agree with my conclusions, the following is a nice example of how philosophy[12] can lead to concrete scientific output[13, 14]. Specifically, the search for cosmological models exhibiting what I called observer-obvserved duality or 'quantum Mach principle' led to what remains today one of the two main classes of quantum groups.

Briefly, suppose that the universe is simply modelled by a homogeneous space; this is to simplify the problem somewhat by focussing on its large scale structure. Now consider a quantum particle moving in this space quantised in a normal way (say by Mackey quantisation). In my 1988 PhD thesis I showed that there is a class of such homogeneous spaces M with the property that the quantum algebra for the particle is a Hopf algebra. What does that mean? It means among other things that the set of linear expectation states on this algebra themselves generate an algebra, of some other system which in fact turns out to be the quantum algebra of some other homogeneous space $\hat{M}$. So if a quantum particle in our original space M is in some state ϕ, in which according to quantum mechanics one might say that an observable x (say of position or momentum) has expectation value

$$\phi(x)$$

another person could equally well say that ϕ is an observable for a particle on the dual homogeneous space $\hat{M}$ and that x is in fact its quantum expectation state, and that the same numbers above should be written instead as

$$x(\phi)$$

I've glossed over a technicality here: states and observables in quantum theory have positivity and self-adjointness requirements, but this just means that in reversing roles in this way we may go for example from a state to a linear combination of observables. But what it tells us is that in these models there is no true 'observer' and 'observed' in the sense that the same numbers, which I regard as the only hard facts on the ground, can be interpreted either way [13, 14].

It's fair to say that the full physical content of this symmetry has still to be appreciated, but it's central to our philosophy and we shall return to it later. Meanwhile, such 'bicrossproduct quantum groups' as arise here were applied in some related ways notably to 'deformed special relativity' or noncommutative spacetime[19, 22]. One prediction of these is that the speed of light should in fact depend very slightly on energy and this is the effect that, even if a Planck scale one, is theoretically within the realm of detectability by a statistical analysis of cosmological gamma ray burst data to be collected by Nasa's GLAST satellite to be launched later this year. This is therefore a path from the philosophical idea of being able to reverse the roles of observer and observed in quantum theory (within quantum gravity) through new mathematics and then to potential experimental proof using cosmological measurements.

3. Relative Realism

What the bicrossproduct models illustrate is the following general proposition: *there is no absolute physical reality as usually considered* but rather it is we in order to make sense of the Universe, who artificially impose a division into 'abstract laws of nature' (in the form of abstract structures deemed to exist) and 'measurements' made or experiments done to illustrate them. I believe this division is necessary but it is also arbitrary in that the true picture of physical reality should be formulated in such a way as to be independent of this division.

How is it possible to have a view of physical reality that on the one hand allows such relativism but on the other hand is compatible with the 'hardness' or non-arbitrariness of 'hard science'? I'll explain my answer at two levels, first my own route as a pure mathematician doing physics and second in a way that extends this to a wider context.

3.1. **A mathematicians view.** I should say that I was not 'born' a pure mathematician, it was always my vocation to study theoretical physics, quantum theory etc and I always disliked pure mathematics as too unmotivated. But the problem of quantum gravity for reasons above made me realise in the middle of my PhD that a pure mathematician's mind set was the correct one (as a result I started my PhD in the Physics department at Harvard and ended it in the Mathematics one). Moreover, I do not wish to imply that pure mathematicians typically agree with me on relative realism; most in fact take a more platonic view on mathematical reality, if they choose to think about such things at all. Rather, it's my experience doing pure mathematics that I draw upon.

Here what I think a pure mathematicians does understand deeply is the sense in which definitions 'create' mathematical reality. You define the notion 'ring' and this opens up the whole field of Ring Theory as a consequence of the axioms of a ring (a ring is set with addition and multiplication laws, compatible with each other). Somebody invented the axioms of Ring Theory and if you dont like them you are free to define and study something else. In this sense mathematicians don't usually make a claim on 'reality'.

And yet, in some sense, the axioms of rings were there, waiting to be discovered by the first mathematician to stumble upon them. So in that sense mathematical reality is there, waiting to be discovered, and independent of humans after all.

Are these two statements contradictory? No, provided one understands that all of mathematics is similarly layered as a series of assumptions. Ultimately, if all of mathematics could be laid out before you (there are some technical issues there but I consider them inessential) it would be by definition all things that could be said within a set of assumptions known as being a mathematician (presumably some form of logic and certain rules as to what it entails to make a proof in any reasonable sense). So even the totality of the mathematical reality being uncovered by mathematicians is based on certain implicit axioms. In between this extreme and the example of Ring Theory, are a whole heirarchy of assumptions that you could accept or not accept. Not accepting the associative law of multiplication takes you to a more general theory than Ring Theory (namely to the theory of 'nonassociative rings' by which is meant not necessarily associative rings). At that level there are facts, constraints, the hard reality of nonassociative ring theory, and it includes as a sub-reality the associative special case. And so on, dropping each of the assumptions of Ring Theory takes us to more and more general structures until we are studying very general things indeed, such as the theory of Sets (at least a ring should be a set). And if one does not want to assume even a set, one is doing very abstract and general mathematics indeed.

Here we see a point of view in which it's very much up to you what you want to assume and study, but whatever you assume puts you on a certain 'level' of generality and this is the level of mathematical reality that you are experiencing as you proceed to research. Arguably the more general level is the more 'real' as the least is put in by an assumption, but equally convincing, the less general level is the more real as having 'real substance'. What we see here is *two points of view* on reality, on what really matters. If you are 'in the room' of Ring Theory looking around you see the nuggets or reality and hidden treasures that are Ring Theory. You might look out through the doorway and wonder what lies outside but you dont perceive it directly. On the other hand if you step outside you see the room of Ring Theory as one of many rooms that you might enter. You do don't directly perceive the hidden treasures of Ring Theory but you are aware that you could if you went inside the room. Rather, the room itself is one of the nuggets of reality in your world at this 'level'. This is part of the paradox between the general and the specific that inspires pure mathematics research. Quite often it's much easier to prove a general theorem that applies to a whole class of objects than to prove the same thing for a specific object, conversely, it is the specific examples that ultimately make a general theorem of interest. A pure mathematician does not take one view or another but sees all the structure of (overlapping) nested boxes in its entirety as of interest, and recognises the choice in where to think within it.

In short, what is usually called reality is in fact a confusion of two mental preceptions. Wherever we are in the world of mathematics we have made a certain number of assumptions to reach that point. If we put them out of our mind we perceive and work within the 'reality' that they create. If we become self-aware of an assumption and consider it unecessary we transcend to a slightly more general reality created by the remaining assumptions and look back on the aspects of reality created by the previous assumption as an arbitrary choice. Thus what you consider real depends on your attitude to a series of choices.

This is a model of reality that is adapted to the fundamental nature of knowledge, achieves the *minimal goal* of explaining how we perceive it and also has at its core notions of consciousness or self-awareness and free will. These last are arguably missing ingredient of any quantum gravity theory. As for the measurement problem we don't need to speculate further as to what these 'really' are in order to have an operationally useful philosophy but we do put them in pole position.

3.2. **In everyday life.** Everything I said above could also be applied in everyday life. The example I like to give[12] is playing chess. If you accept the rules of chess then you are in the reality of chess, you can experience the frustration and anguish of being in a check-mate. But if you step back and realise that it's just a game, the rules of which you dont have to take on, your anguish dissolves and you transcend to a more general reality in which the fact that chess is possible is still a reality but not one that you are immersed in. It is a nugget of reality at a higher level, the reality of board games for example, along side other board games of interest for their rules rather than for actually playing.

The work [12] was originally submitted to the Canadian Philosophy Journal in 1988 where an enthusiastic referee told me to read Kant and rewrite the paper. I spent the summer doing that (after which the paper was rejected) but I now think that a more down to earth comparison is in fact not with Western philosophers such as Mach, Kant or Hegel but with the philosophy of Buddism. I am not an expert on Buddism but its central tenet surely is that suffering, anguish, can be transcended by thought alone (meditation). In Tibetan Buddism, by stages of contemplation, one progresses to greater and greater levels of enlightenment in which successive veils are lifted, which I understand as proceeding up to higher levels of generality in which more and more assumptions are become aware of as optional. In all forms of Buddism this appears as the central role of compassion as a guide to dealing with others. In mathematics it appears as a compassion towards other researchers who have taken on different axioms. Ring theorists don't fight with group theorists about who is 'right'. Neither is 'true' and likewise Buddism generally rejects our usual concrete reality or *samsara* as an illusion.

I should clarify that I am not speaking here about religion (as a strident atheist that would be highly problemmatic for me). I am speaking about a particular non-theistic philosophy and, moreover, I would insist that the comparison only works up to a point and is ultimately a bit misleading. If relative realism coming out of pure mathematics ends up with things in common with the teachings of an apparently historical figure from 500 B.C. this tells me that that person and his followers had a glimpse of some of the same insights but does not imply that anything else of the dogma or trappings of Buddism have any relevance. I will therefore devote a few lines to the key *differences* as I see it, and leave it at that.

First of all, in Buddism, the term ‘relative reality’ is used but not at all to mean what we have meant above (it refers to the reality contained in the transformation from one state to another, as a form of relationism). In Buddism what is truly ‘real’ is what is left after letting go of all assumptions, even the very notions of ‘is’ and ‘is not’. This is why, in the Zen tradition, the truth can only be seen in the smile that occurs when perceiving the contradiction in a Zen riddle. In this way Buddism claims to avoid nihilism. In mathematical reality there is likewise something that remains after letting go of all assumptions, but it’s just the entirety of the fine structure of all of mathematics.

There is also a key difference in conclusion. In Buddism the attitude to *samsara* is perjorative, as an illusion it is to be dismissed. In relative realism the boxes that you can find yourself in *are* all there is, they are the nuggets of reality, the heart and soul of science, the whole point so to speak and very much to be celebrated. Implicit here is the claim, which goes far beyond buddism, that *just because they are the product of assumptions does not make them arbitrary and worthless*. That would be like saying that pure mathematics was arbitrary and worthless which is not really the case. The idea of relative realism as we have explained above is that the arbitrariness is always pushed up to a higher level where a given choice is part of the hard reality or structure of possible such choices, and that the entire structure of choices is ultimately the hard reality.

But there is also something else going on concerning the nature of representation which we come to in Section 4 and in its purest form, let us call that pure form ‘physical reality’ or ‘hard science’. Everyday life is somewhere in between, our thoughts and perceptions are ‘hard’ to the extent that we think about them rigorously or less hard if not. In Buddism there is a vague idea that the illusion of *samsara* is a collective one (but no less an illusion for it) due to interaction between consciousnesses . In relative realism we are going to go beyond this and argue for a specific mechanism for how this comes about, an equation so to speak, and in this way things will be far from arbitrary. It is still a collective thing but not fundamentally different from thermodynamics where general principles for interaction lead to quantifiable rules, to science. Related to this, the notions of ‘interpentration and interbeing’ in Buddism are all very well but to a physicist they just assert trivially that everything tends to affect everything else. The whole point of science is to study, make precise and quantify these interactions, so there is also a fundamental difference in approach.

