# Note on Carnap's "On the Dependence of the Properties of Space Upon Those of Time"*

Version 1.1

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Carnap's goal in the paper is to make precise a sense in which, if relativity theory is correct, statements about the topological structure of physical space can be reduced to statements about temporal or causal order. In what follows, I will reconstruct Carnap's account, indicate a number of technical problems, suggest how they might be fixed and, finally, contrast Carnap's work here with that done earlier by the British mathematician A. A. Robb [3].

## 1 Carnap's Construction

It may help to proceed in two stages. In this section, I'll give a summary description of Carnap's construction using, more-or-less, the same terms that he employs. Then, in the following section, I'll revisit certain points in the con-

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struction and recast them using the more precise language of modern relativistic spacetime geometry.

Carnap starts by considering the class of all *world points (Weltpunkte).* He understands these to be the basic elements that enter into spatiotemporal relations, but leaves open exactly how we are think about them. In particular, he declines to take a stand on whether world points can be coincident without being identical. (At issue here, it seems, is whether we take them to be "point events" (or, perhaps, "possible point events") or rather "point event locations".) Instead, he considers alternative languages, side-by-step, in which one does and does not have a special relation symbol for coincidence. So far as Carnap's actual construction is concerned, it makes little difference which of these interpretations, or which of these languages, one adopts. For convenience, I will (in effect) work with the latter, and so avoid the need to consider coincidence as a relation distinct from identity.

Carnap considers a second choice that is more important for our purposes. At issue here is just what order structure on the set of world points is taken as primitive. He considers two possibilities. On the first, one takes *worldlines (Weltlinie)* themselves as primitive elements in the construction. (These worldlines are understood to carry a temporal orientation; there is a future-direction at each point.) On the second, one makes do with less. One takes as primitive a two-place relation on the set of world points. Carnap uses the symbol '$W$' for it. Two world points stand in the relation $W$ if there *exists* a worldline that runs from the first to the second.

When Carnap talks about reducing spatial topology to "temporal order", he has in mind that version of the construction in which worldlines are taken as primitive. When he talks instead about reducing it to "causal order", he has in mind the alternative version in which the relation $W$ is taken as primitive.

The reduction itself is realized with two definitions. First, using only the allowed primitive relations, he introduces the notion of a *spatial class* (*Raumklasse*). We are to think of any one spatial class, intuitively, as a set of world points that qualifies as a candidate for constituting "space at a given time" or being a "simultaneity slice". We may think of spatial classes, in particular cases, as determined relative to some particular worldline or family of worldlines or, perhaps, determined in some other way. But "where they come from" is not important for Carnap here and he does not discuss the matter. He is only interested in structural features of spatial classes that make them appropriate for their assigned role. Carnap's official definition is the following.

> (Definition 1) A set $S$ of world points is a *spatial class* if (i) no two (distinct) points in $S$ stand in the relation $W$, and (ii) every worldline intersects $S$.

One could, equivalently, require that every worldline intersect $S$ in a unique point – for if a worldline intersected $S$ in two points, those two *would* stand in the relation $W$.

The intuition behind the definition should be clear. Spatial classes are "partition sets" of a certain kind. They serve to partition every worldline into regions of "before", "now", and "later". Equivalently, they are aggregations of "now" points on different worldlines that fit together properly, i.e., fit together so that no one point is $W$-related to any other.

As it stands, the definition makes reference both to "wordlines" and to the relation $W$. One can always eliminate the latter in terms of the former. But it is not evident that one can recast the definition making reference only to $W$. Carnap does not propose any version involving just $W$ and does not comment on the matter. This is puzzling because he does seem to claim (e.g., in the penultimate paragraph) that $W$, by itself, is an adequate primitive for the reduction he proposes. And he does seem to consider it essential (as part of that reduction) that one be able to characterize spatial classes in terms of the available primitives.

