Interpolation and Beth Definability over the Minimal Logic

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Abstract

Extensions of the Johansson minimal logic J are investigated. It is proved that the weak interpolation property WIP is decidable over J. Well-composed logics with the Graig interpolation property CIP, restricted interpolation property IPR and projective Beth property PBP are fully described. It is proved that there are only finitely many well-composed logics with CIP, IPR or PBP; for any well-composed logic PBP is equivalent to IPR, and all the properties CIP, IPR and PBP are decidable on the class of well-composed logics..

Keywords: Interpolation, Beth property, minimal logic.

1 Superintuitionistic logics and J-logics

In this paper we consider extensions of the Johansson minimal logic J; this family extends the class of superintuitionistic (s.i.) logics. The main variants of the interpolation property are studied. In [4] we have proved that the weak interpolation property is decidable over J. There are only finitely many superintuitionistic logics with CIP, IPR or PBP, all of them are fully described [1,3], and CIP, IPR and PBP are decidable on the class of s.i. logics. Here we extend these results to the class of well-composed J-logics.

The language of J contains $\&, \vee, \rightarrow, \bot$ as primitive; negation is defined by $\neg A = A \rightarrow \bot$. The logic J can be given by the calculus, which has the same axiom schemes as the positive intuitionistic calculus Int^{+}, and the only rule of inference is modus ponens. By a J-*logic* we mean an arbitrary set of formulas containing all the axioms of J and closed under modus ponens and substitution rules. We denote

$\text{Int} = \text{J} + (\bot \rightarrow A)$, $\text{Neg} = \text{J} + \bot$, $\text{Gl} = \text{J} + (A \vee \neg A)$,
$\text{Cl} = \text{Int} + (A \vee \neg A)$, $\text{JX} = \text{J} + (\bot \rightarrow A) \vee (A \rightarrow \bot)$.

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A J-logic is *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if contains Neg. A J-logic is *well-composed* if it contains JX. For a J-logic L, the family of J-logics containing L is denoted by $E(L)$.

If $\mathbf{p}$ is a list of variables, let $A(\mathbf{p})$ denote a formula whose all variables are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic. *The Craig interpolation property CIP, the restricted interpolation property IPR* and *the weak interpolation property WIP* are defined as follows (where the lists $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are disjoint):

CIP. If $\vdash_L A(\mathbf{p},\mathbf{q}) \rightarrow B(\mathbf{p},\mathbf{r})$, then there is a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \rightarrow C(\mathbf{p})$ and $\vdash_L C\mathbf{p}) \rightarrow B(\mathbf{p},\mathbf{r})$.

IPR. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

WIP. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$ are disjoint lists of variables that do not contain x and y, $\mathbf{q}$ and $\mathbf{q}'$ are of the same length, and $A(\mathbf{p},\mathbf{q},x)$ is a formula. We define *the projective Beth property*:

PBP. If $A(\mathbf{p},\mathbf{q},x), A(\mathbf{p},\mathbf{q}',y) \vdash_L x \leftrightarrow y$, then $A(\mathbf{p},\mathbf{q},x) \vdash_L x \leftrightarrow B(\mathbf{p})$ for some $B(\mathbf{p})$.

The weaker *Beth property BP* arises from PBP by omitting $\mathbf{q}$ and $\mathbf{q}'$.

All J-logics satisfy BP, and for these logics the following hold:

- CIP $\Rightarrow$ PBP $\Rightarrow$ IPR $\Rightarrow$ WIP, PBP $\not\Rightarrow$ CIP, WIP $\not\Rightarrow$ IPR.

It is proved in [4] that WIP is decidable over J, i.e. there is an algorithm which, given a finite set Ax of axiom schemes, decides if the logic J+Ax has WIP. The families of J-logics with WIP and of J-logics without WIP have the continuum cardinality.

The logics J, Int, Neg, Gl, Cl and JX possess CIP and hence all other above-mentioned properties. It is known [3] that

- IPR $\Leftrightarrow$ PBP over Int and Neg.

It is known that there are only finitely many s.i. and negative logics with CIP, IPR and PBP [1,3]. Here we extend this result to all well-composed logics. Also we prove that IPR is equivalent to PBP in any well-composed logic, and CIP, IPR and PBP are decidable over JX.