3.3. **Is this a chair?** To illustrate these ideas a little further at the level of everyday life, let us ask what does it mean to say that here in front of me is a chair? This is something solid and concrete, surely there is nothing illusory about this. From an operational point of view it’s enough to be able to say how someone could tell

FIGURE 2. Is this a rocking chair (as claimed by its designer Frederick Kiesler)? What does it mean to say that an actual chair stands before you? Photo from the V&A museum shop, London.

if there was a chair here. We could try to specify a recognition algorithm, which is in fact a quite difficult task as anyone involved with AI will testify. Suffice it to illustrate some of the difficulties in Figure 2. Is the item shown a (rocking) chair as claimed by its designer and Austrian manufacturer of surrealist furniture? It does not have legs, for example. At the end of the day I think the best definition which we in fact all use is *a chair is whatever you and I agree to be a chair when you and I use this word in a particular communication.* To the extent that we agree, that is the extent to which there is a chair in the room in the case of the figure. It is not even a matter of function over form, it's a matter of the 'handshaking' agreement that is involved in any act of communication and the choice to define something like this and then to use that definition.

In relative realism then we sort of 'swim' in a sea of definitions of which a crude approximation might be the choices to use a few hundred thousand words or concepts. Some of us who use more have a richer experience of reality while I believe the minimum for getting by in everyday life is judged by linguists at about 600 which would be a sterile and angst-ridden experience. But does something exist if we don't have a word for it but someone else does? It exists for them and not for us, provided one understands that we are using the word 'exist' here in a certain way. This is why it is relative realism; that there should be a single answer is the old materialism that it replaces. Note that in practice we would also need to take account of the baggage of a whole sea of other assumptions that would be likely in practice to play the role of a missing word. Moreover, while one could say that what replaces mathematical reality in everyday life is crudely approximated by linguistic reality, I don't really mean just the words but nonverbal concepts too, as well as the entirety of their inter-relations to greater and lesser extents. This is all supposed to be smoothed over and integrated to give an overall 'perception' of solid reality. In mathematical terms, we have talked above about letting go of assumptions, transcending to more and more general points of view. But the other 'limit' of this 'tree of knowledge' is the limit of finer and finer assumptions that one might dimly or less dimly take on and which I once called the 'poetic soup'.

It's the opposite of abstractification, so let us call it a model of direct perception. Moreover, and this is key, our feeling of solidity or grounding exists to the extent that others use the same definitions or (in the limit) perceive the same. It is the extent to which we are 'in touch' with the real world. I am reminded of spy movies from the 1960s in which the first thing before starting off on a mission is for our heros to synchronise their watches. As we communicate we synchronise our terms and concepts to the extent that we communicate successfully and to that extent we then agree on what we are just doing.

The question which we still have not addressed is could such a scheme ever reach the clarity and precision of the physical sciences, where we agree on many things to 13 decimal places so to speak. Part of the answer to this is that assumptions going into it are far more primitive and uniformly accepted than, say, the rules of chess. They are by definition more related to the physical world and might include for example assumptions formed as we first learned to walk. There is also an illustration from physics that may be helpful and which goes back to Bishop George Berkeley commenting on an experiment of Newton, was taken up by Ernst Mach and then influenced Einstein. Would the concavity of the surface of water in a rotating bucket of water still be there if there were no 'fixed stars' up in the sky? The usual explanation is of course that the rotating water experiences centripetal forces that push up the water more where it lies nearer the rim. However, we now know (in view of General Relativity) that an equally good explanation is that the bucket is not (in a suitable frame of reference that rotates with the bucket) rotating, so there are no such forces. Instead the stars in the sky are moving around the bucket in this frame. Their collective gravitational effects presumably conspire to push the water up in the observed way and if they were not present there would be no such effect. Because the stars are now moving in a uniformly circular way around the bucket their accumulated average effect is quite symmetrical and coherent. This is a manufactured example and I'm not sure anyone has ever really done the calculation in GR in this way but a similar one is the Lense-Thirring effect which is well studied (here a rotating spherical shell of mass creates coriolis forces inside the shell which is not quite the same thing but in a similar ballpark). For our purposes it illustrates how things that we interact with conspire to create a bit of physical reality. In this case the stars in the sky create and result in a particular spacetime geometry in which the bucket is located. It is not the act of one star but a collective act known as solving a differential equation over all of spacetime.

This example also warns us to be a bit careful about free will and time here. It is one thing to choose to start rotating the bucket and quite another to chose to start rotating the fixed stars and have their response. These issues also apply to the communication and synchronization scenario above. I suppose sometimes the communications will be literal ones as events in spacetime and there are issues similar to those that already exist in quantum mechanics in the context of the EPR thought experiment. On the other hand, by 'communication' in relative realism here we don't necessarily mean a sharp event, we intend any kind of influence or interaction to a greater or lesser extent by which concepts are transferred and synchronized. Moreover, communication does not necessarily entail a conscious entity doing it. A synonym for each side of a communication could be 'awareness' or 'to be conscious of' and maybe this is all there actually is to consciousness from an operational point of view. This would be analgous to the way that the measurement postulate in quantum theory shelves the issue while remaining operational. Note that 'information' defined as that which is communicated is entering here into the discussion and in physics this links back to thermodynamics and, these days, to gravity.

4. Physics

Within the context of relative realism, I propose then that 'physical reality' for all its apparent concreteness, is not in fundamental essence different in nature from mathematical reality. More precisely, physics and most of 'hard science' today uses mathematics so physical reality whatever it is should be a subset of mathematical reality. If so, there should be an 'equation of physics' that characterises it as a subset. I will give what I think is a first approximation to this equation as well as other predicted insights about the structure of physics[12]. It represents a more specific philosophical position for physical reality that I am tempted to call *self-dualism.* But first, insights from my 2-year old daughter Juliette.

4.1. **My 2-year old's insight into quantum gravity.** If relative realism is right then 'physical reality' is what we experience is a consequence of looking at the world in a certain way, probing deeper and deeper into more and more general theories of physics as we have done historically (arriving by now at two great theories, quantum and gravity) should be a matter of letting go of more and more assumptions about the physical world until we arrive at the most general theory possible. If so then we should also be able to study a single baby, born surely with very little by way of assumptions about physics, and see where and why each assumption is taken on.

Although Piaget has itemised many key steps in child development, his analysis is surely not about the fundamental steps at the foundation of theoretical physics. Instead, I can only offer my own anecdotal observations.

- Age 11 months: loves to empty a container, as soon as empty fills it, as soon as full empties it. This is the basic mechanism of waves (two competing urges out of phase leading to oscillation).
- Age 12-17 months: puts something in drawer, closes it, opens it to see if it is still there. Does not assume it would still be there. This is a quantum way of thinking. It's only after repeatedly finding it there that she eventually grows to accept classical logic as a useful shortcut (as it is in this situation).
- Age 19 months: comes home every day with mother, waves up to dad cooking in the kitchen from the yard. One day dad is carrying her. Still points up to kitchen saying 'daddy up there in the kitchen'. Dad says no, daddy is here. She says 'another daddy' and is quite content with that. Another occasion, her aunt Sarah sits in front of her and talks to her on my mobile. When asked, Juliette declares the person speaking to her 'another auntie Sarah'. This means that at this age Juliette's logic is still quantum logic in which someone can happily be in two places at the same time.
- Age 15 months (until the present): completely unwilling to shortcut a lego construction by reusing a group of blocks, insists on taking the bits fully apart and then building from scratch. Likewise always insists to read a book from its very first page (including all the front matter). I see this as part of her taking a creative control over her world.

- Age 20-22 months: very able to express herself in the third person 'Juliette is holding a spoon' but finds it very hard to learn about pronouns especially 'I'. Masters 'my' first and but overuses it 'my do it'. Takes a long time to master 'I' and 'you' correctly. This shows that an absolute coordinate-invariant world view is much more natural than a relative one based on coordinate system in which 'I' and 'you' change meaning depending on who is speaking. This is the key insight of General Relativity that coordinates depend on a coordinate system and carry no meaning of themselves, but they nevertheless refer to an absolute geometry independent of the coordinate system. Actually, once you get used to the absolute reference 'Juliette is doing this, dad wants to do that etc' it's actually much more natural than the confusing 'I' and 'you' and as a parent I carried on using it far past the time that I needed to. In the same way it's actually much easier to do and teach differential geometry in absolute coordinate-free terms than the way taught in most physics books.
- Age 24 months: until this age she did not understand the concept of time. At least it was impossible to do a bargain with her like 'if you do this now, we will go to the playground tommorrow' (but you could bargain with something immediate). She understood 'later' as 'now'.
- Age 29 months: quite able to draw a minor squiggle on a bit of paper and say 'look a face' and then run with that in her game-play. In other words, very capable of abstract substitutions and accepting definitions as per pure mathematics. At the same time pedantic, does not accept metaphor ('you are a lion' elicits 'no, I'm me') but is fine with similie, 'is like', 'is pretending to be'.
- Age 31 months: understands letters and the concept of a word as a line of letters but sometimes refuses to read them from left to right, insisting on the other way. Also, for a time after one such occasion insisted on having her books read from last page back, turning back as the 'next page'. I interpret this as her natural awareness of parity and her right to demand to do it her own way.
- Age 33 months (current): Still totally blank on 'why' questions, does not understand this concept. 'How' and 'what' are no problem. Presumably this is because in childhood the focus is on building up a strong perception of reality, taking on assumptions without question and as quickly as possible, as it were drinking in the world.
- ... and just in the last few days: remarked 'oh, going up' for the deceleration at the end of going down in an elevator, 'down and a little bit up' as she explained. And pulling out of my parking spot insisted that 'the other cars are going away'. Neither observation was prompted in any way. This tells me that relativity *can* be taught at preschool.

I'm not saying that my daughter understands quantum gravity, she doesnt. But we can see where some of the assumptions that lead us to our current 'mental block' of two incompatible theories enter. To explore the even earlier assumptions, before language, would be very tricky as one would need to disentangle assumptions from genetic aspects of the underlying biology. The 15, 29 and 33 month anecdotes could perhaps be viewed as signs of relative realism itself and are mentioned for this reason. It seems to me that there is a striking similarity between pure mathematics and the way that toddlers have a generous attitude to reality in being willing to take on and play within fairy tales they are told, rules picked up from parents, and rules that they create for themselves, with equal conviction.