We will say more about this first definition in the next section, but let us move on. Carnap's second task is to define a topology on spatial classes, again using only the allowed primitives. This amounts to specifying which subsets are to qualify as "open". Once the definition is in place, every statement about (this) topology can, in principle, be translated into a longer statement about the primitives alone. Here is a lightly paraphrased version of Carnap's definition. (It uses one bit of notation that is not in the paper. Given any world point $p$, let $W^+(p)$ be the set of all world points $q$ such that $pWq$. It is the "future set" of $p$ relative to the relation $W$.)

(Definition 2) Let $S$ be a spatial class and let $O$ be a subset of $S$. Then $O$ is *open* if, for all world points $q$ in $O$, there is a world point $p$ such that $p \neq q$, $pWq$, and $(W^+(p) \cap S) \subseteq O$.

(See figure 1.) The idea is that we start with all future sets $W^+(p)$, where $p$ to the $W$-past of $S$, and then restrict them to $S$. In this way we generate a set of what might be called "$W$-neighborhoods". Then we take a set to be open if it can be realized as a union of $W$-neighborhoods.[1]

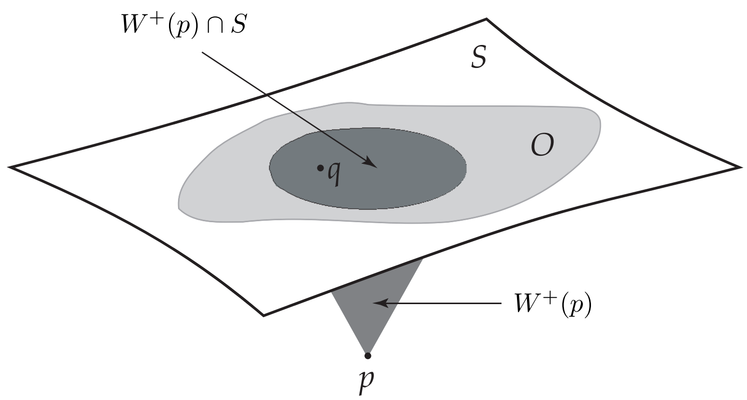


Figure 1: In definition 2, the "open" subsets of a spatial class are defined in terms of $W$.

Of course, a question arises. How do we know that the set of open sets as characterized here, in the case of any one spatial class, *does* qualify as a topology? In particular, how do we know that it is closed under finite intersections? Carnap offers no details. He asserts that "it can be shown" that all the requisite conditions for a topology are satisfied – indeed, he specifies "Hausdorff topology" – but leaves further argument for another occasion. He takes himself to be offering only a sketch of what would have to be a long presentation.

We will return to this question in the next section. It is hard to get a grip on it until we further pin down what geometrical objects are to count as "worldlines".

[1] We include the requirement that $p \neq q$ in the definition because no convention has been specified as to whether singleton sets count as worldlines. If they do, then $W$ is reflexive, and $W^+(q) \cap S = \{q\}$. So, without the requirement, every subset of $S$ would qualify as open.

## 2 Carnap's Construction and Spacetime Geometry

Let us now adopt the framework of relativistic spacetime geometry and revisit Carnap's account. We need to consider three issues.

(1) How do we represent "worldlines"? (Once that question is answered, it will be determined, derivatively, how we represent the relation $W$.) Presumably, we should represent them as curves of some type on the underlying spacetime manifold.[2] But two options arise as to what that type should be, and both pose problems for Carnap's definitions.

(2) How general is the account supposed to be? Carnap takes himself to be discussing relativistic spacetime structure quite generally (in contrast to Robb who limits attention to one particular spacetime model, namely Minkowski spacetime). But Carnap takes things for granted that are not true in all relativistic spacetime models. In particular, he takes for granted that spatial classes *exist*, and that is not always the case.

(3) How well motivated is the definition of spatial classes? (This will have to be explained.)

We can understand relativity theory to determine a class of geometrical models for the spacetime structure of the universe. Each is an ordered pair $(M, g_{ab})$, where $M$ is a smooth, connected, four-dimensional differential manifold without boundary, and $g_{ab}$ is smooth, pseudo-Riemannanian metric on $M$ of Lorentz signature.[3] What is most important for present purposes is that the metric determines a "null cone structure" in the tangent space at each point of $M$, i.e., a partition of tangent vectors there into three classes: timelike vectors (that fall inside the cone), null vectors (that fall on its boundary), and spacelike vectors (that fall outside the cone). We say that a smooth curve is timelike (respectively null or spacelike) if its tangent vector at every point is of that type.