2 Interpolation and amalgamation

The considered properties have natural algebraic equivalents. There is a duality between J-logics and varieties of J-algebras [6].

Algebraic semantics for J-logics is built via J-*algebras*, i.e. algebras $\mathbf{A} =< A; \&, \vee, \rightarrow, \bot, \top >$ such that A is a lattice w.r.t. $\&, \vee$ with the greatest element $\top$, $\bot$ is an arbitrary element of A, and

$z \leq x \rightarrow y \iff z \& x \leq y.$

A J-algebra $\mathbf{A}$ is a *Heyting algebra* if $\bot$ is the least element of A, and *a negative algebra* if $\bot$ is the greatest element of A; the algebra is *well-composed* if every its element is comparable with $\bot$. For any well-composed J-algebra $\mathbf{A}$, the set $\mathbf{A}^l = \{x|\ x \leq \bot\}$ forms a negative algebra, and the set $\mathbf{A}^l = \{x|\ x \geq \bot\}$ forms a Heyting algebra. If $\mathbf{B}$ is a negative algebra and $\mathbf{C}$ is a Heyting algebra, we denote by $\mathbf{B} \uparrow \mathbf{C}$ a well-composed algebra $\mathbf{A}$ such that $\mathbf{A}^l$ is isomorphic to $\mathbf{B}$ and $\mathbf{A}^u$ to $\mathbf{C}$. For a negative algebra $\mathbf{B}$, we denote by $\mathbf{B}^\Lambda$ a J-algebra arisen from $\mathbf{B}$ by adding a new greatest element $\top$.

A J-algebra $\mathbf{A}$ is *finitely indecomposable* if for all $x, y \in \mathbf{A}$:

$x \vee y = \top \Leftrightarrow (x = \top$ or $y = \top)$.

If A is a formula, $\mathbf{A}$ a J-algebra, then A is *valid in* $\mathbf{A}$ (in symbols, $\mathbf{A} \models A$) if the identity $A = \top$ is valid in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$. Let $V(L) = \{\mathbf{A}|\mathbf{A} \models L\}$. Each J-logic L is characterized by the variety $V(L)$.

We recall the definitions. A class V has *Amalgamation Property* if it satisfies

AP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\epsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \epsilon(x)$ for all $x \in \mathbf{A}$.

Super-Amalgamation Property (SAP) is AP with extra conditions:

$$\delta(x) \leq \epsilon(y) \Leftrightarrow (\exists z \in \mathbf{A})(x \leq z \text{ and } z \leq y),$$

$$\delta(x) \geq \epsilon(y) \Leftrightarrow (\exists z \in \mathbf{A})(x \geq z \text{ and } z \geq y).$$

Restricted Amalgamation Property (RAP) and *Weak Amalgamation Property (WAPJ)* are defined as follows:

RAP: for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and the restriction of g onto $\mathbf{A}$ is a monomorphism.

WAPJ: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\epsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \epsilon(x)$ for all $x \in \mathbf{A}$, and $\bot \neq \top$ in $\mathbf{D}$ whenever $\bot \neq \top$ in $\mathbf{A}$.

A class V has *Strong Epimorphisms Surjectivity* if it satisfies

SES: For each $\mathbf{A}, \mathbf{B}$ in V, for every monomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ and for every $x \in \mathbf{B} - \alpha(\mathbf{A})$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta : \mathbf{B} \to \mathbf{C}$, $\gamma : \mathbf{B} \to \mathbf{C}$ such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \neq \gamma(x)$.

Theorem 2.1 ([2]) *For any* J*-logic L:*

(1) L has CIP iff V(L) has SAP iff V(L) has AP,

(2) L has IPR iff V(L) has RAP, (3) L has WIP iff V(L) has WAPJ,

(4) L has PBP iff V(L) has SES.

In varieties of J-algebras: SAP $\iff$ AP $\Rightarrow$ SES $\Rightarrow$ RAP $\Rightarrow$ WAPJ.

3 Weak interpolation and negative equivalence

For $L_1 \in E(\text{Neg})$, $L_2 \in E(\text{Int})$ we denote by $L_1 \uparrow L_2$ a logic characterized by all algebras of the form $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A} \models L_1$, $\mathbf{B} \models L_2$; a logic characterized by all algebras $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A}$ is a finitely decomposable algebra in $V(L_1)$

and $\mathbf{B} \in V(L_2)$, is denoted by $L_1 \Uparrow L_2$. Say that a J-logic is *primary* if it is of the form $L_1 \uparrow L_2$ or $L_1 \Uparrow L_2$.