4.2. **Meta-equation of Physics within Mathematics.** How can we turn philosophy into equations and ultimately, we hope one day, find the actual fundamental equations of physics as a consequence. The work [12] made a first attempt at this, by no means the last word but in my view 'of the essense'. For if physical reality is created by the assumptions known as looking at the world as a hard scientist, discovering the ultimate equations of physics should (we proposed) be in fact rediscovering this fact, in effect rediscovering our scientific selves. In that case a little thought about the nature of science should lead us to the answer (or more realistically, provide insight to help us in quantum gravity). In this approach physical reality is only as well-defined as the extent that physicists are doing something coherent in the first place. This is a circular definition but that is because science is being formulated as, like chess, an activity the rules of which one does not need to take on but which one would not wish either to modify in order to still be called a scientist.

What then is the fundamental nature of science? In the first approximation I consider the key part of this to be the concept of 'representation'. Aside from mathematics (logical thinking) is, as mentioned above, the idea of theory and experiment, of something 'out there' and a 'measurement/image/representation' of it. Plato asked this question: how could prisoners confined to a cave with only one window above their heads deduce the existence of the world outside from shadows of the world projected onto the walls of the cave? As a mathematician I have an answer to this (different from Plato).

A representation of some mathematical structure X generally means a map from it to something considered concrete or self-evident such as numbers or matrices. What is important is not one structure but the set $\hat{X}$ of all such. Generally speaking this $\hat{X}$ has its own abstract structure and we can view each element $x \in X$ as a representation of this, that is we can take the view that $X = \hat{\hat{X}}$. Thus in mathematics one generally has a symmetrical notion in which

$$f(x) = x(f)$$

are two points of view, an element $f \in \hat{X}$ representing the structure of X and here measuring the value of a particular element $x \in X$, or the latter representing the structure of $\hat{X}$ and here measuring the value of a particular element $f \in \hat{X}$. In the context of Plato's cave then one should consider not only one angle of light, which creates one particular pattern of shadows as a representation of the real world X outside, but the set $\hat{X}$ of all such representations corresponding to all possible angles f of the light (to the extent possible). Here I consider one particular geometry of various objects outside the cave as the real structure X, let us say modelled by some point-like objects x in some fixed generic positions, the corresponding bit of the shadow in angle f as their images $f(x)$. In the ideal case of a 100% coverage of angles one would be entirely able to reconstruct X from knowledge of the measurements in all representations $f \in \hat{X}$ (if one could see the shadows from all possible angles). That seems all very reasonable but mathematics and the symmetrical nature of measurement as explained above means that *one could equally well say that the set of angles of the sun $\hat{X}$ was the 'real structure' and* *that each object x outside the cave was in fact a representation of $\hat{X}$.* A bit of the shadow $x(f)$ was the image of the angle f in this particular representation x. It would be quite hard to convince the prisoners that the real world that they longed for outside was nothing but a representation of the angles of the sun coming into their cave but from the mathematical point of view, if these are the only structures in consideration, it is equally valid. This 'reversal of point of view' is an example of observer-observed duality. In a nutshell, while Plato's conclusion was that his cave was an allegory for a pure reality of which we can only ever see a shadow, our conclusion is exactly the opposite, that there is no fundamental difference between 'real' in this platonic sense and the world of shadows since one could equally well consider X as $\hat{\hat{X}}$, in other words as 'shadows' of what we previously thought as shadows, and the latter as 'real' in the platonic sense.

The key question is, is such a reversed point of view physics or is it just a mathematical curiosity? The physical world would have to have the feature that the dual structures $\hat{X}$ would also have to be identifiable as something reasonable and part of it. This could be achieved for example by first of all convincing the prisoners to take $\hat{X}$ seriously, to think about its structure, to take on the view that x was a representation of this structure. Over time they might grudgingly allow that both X and $\hat{X}$ should be considered 'real' and that each represents the other. They would arrive in this way at a self-dual position as to what was 'real', namely $X \times \hat{X}$. This is often the simplest but not the only way to reach a self-dual picture. We take this need for a self-dual overall picture as a fundamental postulate for physics, which we called [12] the principle of representation-theoretic self-duality:

> (Postulate) a fundamental theory of physics is incomplete unless self-dual in the sense that such a role-reversal is possible. If a phenomenon is physically possible then so is its observer-observed reversed one.

One can also say this more dynamically: *as physics improves its structures tend to become self dual in this sense*. This has in my view the same status as the second law of thermodynamics: it happens tautologically because of the way we think about things. In the case of thermodynamics it is the very concept of probability which builds in a time asymmetry (what we can predict given what we know now) and the way that we categorise states that causes entropy to increase (when we consider many outcomes 'the same' then that situation has a higher entropy by definition and is also more likely). In the case of the self-duality principle the reason is that in physics one again has the idea that something exists and one is representing it by experiments. But experimenters tend to think that the set $\hat{X}$ of experiments is the 'real' thing and that a theoretical concept is ultimately nothing but a representation of the experimental outcomes. The two points of view are forever in conflict until they agree that both exist and one represents the other.

To give a simple example, initially one might think in Newtonian mechanics that the structure of flat space X was the 'real thing'. Its structure is that of an Abelian group (there is an addition law) and this is used to define differential calculus, newtonian mechanics and so forth. However, experimenters soon found that the things of particular interest were the plane waves, which mathematically are nothing other than representations of X. They are each determined by a momentum p which expresses the energy of the wave. The set of such waves forms itself an additive group $\hat{X}$ called 'momentum space'. The Irish mathematician Hamilton realised

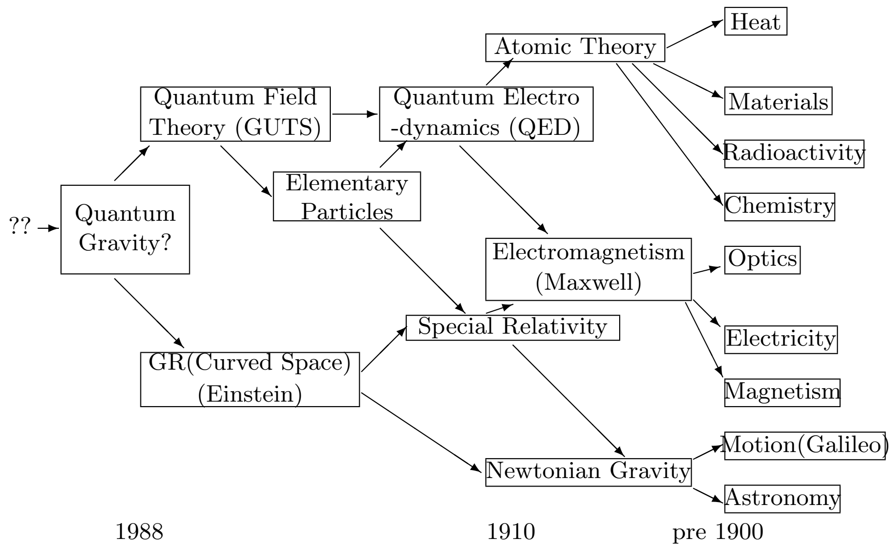

FIGURE 3. Schematic of evolution of theoretical physics. Arrows indicate how the laws of one area may be deduced as a limit of more abstract ones. Assessment of the author [12].

that it was more natural to reformulate Newtons laws in a more symmetrical way in terms of both X and $\hat{X}$ (the combined space is called 'phase space') and by now this is an accepted part of physics. Both X and $\hat{X}$ are equally real and represent each other. The 'reversal of view' mentioned above is called 'Fourier transform'. Most of 19th century mathematical physics revolves around these concepts and tools.

4.3. **Metaphysical dynamics.** What is called physics is not in fact static, there is an element of dynamic as different paradigms or areas are unified as theoretical physics aims to arrive at more fundamental theories which is to say with fewer and fewer restrictions. Old theories of physics don't exactly die, they live on as useful special cases where a further assumption (such as that velocities are small compared to the speed of light) are made. One can relax them for a more general theory of physics or not, and it is the entire edifice from the most general theory (to the extent that we have it) and special cases that is really 'physics'. An illustrative sketch of the sort of thing I have in mind is in Figure 3. The assessment is purely for illustrative purposes and I have in mind only the conceptual foundations of each subject not the subsequent developments. In this context then, we consider the theoretical frontier of physics as moving up this edifice which (in the usual reductionist view) already exists or which in relative realism we create as the solution of a constraint.

I think a key element to the dynamic here is an 'urge' coming from the nature of being a physicist that structures should interact. One is not really happy with X and $\hat{X}$ as independent bits of reality. So long as they are both 'real' they should be part of some more unified structure. This creates a kind of 'engine' that could be viewed as driving the evolution of physics (in an ideal world rather than necessarily how it evolved in practice which will be more haphazard). Thus, along with the above principle is the dynamic urge to then find a new structure $X_1 \supset X, \hat{X}$ as a basis for a more general theory. Experimenters will then construct representations $\hat{X}_1$ and we are back on the same path in a new cycle. Sometimes it can happen that a structure is 'self dual', $\hat{X} \cong X$, in which case the explanation that one is really the other in a dual point of view means that we have no urgency to generalizing the pair $X, \hat{X}$ to a new object. So for self-dual objects the 'engine' stalls and one could consider a theory of physics with such a structure particularly satisfying. For this to be possible the type of object that is $\hat{X}$ has to be the same type that is X, i.e. one has to be in a self-dual category (which is just a self-dual object at a higher up 'level' in our heirerarchy of structures). This gives an exact mathematical formulation of the above principle, namely that:

> The search for a fundamental theory of physics is the search for self-dual structures in a representation theoretic sense.

This self-duality is our proposed 'meta-equation' for the structure of physics. It is obviously an idealisation and hence is not going to be exactly what physics is, but it gives us something concrete to work with which could be used as insight. In physical terms it says that a complete theory of physics should have parts which represent each other hence allowing the reversal of roles of observer and observed, and in the self-dual case the reversed theory has the same form as the original one. The stronger principle of self-duality here says that the reversed theory is indistinguishable from the original up to a relabelling of names of objects and associated units of measurement. The weaker one says that the reversed one may be some other theory but of the same categorical type.