We will restrict attention to spacetime models that are "temporally orientable" and for which a particular temporal orientation has been given. The

[2] I will not bother to distinguish here between curves and their images.

[3] These technical notions and others that follow are defined, for example, in Hawking and Ellis [1] and Wald [4].

latter is a specification, continuous (in an appropriate sense) over the underlying manifold, of which null cone lobe at each point is to count as the "future lobe". This allows us to designate timelike and null curves as "future-directed" and "past-directed". (They are so if their tangent vectors are everywhere pointed in the corresponding direction.)

One more bit of terminology. We say that a vector at a point is "causal" if it is *either* timelike or null. And, derivatively, we say that a smooth curve is of this type if its tangent vector at every point is.

It is one of the basic interpretive principles of relativity theory that massive point particles are represented by timelike curves, and light rays are represented by null geodesics. For this reason, when people discuss the "causal structure" of a relativistic spacetime model, they generally have in mind the following two relations. Given points $p$ and $q$ in the background manifold $M$, we say that

(i) *$q$ is to the temporal future of $p$*, and write $p \ll q$, if there is a smooth, future-directed timelike curve that runs from $p$ to $q$.

(ii) *$q$ is to the causal future of $p$*, and write $p < q$, if there is *either* a smooth, future-directed timelike curve that runs from $p$ to $q$, or a future-directed null geodesic that does so.

One can prove that the condition in (ii) holds iff there is a smooth, future-directed causal curve that runs from $p$ to $q$. This alternate formulation is a bit more compact.

These two relations, $\ll$ and $<$, seem the most natural candidates for Carnap's relation $W$. Equivalently, it seems most natural to take his "worldlines" to be either smooth future-directed timelike curves (narrow option) or, more inclusively, smooth future-directed causal curves (broad option).[4]

There is evidence in the text that Carnap, at least officially, has the broad option in mind. Indeed, he says explicitly:

[4] We could consider piecewise smooth curves here. (And Carnap does sometime talk about "chains" of worldline segments.) But doing so would not change much. One can show that if there exists a piecewise smooth timelike (respectively causal) curve running from $p$ to $q$, then there is also a smooth timelike (respectively causal) curve running from $p$ to $q$.

> ... it should be remembered that the world lines represent the lines not only of material elements, but also of energetic ones – for example, light rays.

But if we understand $W$ to be the relation $<$, then Carnap's second definition breaks down. It is not true in this case then the the collection of "open" subsets of a spatial class must be closed under finite intersections. For example, suppose we consider the simplest possible case – that in which the spatial class $S$ is a flat spacelike submanifold in Minkowski spacetime. Then every "closed spherical ball" in $S$ qualifies as "open".[5] But the intersection of two such balls can be a singleton set, and singleton sets do not qualify as "open".

Let $I^+(p)$ and $J^+(p)$ be the future sets of a point $p$ associated with the relations $\ll$ and $<$ respectively, i.e.,

$$I^+(p) \;=\; \{q:\; p \ll q\} \qquad\qquad J^+(p) \;=\; \{q:\; p < q\}.$$

Further, let $I^-(p)$ and $J^-(p)$ be the corresponding past sets. (This is standard notation.) The sets $I^+(p)$ and $I^-(p)$ are open in the manifold topology – for all points $p$ in all relativistic spacetime models (Wald [4, p.190]). In contrast, the sets $J^+(p)$ and $J^-(p)$ are, in general, neither open nor closed in the manifold topology. (In Minkowski spacetime, specifically, they are closed.) For this reason, it is more convenient to work with the "narrow option". If we understand $W$ to be the relation $\ll$, then Carnap's second definition *does* successfully characterize a "$W$-topology" on all spatial classes, in all relativistic spacetimes models.