In [2] an axiomatization was found for logics $L_1 \uparrow L_2$ and $L_1 \Uparrow L_2$, where L_1 is a negative and L_2 an s.i. logic.

All s.i. and negative logics have WIP. On the contrary, there are only finitely many s.i. and negative logics with CIP, IPR and PBP [1,2,3]. We give the list of all negative logics with CIP:

$$\mathrm{Neg},\ \mathrm{NC} = \mathrm{Neg} + (p \to q) \vee (q \to p),\ \mathrm{NE} = \mathrm{Neg} + p \vee (p \to q),\ \mathrm{For} = \mathrm{Neg} + p.$$

It is proved in [4] that WIP is decidable over J, i.e. there is an algorithm which, given a finite set Ax of axiom schemes, decides if the logic J$+Ax$ has WIP. A crucial role in the description of J-logics with WIP [4] belongs to the following list SL of eight *etalon logics*:

$$\{\mathrm{For}, \mathrm{Cl}, (\mathrm{NE} \uparrow \mathrm{Cl}), (\mathrm{NC} \uparrow \mathrm{Cl}), (\mathrm{Neg} \uparrow \mathrm{Cl}), (\mathrm{NE} \Uparrow \mathrm{Cl}), (\mathrm{NC} \Uparrow \mathrm{Cl}), (\mathrm{Neg} \Uparrow \mathrm{Cl})\}.$$

We say that a J-algebra is *central* if $\bot \neq \top$ and $x \leq \bot$ for any $x \neq \top$. For a J-logic L define the *center* $\Lambda(L)$ as the class of all central algebras validating L. Let a *central companion* L_{cn} of L be a logic generated by $\Lambda(L)$.

All etalon logics are generated by their centers, finitely axiomatizable, and finitely approximable [4]. A center of an etalon logic is said to be an *etalon center.*

Proposition 3.1 *For each etalon logic L_0 there is an algorithm which, given a finite set Ax of axiom schemes, decides if the logic* J $+ Ax$ *is equal to L_0.*

Theorem 3.2 ([4]) *For any* J*-logic L the following are equivalent:*

(i) *L has WIP,*

(ii) *$\Lambda(L)$ has the amalgamation property.*

(iii) *L has an etalon center.*

Two J-logics L and L' are *negatively equivalent* [6] if for any formula A

$$L \vdash \neg A \iff L' \vdash \neg A.$$

Theorem 3.3 *Two* J*-logics are negatively equivalent iff they have the same center.*

Theorem 3.4 *A J-logic has WIP iff it is negatively equivalent to one of the etalon logics.*

Theorem 3.5 ([4]) *WIP is decidable over* J.

4 Interpolation in well-composed J-logics

For any J-logic L define the *negative* and *intutionistic companions*:

$$L_{neg} = L + \bot,\ L_{int} = L + (\bot \to A).$$

The following theorem describes all well-composed logics with CIP.

Theorem 4.1 ([5]) *Let L be a well-composed logic. Then L has CIP if and only if L_{neg} and L_{int} have CIP, and L is representable as $L = L_{neg} \cap L_1$, where L_1 is a primary logic with an etalon center.*

The following theorem gives a full description of well-composed logics with IPR and PBP.

Theorem 4.2 ([5]) *For any well-composed logic L the following are equivalent:*

(i) *L has IPR,*

(ii) *L has PBP,*

(iii) *the companions L_{neg} and L_{int} have IPR, the central companion L_{cn} is an etalon logic, and L is representable as*

$$L = L_{neg} \cap L_{cn} \cap L_1,$$

where L_1 is a primary logic with an etalon center.

Corollary 4.3 *There are only finitely many well-composed logics with IPR; all of them are finitely axiomatizable and finitely approximable.*

Theorem 4.4 ([5]) *CIP, IPR and PBP are decidable on the class of well-composed logics.*

The following problems are still open.

Problem 1. How many J-logics have CIP, IPR or PBP?
Problem 2. Are IPR and PBP equivalent over J?
Problem 3. Are CIP, IPR and/or PBP decidable over J?

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