4.4. **Why is there quantum mechanics and why is there gravity?** In the example above of Newtonian mechanics and plane waves, the next step after classical mechanics was to include position X and momentum $\hat{X}$ into a more interacting structure defined by the Heisenberg commutation relations, and so quantum mechanics was created. Actually the usual Heisenberg commutation relations of flat space quantum mechanics does not give a self-dual object but by slightly deforming it to

$$[p, x] = \imath\hbar(1 - e^{-\gamma x})$$

we can obtain a self-dual structure X_1 in a self-dual category, denoted $\mathbb{C}[p]\triangleright\!\!\blacktriangleleft\mathbb{C}[x]$. As well as the algebra product there is a certain 'coalgebra coproduct'

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta p = p \otimes e^{-\gamma x} + 1 \otimes p, \quad \epsilon x = \epsilon p = 0$$

going the other way which makes the dual an algebra. It expresses how momenta and positions should be combined but expressed in Δ in a latent 'potential' form. The dual $\hat{X}_1 = \mathbb{C}[x]\blacktriangleright\!\!\triangleleft\mathbb{C}[p]$ has a left-right reversed but otherwise analogous structure with different parameter values:

$$\hbar' = 1/\hbar, \quad \gamma' = \hbar\gamma.$$

The deformation expressed by γ could be viewed roughly as an element of gravity in the system (so one needs both). The structures here are not Abelian groups as they were in the previous cycle of the 'engine' but quantum groups as in Section 2, but the ideas are the same and the duality between $X_1, \hat{X}_1$ can be expressed physically now by a 2D quantum group Fourier transform. In the process, *we see that our* *philosophical setting provides an answer to why quantum and why gravity.* One can make this more precise:

Theorem 4.1. *The moduli space of all quantum groups X_1 that contain the Hopf algebra of functions in one variable x and map onto the Hopf algebra of functions in another variable p (in the sense of a quantum group extension) has two free parameters $\hbar, \gamma$ and relations as above.*

The technical definition of a (cleft) quantum group extension includes the assertion that the functions of both x, p are subalgebras and that X_1 as a vector space is the space of functions in these two variables. See [20]. The theorem says that if we follow our principle as a constraint within mathematics we will be led to the above models with two parameters coming out of the mathematics. They could be viewed as normalisations of x, p but if we fix these by our physical intention that they shall be position and momentum respectively, then our parameters have physical dimensions. It remains to identify their physical role. At large x the algebra clearly becomes the usual flat space quantum mechanics algebra, so we identify $\hbar$ on this basis as Planck's constant at large x. Now consider an infalling particle of mass m with fixed momentum $p = mv_\infty$ (in terms of the velocity at infinity). By definition p is intended to be the free particle momentum and the free particle Hamiltonian $\frac{\hat{p}^2}{2m}$ induces the motion

$$\dot{p} = 0, \quad \dot{x} = \frac{p}{m}(1 - e^{-\gamma x}) = v_\infty(1 - \frac{1}{1 + \gamma x + \cdots})$$

at the classical level. We see that the particle takes an infinite time to reach the origin, which is an accumulation point. This can be compared with the formula in standard radial infalling coordinates

$$\dot{x} = v_\infty(1 - \frac{1}{1 + \frac{c^2 x}{2GM}})$$

for distance x from the event horizon of a black hole of mass M (here G is Newton's constant and c the speed of light). So $\gamma \sim c^2/GM$ and for the sake of further discussion we will use this value. With a little more work one can see then that

$$\begin{tikzcd}[row sep=small]
 & \mathbb{C}[x]>\!\!\!\triangleleft\mathbb{C}[p] \text{ usual flat space quantum mechanics algebra} \\
\mathbb{C}[x]\blacktriangleright\!\!\!\triangleleft\mathbb{C}[p] \arrow[ur, "mM \ll m_P^2"] \arrow[dr, "mM \gg m_P^2"'] & \\
 & \mathbb{C}(\mathbb{R}\triangleright\!\!\!<\mathbb{R}) \text{ functions on a classical curved geometry}
\end{tikzcd}$$

where m_P is the Planck mass of the order of 10^{-5} grams and $\mathbb{R}\triangleright\!\!\!<\mathbb{R}$ is a nonAbelian Lie group which we think of as a (marginally) curved space. In the first limit the particle motion is not detectably different from usual flat space quantum mechanics outside the Compton wavelength from the origin. In the second limit the estimate is such that quantum noncommutativity would not show up for length scales much larger than the background curvature scale. In short, starting with nothing other than functions in x our philosophical machinery implies something like quantum mechanics combined with something a bit like a black hole[14]. Our analysis here is nonrelativistic as we were speaking about newtonian classical mechanics to begin with; it would be interesting to have a relativistic version.

In proving the theorem one has to consider the moduli of solutions of a pair of cross-coupled first order differential equations which we consider as a 'toy version' of Einstein's equations for matter coupled to gravity, complete with a coordinate singularity at $x = 0$. The same applies in higher dimensional bicrossproduct quantum group models governed by cross-coupled 'matched pair' equations typically with coordinate singularities when the domains are noncompact. As alluded to at the end of Section 2, aside from our quantum algebra combined with gravity interpretation, such quantum groups X_1 can also be viewed as deformed Poincaré groups for models of quantum spacetime, again with Planck scale predictions, see [22]. Also, solving the equations is equivalent to a local group factorisation containing the original classical groups (so in the example above it means a locally defined nonAbelian group $\tilde{X}_1 = \mathbb{R}\bowtie\mathbb{R}$) related to the quantum group X_1 by a process that we call 'semidualisation'. In this way the construction can be applied starting from quantum groups not groups, so from $\tilde{X}_1$ a quantum group factorisation and this turns out to be central to 3D quantum gravity with cosmological constant. We shall explain this further in Section 5.4. Finally, the observable-state duality in the models has been physically related at the semiclassical level to Poisson-Lie T-duality for certain integrable sigma-models[2], of interest in string theory. This makes manifest the sense in which the observable-state duality is also part of a 'micro-macro' duality interchanging small size with large, and is in the spirit if not the detail of a reflection in Figure 1 swapping the line of elementary quantum particles with the line of black holes. Thus the connections between the above ideas and Planck scale physics have continued to run quite deep even if they remain attached so far to specific models.

5. Solutions of the self-duality meta-equation

Here we look at the overall structure of the solutions of our 'meta-equation' which we proposed as what singles out the structure of physics from within mathematics (to first approximation). If our philosophy is right then *this is physics* (to first approximation). So to what extent does it resemble physics? Note that the 'equation' we have in mind is not about constructing one algebra or solving a differential equation but rather in first instance its indeterminates are axiom-systems for mathematical objects and the equation is that these axioms are self-dual in a representation-theoretic sense. Our postulate requires a complete theory of physics to have such a mathematical structure, so at the very least this narrows down our search. The 'strong' version of the postulate is that this is actually *the* main equation of physics in the sense that all that is mathematically possible as solution is (within some other broad requirements) physics. If Einstein's equation and other laws of theoretical physics could indeed be deduced from such a postulate alone, we would have achieved a Kantian or Buddist view of the nature of physical reality as a consequence of the choice to look at the world in a certain way (namely, the scientific method itself). We have had a taste of it in the example above and our 'programme' is to achieve the same more generally and call it quantum gravity.

The first thing to note about the meta-equation is that as an 'equation of physics' within the space of mathematical axioms it has more than one solution, i.e. it predicts that theories of physics form more or less complete paradigms solving the self-duality constraint, that need be perturbed only when we wish to extend them to include more phenomena. It turns out the meta-equation is also highly restrictive. There are not that many self-dual categories known. My own assessment of the mathematical scene is in Figure 4 with the self-dual ones along the central axis.

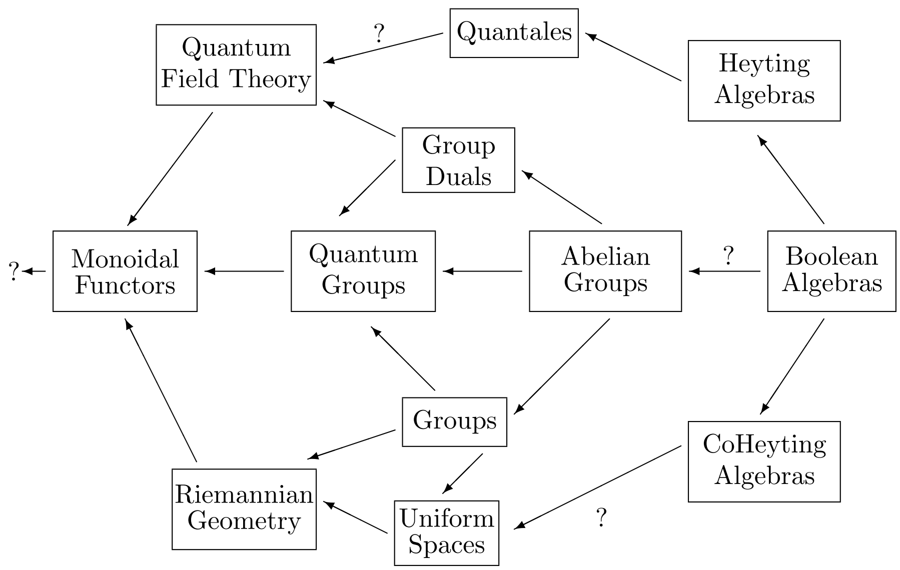

FIGURE 4. Representation-theoretic approach to quantum gravity[12]. Arrows are functors between different categories of objects.

According to our postulate above, the self-dual are 'sweet spots' as paradigms for complete theories of physics while straying off the axis breaks the self-duality and requires also to have the dual or mirror theory. This will not be exactly the historical paradigms of physics but it more like how the paradigms should have developed in an ideal world without the whims of history, according to our theory (which we only intend as a first approximation). The arrows in Figure 4 are functors or maps between different categories of structure whereby one type of structure can be generalised to another or conversely where the latter can be specialised to the former. These express how different paradigms in physics in so far as we identify them can evolve one into the other or conversely remain as a special case.

On the axis we see the 'sweet spot' of Abelian groups (spaces with an addition law) as the setting for classical mechanics as discussed above. Below the axis are nonAbelian groups, which can be considered as the first examples of curved spaces. Above the axis are their duals, which in this case are not groups but are defined by Hilbert spaces and operator representations, i.e. the simplest quantum systems. So above the axis is quantum, below is gravity and they represent each other in some broad sense as illustrated in our discussion of Plato's cave. To unify them we need to put them into a single self-dual category namely that of quantum groups and there we could restore an observer-observed duality as we saw in Sections 2 and 4.4. We see that quantum theory does not have observer-observed symmetry and nor does gravity, only in quantum gravity (by which we mean any form unification) can this be restored and we propose that this should be possible should in fact be a key feature of quantum gravity helping us to find it. The 'sweet spot' of quantum groups allowed certain toy models for this. But general curved spaces are not necessarily groups and real quantum systems are not necessarily group duals, for quantum gravity we need therefore a more general self-dual category and our proposal for this (from two decades ago) is shown in Figure 4 as the category of 'monoidal functors', as the next more general 'sweet spot'. As of last year, this setting is now beginning to be used. One can also reflect on logic as the birth of physics and consider how this evolves into geometry and quantum theory in the first place, which ties up with quantale and topos approaches to quantum gravity. Let us now look at our various paradigms in more detail.

I would like to make one more general observation about our 'meta-equation'. This is that inherited from the relative realism setting is a kind of 'fractal-like structure' to solutions. What I mean is that one can apply the self-duality at several levels, which in some sense are the same concept in a different setting. Thus one can look for a self-dual category, but in that one can look for and perhaps find an actual self-dual object say some quantum group isomorphic to its dual. Within that object, assuming it has elements, one can look for and perhaps find an actual self-dual element invariant under the relevant Fourier transform, while going the other way the self-dual axioms of our first category could perhaps be viewed as self-dual objects at a higher level still where objects were axiom-systems. We shall hint at the role of these different levels without at the moment having a general theory about it.