But a problem arises with this option as well. If we understand $W$ to be the relation $\ll$, then spatial classes (as characterized in the first definition) can contain distinct points that are null related.[6] As a result, the $W$-topology need not be Hausdorff and need not agree with the topology that is induced on the spatial class by the manifold topology. Consider another example in Minkowski spacetime (figure 2). (For ease of representation only, we will work here with

[5] We can understand a "closed spherical ball in $S$ of radius $r > 0$" to be the set of all points in $S$ whose Minkowski distance from some central point is less than or equal to $r$".

[6] One might take this to be a problem all by itself. It seems natural to require that if a set of world points is a candidate for representing "space at a given time", then the points should be *neither* timelike nor null related. We will return to this concern later.

a two-dimensional version of Minkowski spacetime.) It shows a spatial class $S$ that consists of a collection of "jointed" null geodesic segments. Given any "open" subsets $O_1$ and $O_2$ of $S$, if $O_1$ contains $p$ and $O_2$ contains $q$, then their intersection $O_1 \cap O_2$ contains $r$.[7] So the Hausdorff condition is violated.

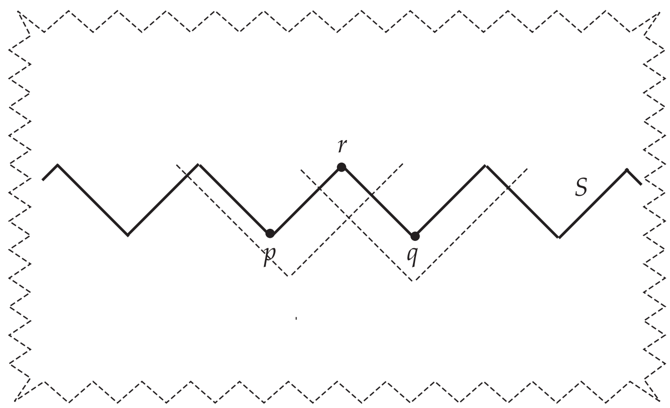


Figure 2: Even if we understand $W$ to be the relation $\ll$, the $W$-topology need not be Hausdorff. Here given any "open" subsets of $S$ containing $p$ and $q$ respectively, their intersection contains $r$.

But an easy fix is available. Carnap formulated the second definition solely in terms of future sets. We can avoid the two problems mentioned so far if we understand $W$ to be $\ll$ – which we will do in what follows – *and* we work with intersections of past and future sets. Here is a reformulation.

> (Definition 2, second version) Let $S$ be a spatial class, and let $O$ be a subset of $S$. Then $O$ is *open* if, for all world points $q$ in $O$, there are exist world points $p$ and $r$ such that $p \neq q$, $r \neq q$, $p \ll q$, $q \ll r$, and $(I^+(p) \cap I^-(r) \cap S) \subseteq O$.

So characterized, the $\ll$-*topology* on a spatial class *is* equal to the one induced by the manifold topology – for all spatial classes, in all relativistic spacetime models.[8]

[7]This follows because given any point $s$, if $p \in I^+(s)$, then $r \in I^+(s)$; and similarly, if $q \in I^+(s)$, then $r \in I^+(s)$.

[8]The clauses $p \neq q$, $r \neq q$ in the definition are redundant if one accepts the standard convention that singleton sets qualify as null curves, but not as timelike curves. For in that

Let us now consider a somewhat more significant problem. The revised version of the second definition works fine when we have a spatial class in hand. But there simply *are* no spatial classes in some relativistic spacetime models.

We have to be a bit careful here. If in the definition of a spatial class, we take a "worldline" to be represented by *any* smooth timelike curve, then we can never expect to find spatial classes. It is then just too hard to meet the requirement that *all* worldlines intersect a candidate set $S$. Given any smooth timelike curve that *does* intersect $S$, we can always move to a small segment of it that does not do so. Clearly, we have to restrict attention here to smooth timelike curves that, intuitively, "do not come to an end before they have to". There is a standard way to make this idea precise. One introduces the notion of an "endpoint" to a curve, and then restricts attention to curves that have no endpoints.[9]