5.1. **The Abelian groups paradigm.** Thus, vector spaces are a self-dual paradigm or 'category' in mathematical terms. In the finite-dimensional case we mean $\mathbb{R}^n$ and one can consider that this as position space is the paradigm of classical mechanics. Its additive group structure enters mathematically into the notion of differentiation, which is the main ingredient of the kinematical set-up of classical mechanics. Representations of this additive group are the plane waves labelled by p in momentum space. The dual space where p lives is also an vector space and its additive structure is the composition of plane wave representations. One could however view a wave $e^{\imath xp}$ as a wave in momentum space labelled by x, reversing the roles of position and momentum. Finally, in a vector space one has different bases, the delta-function (labelled by points in position space) and the plane-wave basis are both used and provide complementary 'wave-particle' points of view on any phenomenon. According to the principle above, a theory of mechanics based just on point particles in position space would not be complete from a phenomenological point of view, one needs both, connected by Fourier transform.

Next, we should look for self-dual objects. Indeed, gaussians are self-dual under Fourier transform and play a fundamental role in dynamical situations. Likewise, as recalled above, dynamics in classical mechanics is best written symmetrically with respect to position and momentum, in Hamilton-Jacobi form on the product of the two. This is a second layer of the self-duality postulate that dynamics is connected with self-duality of objects (and self-dual equations on them) that is made possible when the paradigm is self-dual. In the vector space paradigm a vector space X is not self-dual until equipped with more structure, but for a suitable notion of dual, $X \times X^*$ is self-dual (using $X{\cong}X^{**}$) in a canonical way. And if we want X itself to be self-dual we need an inner product of some kind, which is to say the main ingredient of a free particle Hamiltonian. One can add interaction terms to the Hamiltonian and of these the harmonic oscillator with Hamiltonian $p^2/2m + \omega^2 x^2$ has a self-dual form which singles it out as a fundamental classical mechanical system in 'pole position'.

Next, we recall that the main ideas of Fourier transform generalise easily to any (locally compact) Abelian group. If X is such a group the set of irreducible (1-dimensional) representations $\hat{X}$ is another in a canonical way and (the theorem of Pontryagin) $X \cong \hat{\hat{X}}$. If ϕ, ψ are two representations of X, define their product as

$$(\phi\psi)(x) = \phi(x)\psi(x)$$

and check that this remains a representation. The only self-dual objects will be of the form $X \times \hat{X}$, $\mathbb{R}$ or $\mathbb{Z}/m\mathbb{Z}$ or products of these. Thus classical particles on circles again have position space and momentum space in the same paradigm of Abelian groups, namely S^1 and $\mathbb{Z}$ and are therefore part of the theory of physics at this level, but if one considers them, one should also consider phenomena where position is $\mathbb{Z}$ and momentum is S^1 as also, a direction more possible to develop today (using noncommutative geometry) than in the time of Hamilton and Jacobi but even then possible (as discrete mechanics). These ideas were of course precursors to quantum theory and we have used them in our illustrations in Section 4, while the last point about discrete spacetime is relevant to some approaches to quantum gravity.

5.2. **The quantum groups paradigm.** The next most general self-dual category is that of quantum groups or 'Hopf algebras' which we have touched upon in Sections 2 and 4. We defer details to [22] but the main features of a quantum group are an algebra H, so there is a product map $\cdot : H \otimes H \to k$ (where we work over some field k), which is also a coalgebra in the sense of a 'coproduct' map $\Delta : H \to H \otimes H$, compatible with the product in the sense of an algebra homomorphism. These and the full axioms are self dual both in the sense that there is an input-output symmetry (so for every structure map there is another one going in the reverse direction and the axioms are invariant under such arrow-reversal) and in that the dual space H^* (defined suitably in the infinite-dimensional case) is again a Hopf algebra, denoted $\hat{H}$ in our previous notation above. The relationship between elements of H^* and representations is a bit more indirect but if $\phi, \psi \in H^*$ their product is

$$(\phi\psi)(x) = (\phi \otimes \psi)(\Delta x),$$

which is to be compared with the formula above for the dual of an Abelian group. Quantum groups also have a Fourier transform precisely generalising Abelian groups. So they fulfill our requirements and should be a paradigm for more general theories of physics. In fact quantum groups contain both nonAbelian groups and nonAbelian group duals as special cases and quantum group Fourier transform contains non-Abelian Fourier transform as a special case. For example, if X is a (possibly) nonAbelian group its span $H = kX$ is a Hopf algebra (called the 'group algebra') of a particularly special form where the output of Δ is symmetric. As a vector space it consists of formal linear combinations of elements of X. The product is the group product law of X extended linearly, and so forth so for a nonAbelian group kX is not commutative. The dual Hopf algebra denoted $k(X)$ is (some form of) algebra of functions on X and is commutative, so again of a classical 'not strictly quantum' form. Its Δ is not symmetric when X is nonAbelian.

A true quantum group has to go beyond such special cases and even though the axioms for quantum groups had been proposed by the mathematician E. Hopf in the 1940's, it was not clear even in the mid 1980s if there were large classes of genuine examples, i.e. if this was a useful definition or mostly empty. In view of the above philosophy, we morally equated the problem of constructing true quantum groups with a toy version of quantum gravity. Thus, we view nonAbelian Lie groups as the first examples of curved geometry, and nonAbelian group duals which usually means irreducible representations of Lie algebras, as the first examples of quantum theory. They are shown below and above the self-dual axis because nonAbelian groups are not a self-dual category (the dual of a nonAbelian group is not a group) and nor are group duals for the same reason. But by viewing both in the same category, namely that of quantum groups, we can hope to reconcile them to obtain true quantum groups that are both modified nonAbelian group duals and modified nonAbelian group algebras. Both have to be modified in view of the mutually incompatible features of commutativity and symmetry of Δ in the two special cases above. Moreover, one should look particularly at quantum groups of self-dual form which if nontrivial will have equally strong geometry and quantum aspects.

We have seen by our example in Sections 2 and 4.4 how such a constraint leads to what we called 'Einstein-like equations' and bicrossproduct quantum groups, which we also explained as to be understood in terms of Lie group factorisation data. Recall that Einstein's equation does indeed equate a geometrical object (the Einstein tensor which measures the curvature of spacetime) to a quantum-mechanical object (the vacuum expectation of the stress energy tensor which measures the matter content). While we do not have such a clear physical picture (mainly due to not knowing how to define these objects in general) we do have some rough similarities. At the time of [13, 14] there were also found independently a different class of true quantum groups, the Drinfeld-Jimbo q-deformations $U_q(g), C_q(G)$ of simple Lie groups (in some sense) from another context (that of generalised symmetries and integrable systems). They are not self-dual in the quantum group sense, although their 'Borel subalgebra' parts $U_q(b_\pm)$ are[4]. But later on (in my work from the 1990's) it was realised that the true geometry of these quantum groups is better seen in their braided versions and that these are in fact essentially self-dual as braided-quantum groups, as a reflection of the linear isomorphism $g \cong g^*$ for a (semi)simple Lie algebra provided by the Killing form. They led to deep results in knot theory which we will touch upon also below.

Finally, even though it is Lie group bicrossproducts which have the interpretations above and which are the main outcomes of [13], the finite group factorisation case is also interesting. This case had been anticipated independently in the Russian literature in the work of G.I. Kac[6] in the 1960s and rediscovered in a Hopf algebra paper by M. Takeuchi in the early 1980's, as well as by myself a few years later. It is quite instructive to see the quantum group structure in further detail, as follows [20]. The key idea is that if a classical group X factorises as $X = G{\bowtie}M$, say, then there is an action $\triangleright : G \times M \to M$ and a 'matching back-reaction' $\triangleleft : G \times M \to G$ defined by $s.u = (s \triangleright u)(s \triangleleft u)$ for all $u \in G, s \in M$ on making use of the unique factorisation. This is equivalent to the matched pair equations that one otherwise has to solve for $\triangleright, \triangleleft$. Now consider squares labeled as in Figure 5(a) by elements of M on the left edge and elements of G on the bottom edge. We can fill in the other two edges by thinking of an edge transformed by the other edge as it goes through the square either horizontally or vertically, the two together is the 'surface transport' $\Rightarrow$ across the square. The matched pair equations have the meaning that a square can be subdivided either vertically or horizontally as shown in Figure 5(b), where the labeling on vertical edges is to be read from top down. There is also a

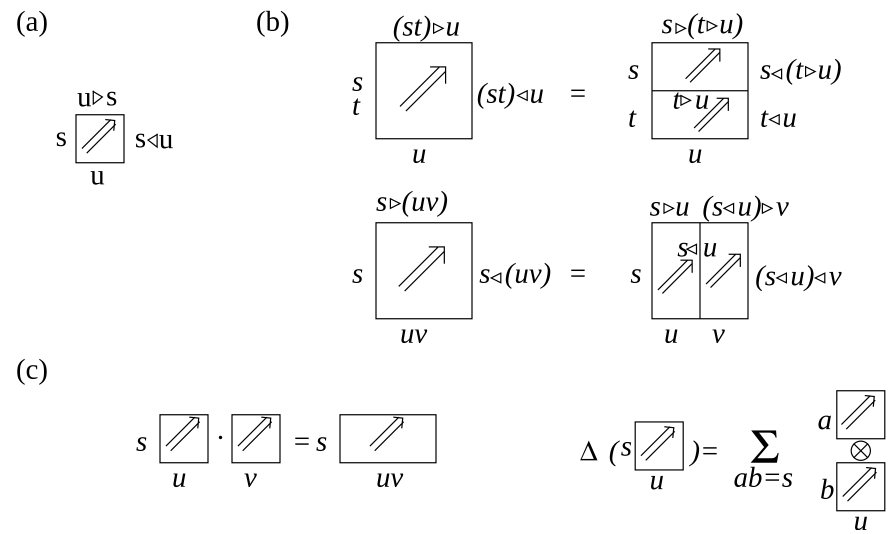

FIGURE 5. (a) Square labeled by elements of a group factorisation, (b) matched pair condition as a subdivision property, and (c) product and coproduct of bicrossproduct quantum group

condition for the group identity which we have omitted. Incidentally, the subdivision property suggests that by infinitely subdividing one should have 'density' versions of the data for $\triangleright, \triangleleft$ which form some new kind of differential geometric '2-dimensional connection' such that infinitesimal squares can be glued back together to define a global 'surface transport' around a piece of surface. The 'exponentiated' transport operation here is nothing other than normal ordering in the factorising group. If one considers solving the quantum Yang-Baxter equations on groups, they appear in this notation as an equality of surface transport going two ways around a cube, and the classical Yang-Baxter equations as zero curvature of the underlying '2-connection'. This appears different from, but may tie up with, more recent ideas for surface transport using 2-categories and of relevance to quantum gravity.