The present point is that even if one *does* impose this restriction – and so makes it easier to satisfy clause (ii) in the definition of a spatial class – there still need not exist any spatial classes. For one thing, it can be the case that $p \ll q$ for all world points $p$ and $q$ (and so no non-empty set can satisfy clause (i) in the definition). This is the case in Gödel spacetime, for example. But the problem does not go away if we restrict attention to spacetimes in which there are no closed, or almost closed, timelike curves. It can still be the case that no spatial classes exist. The simplest example is Minkowski spacetime with one point $p$ removed. In this case, every candidate for a spatial class $S$ will be disqualified since there exist smooth timelike curves without endpoint that fail to intersect $S$, namely ones that, intuitively, run into the the excised point before reaching $S$.

A more interesting example is (the covering space of) anti-de Sitter spacetime. A two-dimensional version is given in figure 3. Here one can find smooth

case, $q \ll q$ can only hold if there is a non-trivial smooth timelike curve that begins and ends at $q$. And *that*, one can show, is incompatible with the assumption that $S$ is a spatial class.

[9]More precisely, let $\gamma\colon I \to M$ be a smooth timelike (or just causal) curve on the underlying manifold $M$. Here $I$ is a (possibly infinite or half infinite) interval on the real line. We way that $p$ is a *future end point* (resp. *past end point*) of $\gamma$ if, given any open set (in the manifold topology) $O$ containing $p$, there is a number $x_0 \in I$ such that, for all $x \in I$, if $x > x_0$ (resp. $x < x_0$), then $\gamma(x) \in O$. A point is an *end point* to $\gamma$, of course, it it is either a future or past end point to the curve. The definition is slightly subtle because an endpoint to $\gamma$ need not belong to the image set $\gamma[I]$.

spacelike submanifolds that seem like plausible candidates for representing "space at a given time". One is displayed in the figure. But they fail to qualify as spatial classes because there are smooth timelike curves without endpoint that fail to intersect them. They "run off to spatial infinity" before doing so.

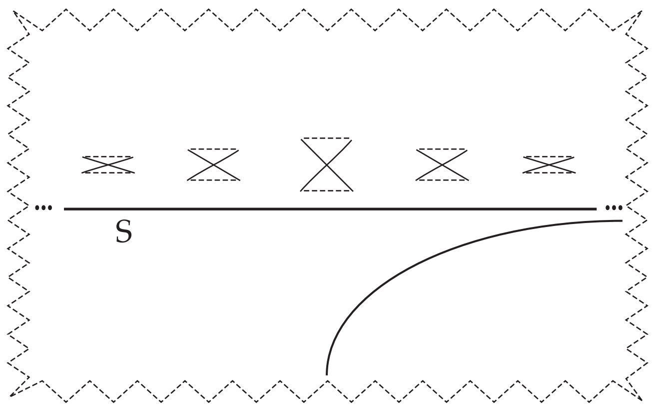


Figure 3: A two-dimensional version of (the covering space of) anti de-Sitter spacetime. The indicated set of world points $S$ seems a plausible candidate for representing "space at a given time". But it fails to qualify as a spatial class because there are smooth timelike curves without endpoint that "rush off to spatial infinity" before intersecting it.

Of course, one might just decide not to worry about examples of this type.[10] But there is another option. We can consider relaxing the defining conditions on a spatial class. And, in fact, there is a good reason to do so. There is a sense in which those conditions are too stringent.

Consider the situation in Newtonian spacetime physics. Here we have invariant simultaneity slices and, presumably, there is no doubt that they serve to represent "space at a given time". But if we formulate a notion of "spatial class" in this Newtonian context, one that is a close analogue of the one we have been considering in relativity theory, then the invariant simultaneity slices do *not* qualify! The analogue to a timelike curve here is one that is nowhere tangent to a simultaneity slice, i.e., one that "crosses" every simultaneity slice

[10] What the examples have in common is that they are relativistic spacetime models in which there are no "Cauchy surfaces" (Wald [4, p. 201]). Indeed, we can view Carnap's definition of a spatial class as just a variant definition of a Cauchy surface.

that it intersects. But given any simultaneity slice $S$, it is simply not true that all smooth (Newtonian-)timelike curves without endpoint intersect it. Some, "rush off to spatial infinity" before crossing $S$ (or "rush in from spatial infinity" without having done so). The situation with these "space-evader" or "space invader" worldlines is much the same as in anti-de Sitter spacetime. Why ask for more in the relativistic context than one has in the Newtonian context?