The main structure of the bicrossproduct quantum group $k(M){\blacktriangleright\!\!\triangleleft} kG$ is shown in this notation in Figure 5(c). It has such labeled squares as basis with product zero unless the edges of the squares match up horizontally as shown, in which case the result is the composite square. The coproduct has a similar form but 'unmultiplies' in the sense that Δ consists of all pairs of tensor products which *would give* the initial square when composed vertically. So the role of the coproduct is to do with 'possibility' or 'inference' in contrast to the product which has more of a deductive flavour. Moreover, noncommutativity of the product is interpreted as quantum mechanics, nonsymmetry of the coproduct is linked to curvature (albeit on phase space) which is part of their dual relationship. The dual quantum group $kM{\triangleright\!\!\blacktriangleleft} k(G)$ has the same labeled squares as basis but with the role of vertical and horizontal interchanged. By now several other quantum groups are known, and ones of certain low dimensions over certain fields k have been classified, but the bicrossproduct and the q-deformation ones (defined by braiding matrices) remain the main classes of interest. More on quantum groups can be found in [20, 21].

5.3. **The Boolean logic paradigm.** Before coming to the arguable 'end' of theoretical physics in the form of quantum gravity, lets us look on the right side of Figure 4 and comment on its 'birth'. We take the view that the simplest theories of physics are based on classical logic or, roughly speaking, Boolean algebras. A Boolean algebra can be defined abstractly as a set with two operations $\cap, \cup$ standing for 'and' (conjunction) and 'or' (disjunction) of propositions, an operation $\bar{}$ for the negation of propositions, a zero element 0 for the always false proposition and a unit element 1 for the always true proposition, and some algebraic rules. Both $\cap$ and $\cup$ are commutative, associative and obey

$$A \cap (B \cup C) = (A \cup B) \cap (A \cup C), \quad A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$$

$$A \cap (A \cup B) = A, \quad A \cap \bar{A} = 0, \quad A \cup (A \cap B) = A, \quad A \cup \bar{A} = 1$$

from which one may deduce that

$$A = A \cap A = A \cup A = A \cap 1 = A \cup \phi, \quad A \cap 0 = 0, \quad A \cup 1 = 1$$

$$\bar{1} = 0, \quad \bar{0} = 1, \quad \bar{\bar{A}} = A, \quad \overline{A \cap B} = \bar{A} \cup \bar{B}, \quad \overline{A \cup B} = \bar{A} \cap \bar{B}$$

The last assertions here are known as de Morgan duality and say that $\cap$ and $\cup$ are not independent, the complementation operation takes any boolean algebra expresson (a proposition in propositional logic) over to the similar expression with their negatives and the $\cap, \cup$ interchanged. For example 'not the case that apples are round *and* square' = 'apples are not round *or* apples are not square'. In any Boolean algebra we have a partial ordering $A \subseteq B$ (standing for entailment or implication in propositional logic) defined as holding whenever $A \cap B = A$, which makes a Boolean algebra into distributive complemented lattice (a point of view of relevance perhaps to poset approaches to quantum gravity).

The most concrete way of thinking of a Boolean algebra is as the algebra 2^E of subsets of some 'universe' set E and every finite Boolean algebra is of this form. One can think of a proposition in logic as modelled concretely by the set of things for which it holds, and $\cap, \cup$ are then the intersection and union of subsets, $\bar{A} = E - A$ is complementation of subsets and $\subseteq$ is inclusion. Here $1 = E$ the universe set itself and $0 = \phi$ the 'empty set', both viewed as subsets of E. For simplicity we will focus on this as the main example here. Also, being a bit more precise, the collection of open and closed sets in any topological space form a Boolean algebra so these stand exactly as the starting point for geometry at the level of topology. The link with propositional logic is a dual relationship in which instead of the set E we work with functions $C(E)$ on it. In any unital ring or algebra the set of elements that are central and projections (so $A^2 = A$ and A commutes with all other elements) forms a Boolean algebra with $A \cap B = AB$ and $A \cup B = A + B - AB$ among such elements, which include $0, 1$. In the 'classical algebra of observables' $C(E)$ such functions are precisely the projections or characteristic functions $C(E, \mathbb{Z}/2\mathbb{Z})$ with values in $\{0, 1\}$. Specifying one of these characteristic functions p_A is equivalent to specifying the subset A where its value is 1. When the algebra of observables is noncommutative as in quantum theory such projections still play a role in 'quantum logic' even though they may now not commute. In this way, from a dual perspective, Boolean algebras are also at the starting point for quantum theory. This justifies the placement of Boolean algebras on the axis in Figure 4.

We would still like to make this placement more precise, particularly to explore the sense in which de Morgan duality is a representation-theoretic self-duality. Certainly, one can map any boolean algebra over to an Abelian group (as in Figure 4) by considering the 'exclusive or' operation

$$A \oplus B = (A \cup B) \cap \overline{A \cap B}.$$

Every element of the group is its own inverse, $A \oplus A = 0$ where $A \oplus 0 = A$ identifies 0 as the group identity. In the case of subsets of a set E of order $|E|$ the group here is isomorphic to one of the self-dual groups of the form $(\mathbb{Z}/2\mathbb{Z})^{|E|}$ as discussed above in the Abelian groups paradigm. This suggests that Fourier transform for functions f on the group $(\mathbb{Z}/2\mathbb{Z})^{|E|}$, namely

$$\tilde{f}(A) = \sum_{B \subseteq E} f(A)(-1)^{|A \cap B|}$$

should play some role in the relevant duality. One should think of a subset $A \subseteq E$ here as a $\mathbb{Z}/2\mathbb{Z}$-valued vector of length $|E|$ and $|A \cap B|$ mod 2 as the analogue of the dot product of two vectors in a usual Fourier transform. We think of $f_A(B) = (-1)^{|A\cap B|}$ here as a plane wave on 'position' $(\mathbb{Z}/2\mathbb{Z})^{|E|}$ labelled by elements of the dual 'momentum' copy of $(\mathbb{Z}/2\mathbb{Z})^{|E|}$ and we note that such plane waves are invariant under Fourier transform up to a normalisation.

Note that not all the information of the Boolean algebra is contained in this Abelian group, one needs also the product $\cap$, as used above to define the 'plane waves'. From another point of view this $\cap$ behaves well (distributes) with respect to $\oplus$ and together they form a ring. One can recover the rest of the structure from this ring by

$$A \cup B = A \oplus B \oplus A \cap B, \quad \bar{A} = 1 \oplus A.$$

Note that if we began with a ring then $A \oplus B = A + B - 2AB$ is not necessarily the original ring addition $+$ but is still an associative operation on the set of central projections. However, if $2 = 0$ in the ring then the two coincide. This is true for example if every element of the ring squares to itself. Indeed, one can show in this way that a Boolean algebra is exactly the same thing as a ring where every element squares to itself. From the ring theory point of view de Morgan duality then amounts to saying that for any ring of this type there is another ring with the same features (another Boolean algebra) with new product and sum defined with respect to the old by

$$A \cap' B = A \oplus B \oplus A \cap B, \quad A \oplus' B = 1 \oplus A \oplus B.$$

Equally, one can say that Boolean algebras are a theory of algebras over the field $\mathbb{Z}/2\mathbb{Z}$ in which every element is invariant under the natural 'Frobenius' squaring map defined for any such algebra. From this point of view we can identify $(\mathbb{Z}/2\mathbb{Z})^{|E|} = C(E, \mathbb{Z}/2\mathbb{Z})$ as the algebra of characterisitc functions on the set E with $\cap$ their pointwise multiplication and $\oplus$ their point-wise addition. The dual vector space of this algebra is again identified with $(\mathbb{Z}/2\mathbb{Z})^{|E|} = \mathrm{span}_{\mathbb{Z}/2\mathbb{Z}} E$, this time viewed as the coalgebra over $\mathbb{Z}/2\mathbb{Z}$ with basis E. An element here is the same thing as a list of elements of E, i.e. once again a subset. The coproduct on singleton sets (basis elements) is $\Delta\{a\} = \{a\} \otimes \{a\}$ and this is extended linearly. The pairing between a characteristic function and this dual space is $\langle p_A, B\rangle = |A \cap B|$ modulo 2. This gives some insight but does not solve the precise mathematical sense in which de Morgan duality is some kind of 'representation theoretic' self-duality or Fourier transform. Neither the algebra or coalgebra are Hopf algebras. For the former to be, one would need something like a group structure $\bullet$ on the set E then one could define a coproduct on $C(E, \mathbb{Z}/2\mathbb{Z})$ by

$$\Delta p_A = p_{\Delta A}, \quad \Delta A = \{(b,c) \in E \times E \mid b \bullet c \in A\}$$

defined in terms of the characteristic function of the subset $\Delta A \subseteq E \times E$. For example, for a singleton set $\{a\}$ we have $\Delta p_{\{a\}} = \sum_{b \bullet c = a} p_{\{b\}} \otimes p_{\{c\}}$. In summary, $\cap$ provides the algebra product of $C(E, \mathbb{Z}/2\mathbb{Z})$ while the coproduct if it exists is some kind of sum over the possible elements which *would* multiply to a given one under some extended (not pointwise) operation on the set E. This has a 'time-backwards' flavour of inference as opposed to deductive logic expressed in the product. We have seen such a flavour extending to our bicrossproduct Hopf algebras in the quantum groups paradigm. Equivalently, from the coalgebra point of view we would need something like a product on E to define a product $A \bullet B$ of subsets and make $\mathrm{span}_{\mathbb{Z}/2\mathbb{Z}} E$ into a Hopf algebra. Here $A \bullet B$ has something of the flavour of $A \cup B$ so even when there is no group structure on E we can think of the latter as defining 'something like' a Hopf algebra structure. So $\cap$ and $\cup$ are in 'something like' a dual Hopf algebra relationship. One cannot take this too literally, for example $\cup$ is not bilinear over $\oplus$ (one should use $\oplus'$ for that).

Although none of these points of view is conclusive, they are part of our intuition behind the placement in Figure 4. Next, above the axis moving to Heyting algebras and beyond takes us into intuitionistic logic and ultimately into an axiomatic framework for quantum field theory. A Heyting algebra describes logic in which one drops the familiar 'law of the excluded middle' that $A \cup \bar{A} = 1$ (either a proposition or its negation is true). This generalisation is also the essential feature of the logical structure of quantum mechanics and fits in with the distributive lattice point of view. One can go on to study quantales and projections inside C*-algebras and so forth. *Less familiar* but dual to this is the notion of co-Heyting algebra and co-intuitionistic logic in which one drops the axiom that the intersection of a proposition and its negation is empty. It has been argued by F.W. Lawvere[8] and his school that this intersection is like the 'boundary' of the proposition, and, hence, that these co-Heyting algebras are the 'birth' of geometry. Indeed,

$$\mathrm{d}(A) = A \cap \bar{A}$$

has properties like a boundary operation in geometry. For example Figure 6 illustrates how d is a derivation in the appropriate sense

$$\mathrm{d}(A \cap B) = (A \cap \mathrm{d}B) \cup (\mathrm{d}A \cap B).$$

As seen in the figure, the boundary of $A \cap B$ is the union of the part of the boundary of B lying in A and the boundary of A lying in A. The long-term programme at the birth of physics is to develop this geometrical interpretation of co-intuitionistic logic further into the notion of metric spaces and ultimately into Riemannian or Lorentzian geometry.