In any case, there is a simple way to revise Carnap's first definition – one that seems in the spirit of his formulation – that allows the existence of "spatial classes" in punctured Minkowski spacetime and anti-de Sitter spacetime. And this revised definition, when transposed to the Newtonian context, allows standard (invariant) simultaneity slices to qualify as spatial classes. (For any set $A$, let $I^+[A]$ be the set of points $q$ such that $p \ll q$ for *some* point $p$ in $A$. $I^-[A]$ is defined similarly.)

> (Definition 1, second version) A non-empty set of world points $S$ is a *spatial class* if (i) no two points in $S$ are timelike related, and (ii) for all points $p$ and $q$, if $p \in I^-[S]$ and $q \in I^+[S]$, then every smooth timelike curve from $p$ to $q$ intersects $S$.

Now we do not require that every smooth timelike curve without endpoint intersect $S$. In particular, we allow for the possiblity that such curves are fully contained in $I^-[S]$ or fully contained in $I^+[S]$. What is required is that *if* they start in $I^-[S]$, and end up in $I^+[S]$, *then* they must go through $S$ along the way. (One cannot go from "before" to "after" without going through "now".)[11]

This revised formulation is not the last word on the subject of "spatial classes". This is not the place for an extended discussion, but here are a few remarks.

(1) One can recast the definition so that it makes reference only to $\ll$, i.e., makes no direct reference to curves. It suffices to replace clause (ii) by the following condition: for all points $p$ and $q$, if $p \in I^-[S]$ and $q \in I^+[S]$, then

$$I^+(p) \cap I^-(q) \;\subseteq\; I^-[S] \cup S \cup I^+[S].$$

[11] The requirement that $S$ be non-empty has been added because clause (ii) is now formulated in terms of a conditional and, as a result, the empty set would otherwise qualify as a spatial class.

(2) As the definition stands, two points in a spatial class can be null related (though they cannot be timelike related). But it seems natural to require that if a point set is a candidate for representing "space at a given time", then its points should be *neither* timelike nor null related. The intuition here is that, for example, it "takes time" for light to reach us from distant galaxies. In any case, it is certainly possible to strengthen clause (i) in the definition so as to rule out the possibility that two (distinct) points in $S$ are null related. One way is to replace (i) with the following condition: for all for all points $p$ and $q$ in $S$,

$$p \neq q \quad \Longrightarrow \quad I^+(q) \nsubseteq I^+(p).$$

The condition on the right holds iff $q$ is not in the closure of $I^+(p)$ (in the manifold topology). In general, it implies, but is not equivalent to, the assertion that $q$ is not in $J^+(p)$. (That is, $q$ can be on the boundary of $I^+(p)$ without being null related to $p$.) But in the presence of condition (ii) it can fail only if there *are* two distinct point in $S$ that are causally related.

(3) There is one respect in which the definition might still seem too stringent. Carnap's notion of a spatial class is "global" in character. One might also want to consider a local version. So, for example, suppose we start with Minkowski spacetime and a flat, spacelike, three-dimensional submanifold $S$. The latter certainly counts as a spatial class. But we can disqualify it by doing some cutting and pasting that only involves regions of spacetime that are far distant from $S$. We can make it possible for a timelike curve to start in $I^-[S]$, and end up in $I^+[S]$, without passing through $S$. Intuitively, the cutting and pasting opens a new alternate, direct ("wormhole") route for the curve.

Does $S$ deserve to be thought of as representing "space at a given time" in a case like this? There is no clear answer. We are not dealing with a notion that has one clear, unambiguous sense. $S$ is not a spatial class according to our definition, but one might want to say that it is a "local spatial class". We can make the latter notion precise, and when we do we arrive at what Wald calls a *slice* [4, p. 200]. It turns out that the assertion "S is a slice" can be expressed solely in terms of $\ll$, though it requires a bit of work to show this.