Moreover, it has become apparent in recent years that there is a fundamental role of information in physics. If so, then this duality at the birth of physics should tie up with the other dualities above and play a fundamental role in quantum gravity itself. We have seen that de Morgan duality is not exactly representation-theoretic but if, roughly speaking, it plays that role then our claim is that this symmetry of logic in which a set and its complement are interchanged is, ultimately, to be

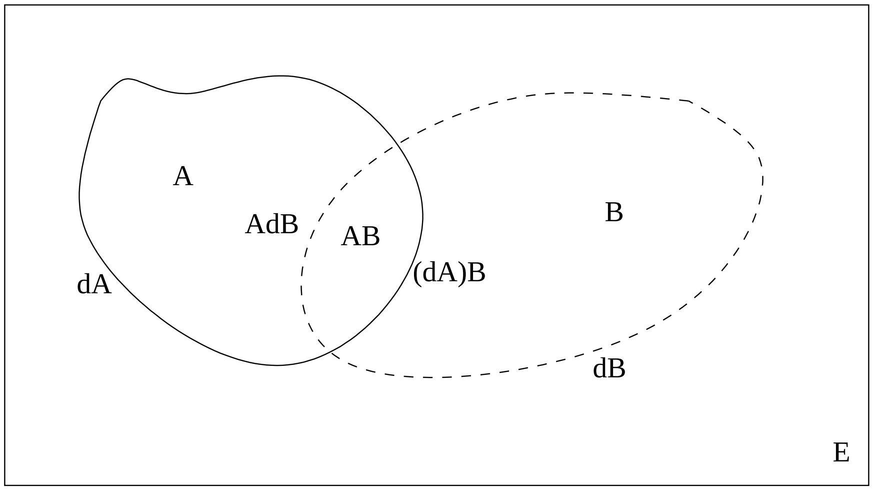

FIGURE 6. Boundary of a set $\mathrm{d}A = A \cap \bar{A}$ is a derivation for the product $\cap$ (denoted here by omission) with respect to $\cup$. This is arguably the birth of geometry.

a symmetry of quantum gravity. Certainly, it is something that is not possible in classical gravity: an apple curves space, but a not-apple which is perfectly equivalent as a concept in classical logic, does not curve space. Concrete matter curves space and this breaks de Morgan duality. On the other hand in a theory of quantum gravity we envision that the symmetry can be restored. When we say for example that one cannot have too many apples in a given volume (the right hand slope in Figure 1) a person with an observer-observed reversed point of view on the same situation might say that space cannot be totally empty of not-apples (which is what that person might regard as the matter content). Indeed, the left slope in Figure 1 expresses quantum theory which says in its field theory formulation that particles are constantly being created in particle-antiparticle pairs out of a vacuum and hence that space is never totally empty[12] . Note that a not-particle here as I envision it should not be confused with an antiparticle which (it is believed) curves space as much as a particle does. Yet the two are also conceptually related in the context of the 'Dirac sea' approach to fermions, a point of view which ignores gravity. This should all be resolved in quantum gravity, wherein we propose that observer-observed symmetry or representation theoretic duality is tied up with a profound extension of de Morgan duality and hence with both time-reversal symmetry and charge conjugation. Some of this was explored in [20, Ch. 5] in the context of quantum random walks and 'coentropy'.

Finally, without knowing quantum gravity one can still look for the kind of logic that might generalise Boolean algebra and might be present there. This may also have a practical application in terms of guiding us to the right logic for quantum computing. The latter is not only about quantum theory, it is also about quantum measurement and about distribution of the processing over physical space. In the next section we will look at the next more general self-dual paradigm, that of monoidal functors between monoidal categories (see below). It is interesting that one attempt to generalise logic is indeed the 'linear logic' of Girard based on something which is a monoidal category in two ways (generalising $\cap$ and $\cup$), see [5] for an introduction. On the other side, topological quantum computers (such as the Kitaev model) have recently been proposed and are based on the monoidal category of representations of quantum groups (or weak quantum groups) at roots of unity. They are also closely related for this reason to 3D quantum gravity with cosmological constant. Recent works of C.J. Isham, F. van Ostaeyen and others on using more general logic and topology in the form of topos theory as an approach to quantum gravity also relates to this setting and is again based on a categorical approach. One also has existing formulations of geometry based on topos theory known as 'synthetic differential geometry' [7].

5.4. **The monoidal functor paradigm.** Finally we come to the left hand side of Figure 4 which, according to our principle of representation-theoretic self-duality should be the setting for the 'end of physics' (as currently understood) in the form of quantum gravity. We need to go much beyond highly symmetrical group manifolds to general (pseudo-Riemannian) geometry on the one hand and beyond group duals to general quantum theory on the other. These are each categories of objects far from the self-dual axis in Figure 4 but again, we believe, in a somewhat dual relationship. Einstein's equation bridges the two and its consistent formulation should therefore be in a self-dual category general enough to contain both general geometry and general quantum theory and deformations of them that would be needed. This is our 'structural approach to quantum gravity'[12, 14].

To this end we introduced in [16, 17, 18] the next most general self-dual category of objects after quantum groups namely the category Mon_* of enriched monoidal categories (or monoidal functors). We proposed this as the paradigm for a more realistic quantum-gravity unification. Briefly, a *monodial category* is a category $\mathcal{C}$ of objects $X, Y, Z, \cdots$ along with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which associates to each pair X, Y of objects a 'tensor product' $X \otimes Y$ with properties similar to that of the tensor product in the category Vec of vector spaces. In particular, for any three objects X, Y, Z one has a specified rebracketting isomorphsm $\Phi_{X,Y,Z}$: $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ and this family of isomorphisms obeys a pentagon identity for the rebracketting of four objects, and is coherent in the sense of being compatible with morphisms between objects. Let me recall that a category is just a clear way of speaking about what objects we consider and what 'maps' (morphisms) are between them and how they are to compose[11]. A functor between categories sends objects of one to objects of the other *and* morphisms of one to morphisms of the other. The reader can just think of $\otimes$ as having the usual properties except that we do not demand that there are isomorphisms $\Psi_{X,Y} : X \otimes Y \cong Y \otimes X$. If these do exist in a coherent manner obeying some hexagon identities (but not necessarily squaring to the identity when applied twice) one has a *braided* monoidal category.

Next, an object of Mon_* is a triple $(\mathcal{C}, F, \mathcal{V})$ where $\mathcal{C}, \mathcal{V}$ are monoidal categories and F is a functor between them that respects their $\otimes$ (a monoidal functor). We regard $\mathcal{V}$ as fixed and consider a morphism between two objects $F : \mathcal{C} \to \mathcal{V}$ and $F' : \mathcal{C}' \to \mathcal{V}$ as a functor $\mathcal{C} \to \mathcal{C}'$ compatible in the obvious way (a commuting triangle) with F, F'. Finally, given an object $F : \mathcal{C} \to \mathcal{V}$ we defined [16] a representation of it as a pair (V, λ) where V is an object of $\mathcal{V}$ and $\lambda_{V,X} : V \otimes F(X) \to F(X) \otimes V$ for all X is a coherent family of isomorphisms subject to

$$\lambda_{V,X \otimes Y} = \lambda_{V,Y} \circ \lambda_{V,X}$$

where isomorphisms $F(X\otimes Y)\cong F(X)\otimes F(Y)$ coming with the functor F and rebracketting isomorphisms are omitted for brevity. If $\otimes$ is some kind of 'associative product' between objects viewed as 'elements of an algebra' then this says that the product is represented by composition of operators 'on' V in some sense. Moreover, the set of all such representations forms itself a monoidal category which we denoted $\mathcal{C}^\circ$. Together with the forgetful functor that forgets λ we have a dual object $F^0:\mathcal{C}^\circ\to\mathcal{V}$ in Mon_*. We also proved that there is a natural morphism $\mathcal{C}\to\mathcal{C}^{00}$ as needed for our self-duality principle. This is not necessarily an isomorphism but it is in the right spirit and we do achieve here a vast generalisation of the Pontryagin theorem for Abelian groups as well a generalisation of quantum group duality.

Specifically, the functor from quantum groups is as follows: for any quantum group H consider its category $\mathrm{Rep}(H)$ of representations. This is tied up with the dual Hopf algebra as we have implied above but one can also consider it abstractly as a monoidal category. Its objects are vector spaces on which H acts with action $\triangleright$ as an algebra. Morphisms are intertwiners. Given two such representations $(V,\triangleright),(W,\triangleright)$, we define the tensor product representation as built on the tensor product $V\otimes W$ of the vector spaces of each one and action given by the action of the elements $\Delta(h)=\sum h_{(1)}\otimes h_{(2)}\in H\otimes H$. Here the coproduct splits $h\in H$ into a sum of parts and the $h_{(1)}$ parts acts on V while $h_{(2)}$ parts acts on W. The forgetful functor completes the triple $F:\mathrm{Rep}(H)\to\mathrm{Vec}$ to an object of Mon_*. Its dual as an object of Mon_* is the collection of 'comodules' of H and under some technical assumptions, this is $\mathrm{Rep}(H^*)$ as given by the dual Hopf algebra.

One application of this duality of monoidal categories is for any monoidal category $\mathcal{C}$ to take the identity functor $\mathrm{id}:\mathcal{C}\to\mathcal{C}$ as an object in Mon_*. In this case the dual $\mathrm{id}^\circ:\mathcal{C}^\circ\to\mathcal{C}$ is a bigger category sometimes nowdays called the 'double category' equipped with a functor to $\mathcal{C}$. In this case, $\mathcal{C}^\circ$ is tautologically braided by $\Psi_{(V,\lambda),(W,\mu)}=\lambda_{V,W}$, as was pointed out to me in a letter by V.G. Drinfeld on coming across the preprint of [16]. Moreover, when $\mathcal{C}=\mathrm{Rep}(H)$ we have $\mathcal{C}^\circ=\mathrm{Rep}(D(H))$ where $D(H)=H{\bowtie}H^{*\mathrm{op}}$ is the Drinfeld double quantum group at least in the finite-dimensional case. Details appeared in [20]. This double $D(H)$ is an example of a special kind of quantum group (called 'quasitriangular') for which representations come equipped with R-matrix operators obeying the braid relations or quantum Yang-Baxter equations [4]. Other quantum groups such as $U_q(su_2)$ are quotients of a certain quantum double and inherit this feature. *We see how this double quantum group construction along with its braiding arises naturally from duality in* Mon_*.