## 3 The Work of A. A. Robb

It appears that Carnap was not aware of the work done by A. A. Robb [3] some years earlier.[12] Robb too showed that, if relativity theory is correct, certain elements of spacetime structure can be characterized directly in terms of a single two-place temporal order relation. (He called it the "after" relation. In our notation, "$q$ after $p$" holds if $p < q$ and $p \neq q$.) But their projects differed in two respects.

(1) Robb dealt only with Minkowski spacetime, i.e., the spacetime structure posited in so-called "special relativity". Carnap explicitly took himself to be analyzing spacetime structure in "general relativity".

(2) Robb's goal was to give a reductive account of the *affine* and *metric structure* of four-dimensional *spacetime*. Carnap's target, instead, was the *topological structure* of three-dimensional *space*.

Among other things, Robb proved a definability theorem in his book.[13] It is easy to formulate. We have characterized relativistic spacetime models as smooth four-dimensional manifolds with Lorentz metrics. But in the very special case of Minkowski spacetime, a great deal of additional structure is present. We can think of it as a metric affine space. Given an ordered pair of points $p$ and $q$, we can associate with it a vector $\overrightarrow{pq}$. And given two such vectors $\overrightarrow{pq}$ and $\overrightarrow{rs}$, we can associate with them an inner product $\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle$. Using these notions, we can define the relations of betweenness, orthogonality, and congruence:

$$\mathrm{Bet}(p,q,r) \longleftrightarrow \overrightarrow{pq} = k\,\overrightarrow{pr} \quad \text{for some } k \text{ with } 0 \leq k \leq 1$$

$$\mathrm{Orth}(p,q,r,s) \longleftrightarrow \langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = 0$$

$$\mathrm{Cong}(p,q,r,s) \longleftrightarrow \langle \overrightarrow{pq}, \overrightarrow{pq} \rangle = \langle \overrightarrow{rs}, \overrightarrow{rs} \rangle.$$

[12] Weyl, for one, did know about it and cited it in [5].

[13] He also sketched a result concerning what we would now call categorical axiomatizability. He showed, in effect, that one can formulate a set of axioms in a formal language with only one non-logical constant – a two-place relation symbol (for $\ll$) – whose only model, up to isomorphism, is Minkowski spacetime (now conceived as just a point set together with the two-place relation $\ll$). The axioms can all be first-order except for one second-order completeness axiom.

Robb, in effect, proved the following theorem.[14]

**Proposition**. *In Minkowski spacetime, the relations Bet, Orth, and Cong are all explicitly, first-order definable in terms of* $\ll$.

With this result in place, not much extra work is needed to extend the discussion to spatial topology. In Minkowski spacetime, there is a natural notion of "space at a given time" and it can be easily characterized in terms of the orthogonality relation. (Indeed, in Minkowski spacetime, the relative simultaneity relation just *is* the orthogonality relation.) We can take a set of world points $S$ to be a *Minkowskian spatial class* if there exist points $p$ and $q$ such that $p \ll q$ and such that, for all $r$,

$$r \in S \iff \langle \overrightarrow{pq}, \overrightarrow{pr} \rangle = 0.$$

So it follows from the proposition that Minkowskian spatial classes can be characterized directly in terms of $\ll$. And their "open" sets can be characterized in terms of $\ll$ just as before (using the second version of Carnap's second definition).

There is no natural way to extend Robb's definability theorem to the larger class of spacetimes considered in general relativity. But it is possible to prove a causal recovery theorem of a somewhat different type that *does* apply to that larger class (Malament [2]). It figures in a certain approach to quantum gravity developed by Rafael Sorkin and co-workers.[15]

[14] In the context of Minkowski spacetime, the three relations "after", $\ll$, and $<$ are interdefinable, so it makes no difference which we take to be our primitive relation.

[15] The article on "causal sets" in *Wikipedia* offers an overview of the program and a comprehensive set of references.

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