5.4.1. *3D quantum gravity and the cosmological constant.* These considerations may seem far removed from quantum gravity but by now (two decades later) it is understood that they do in fact exactly solve quantum gravity in *three* dimensions and with point-source matter. It was already known by the end of the 1980s that most quasitriangular quantum groups provided invariants of knots (such as the Jones invariant provided by $U_q(su_2)$. These were the Reshetikhin-Turaev invariants [25]. One draws the knot on a piece of paper with 'crossings' and reads the result as the trace of the composition of braid operators, one for each crossing. (This then has to be corrected slightly to become an invariant under the first Reidemeister move, using a ribbon structure in the quantum group). From a physical point of view the theory here is a gauge theory for a Lie-algebra valued field α and Chern-Simons action $\mathrm{Tr}\int\alpha\mathrm{d}\alpha+\frac{2}{3}\alpha^3$ as understood by Witten. The quantum theory was also constructed quite explicitly as a conformal field theory (CFT), basically the Wess-Zumino-Witten model. The Kac-Moody Lie algebra of level k there plays the role of the quantum group with

$$q = e^{\frac{2\pi\imath}{k+2}}$$

in the case of $U_q(su_2)$ (in general the 2 here should be replaced by the dual Coxter number of the Lie algebra in question). The physical picture here generalises to gauge theory on any 3-manifold, particularly of the form $\Sigma \times \mathbb{R}$ where Σ is a Riemann surface if one wants to use CFT methods. It is also necessary to consider the Riemann surface as having marked points in order to make an operator product expansion. One obtains invariants of 3-manifolds which can be related to knot invariants if one views the manifold as given by surgery on a knot, a theory at the level of quantum groups also due to Reshetikhin and Turaev. But what could be viewed as at the heart of this construction from an algebraic point of view is the braided Fourier isomorphism between the braided version of the quantum group and its dual [10]. We have mentioned in the quantum groups paradigm section that quantum groups such as $U_q(su_2)$ are essentially self-dual when viewed in a certain braided category, something which is precisely true at q an odd root of unity with $q \neq 1$. In that case the braided Fourier transform becomes an operator which, together with multiplication by the ribbon element generates a representation of the modular group $PSL(2, \mathbb{Z})$ needed to represent the surgery. And why all this talk of 3-manifold invariants? Since quantum gravity is diffeomorphism invariant the true physical observables should be built from invariants.

A particular invariant of 3-manifolds emerging a little later was the Turaev-Viro construction which can be viewed as given by the quantum double, notably $D(U_q(su_2)) = U_q(so_{1,3})$. This was more in the spirit of the Ponzano-Regge state-sum approach to quantum gravity based on 3-manifold triangulation, but as explained it is also a quantum gauge theory based in this case on the Lie algebra $so_{1,3}$ for the gauge group and Chern-Simons action. A useful exposition of these matters appeared in [26]. For simplicity, consider 3-manifolds of the form $\Sigma \times \mathbb{R}$ and a fixed number of marked points. The motion of these marked points provides the world line of our point sources. For gravity we view the theory in a first order form of a dreibein e^a_μ where $a = 1, 2, 3$ and a spin connection with values in so_3 (in, say, the Euclidean version). These data can be combined together into a single $e_3 = so_3 {\triangleright\!\!\!<} \mathbb{R}^3$-valued gauge field. Its Chern-Simons action then reproduces the Einstein-Hilbert action in the first order formalism, as had been observed early on by Witten[27]. Here e_3 is the isometry group of flat $\mathbb{R}^3$. Its role here is as gauge group and the physical variables are the group E_3-valued holonomies recorded if one parallel transports about world lines (there is a conical singularity around each one) viewed up to overall conjugation. The classical system here is one in which Einstein's equation is solved for matter point sources and prescribed such holonomies. We do not need to know the details of the metric; the equations of motion say the curvature vanishes off the world lines so we can and should focus on the prescribed holonomies and the world-lines as the 'physical data'. One can then quantise this phase space as a Poisson manifold and this replaces each copy of the group E_3 by its quantum group version $D(U(su_2))$ (the quantum double of the classical enveloping algebra of the Lie algebra $su_2 = so_3$. One can do this, but one can also first deform the classical situation replacing E_3 by $SO_{1,3}$ (of which it is a certain contraction). This no longer models Einstein's equation but

Einstein's equation *with cosmological constant*. When this is quantized using the natural Poisson-Lie structure on $SO(1,3)$ one has the quantum double $D(U_q(su_2))$. Quantum gravity with cosmological constant in this approximation is exactly described by chosing representations of this quantum double at each of the marked points to define the Hilbert space for the quantisation and indeed amounts to the Turaev-Viro invariant. While not every aspect of this 3D quantum gravity is fully understood, such as precisely how (using noncommutative geometry) the classical limit emerges, it is considered that the theory in this simplified form is completely solvable and largely solved. Moreover, we have seen that the key elements of its structure are the quantum double and a braided Fourier transform operator, both of which have a natural origin in Mon_* and representation-theoretic duality. In principle one could formulate it more naturally in these terms (starting directly with G. Segal's definition of a topological quantum field theory as a functor).

We can also see the role in 3D quantum gravity of bicrossproducts and observable-state duality. We have mentioned at the end of Section 4 that the factorisation construction $H_1 \bowtie H_2$ is canonically related to bicrossproduct Hopf algebra $\hat{H}_2 \blacktriangleright\!\!\triangleleft H_1$ by a process which we called 'semidualisation'. In the case of the quantum double $H \bowtie H^{*op}$ the semi-dual comes out as isomorphic to $H \otimes H^{op}$. In fact this was our own approach to the Drinfeld double in [15] as the semidual of something trivial. On the other hand, $H_1 \bowtie H_2$ acts canonically on $\hat{H}_2$ which we think of as 'Poncaré quantum group' with rotations H_1 and momentum H_2 acting on the algebra of position coordinates $\hat{H}_2$. In this case we have a natural Poincaré-Heisenberg algebra $\hat{H}_2 \rtimes (H_1 \bowtie H_2)$. In the semidual theory the position and momentum are interchanged, so we have 'Poincaré quantum group' $\hat{H}_2 \blacktriangleright\!\!\triangleleft H_1$ and a canonical action on the position algebra H_2. It has the *same Poincaré Heisenberg algebra*

$$\hat{H}_2 \rtimes (H_1 \bowtie H_2) = (\hat{H}_2 \blacktriangleright\!\!\triangleleft H_1) \ltimes H_2.$$

This is an extension of the observable-state duality or 'quantum Born reciprocity' in Section 2 to our current context and we can ask for the physics to be self-dual in the form of the two models having the same content. For example, we saw that the system for 3D quantum gravity without cosmological constant was the quantum group $D(U(su_2))$ viewed as a quantisation of e_3. Its semidual is $U(su_2) \otimes U(su_2)^{op} = U(so_4)$ which is a very different object so this theory has a semidual but is not self-dual. Indeed the semidual of 3D quantum gravity is a classical theory. However, if we *do* have a cosmological constant in the theory, even a very small one, we have the q-deformed versions on quantisation. Then $D(U_q(su_2))$ has semidual $U_q(su_2) \otimes U_q(su_2)^{op}$. But for reasons related to the braided-self-duality of $U_q(su_2)$ that made possible the braided Fouier transform one also has $D(U_q(su_2)) \cong U_q(su_2) \blacktriangleright\!\!\blacktriangleleft U_q(su_2)$ provided $q \neq 1$, see [20], which is equivalent to $U_q(su_2) \otimes U_q(su_2)^{op}$ in the sense of the monoidal category of representations of the one quantum group and the other being equivalent (we called this 'quantum Wick rotation'). Since, as we have explained, the representation category was the main ingredient in the quantum theory we conclude that *the original 3D quantum gravity theory with cosmological constant and its semi-dual are equivalent*. The semidual one is the quantisation of two independent su_2-Chern-Simons theories. We can also look at the semidualisation of the classical theory with cosmological constant. The semidual of the classical enveloping algebra $U(so_{1,3})$ is the bicrossproduct quantum group $\mathbb{C}[\mathbb{R} \ltimes \mathbb{R}^3] \blacktriangleright\!\!\triangleleft U(su_2)$ which is the 3D version of the model[19] mentioned at the end of Sections 2,4 as being testable by gamma-ray burst experiments and reviewed further in [22]. Here a classical theory semidualizes to a quantum one. For self-duality we need both a cosmological constant *and* quantization.

More work is needed to make these ideas precise both physically and mathematically and some of this will be done in the forthcoming work [24], where we explore semidualisation in 3D quantum gravity in detail. What we do already see, however, is that the principle of self-duality applied to 3D quantum gravity not only provides key ingredients in its solutions but requires a nonzero value of the cosmological constant. Why this extra term in Einstein's equation is present as it seems from the experimental observations of astronomers (where it is called 'dark energy') and why in particular it has a very small (compared to the Planck scale) yet nonzero value is an open problem into which our 3D observations do provide some insight.

5.4.2. *4D quantum gravity.* And what can we say really about quantum gravity in four dimensions? Not too much at the moment (I consider the problem still open) although obviously we can get some lessons from the 3D case, fitting in particularly with the spin foam and loop quantum gravity approaches. From the point of view of manifold invariants, however, the 4D case is the hardest and one does not expect to get very far with quantum groups themselves. According to our proposal in [12] we really need to go to more general objects in Mon_*. It is still too early to tell but I would make two remarks here. One approach to 4-manifold invariants is to use 2-categories, and indeed the $F^\circ : \mathcal{C}^\circ \to \mathcal{V}$ construction above was one of the ingredient feeding into Kapranov's formulation these. More recently it has been shown by Fredenhagen and coworkers[1] that the construction of quantum field theory (QFT) on curved spacetime in a covariant manner can be formulated indeed as the construction of a monoidal functor

$$F : \text{Globally hyperbolic manifolds} \to C^* - \text{Algebras}$$

as mentioined at the end of [22]. The functor here expresses QFT on a classical background (it is not yet quantum gravity) and moreover is far from self-dual. Our suggestion now would be to look for more general examples as a deformation of it (i.e. of both QFT and classical geometry) within this paradigm Mon_* and under a self-duality type constraint (this has not yet been looked at at the time of writing). The deformation could include quantisation of the background in the process.

What comes as the next even more general self-dual category of objects? Undoubtedly something, but note that the last one considered is already speaking about categories of categories. One cannot get too much more general than this without running out of mathematics itself. In that sense physics is getting so abstract, our understanding of it so general, that we are nearing the 'end of physics' as we currently know it. This explains from our point of view why quantum gravity already forces us to the edge of metaphysics and forces us to face the really 'big' questions. Of course, if we ever found quantum gravity I would not expect physics to actually end, rather that questions currently in metaphysics would become part of a new physics. However, from a structural point of view by the time we are dealing with categories of categories it is clear that the considerations will be very general. I would expect them to include a physical approach to set theory [18] including language and provability issues raised by Goedel and discussed notably by Penrose.

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