Logic and Logical Philosophy (2023)
DOI: 10.12775/LLP.2023.011
Published online: June 21, 2023

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Relating Semantics for Hyper-Connexive and Totally Connexive Logics

Abstract. In this paper we present a characterization of hyper-connexivity by means of a relating semantics for Boolean connexive logics. We also show that the minimal Boolean connexive logic is Abelardian, strongly consistent, Kapsner strong and antiparadox. We give an example showing that the minimal Boolean connexive logic is not simplificative. This shows that the minimal Boolean connexive logic is not totally connexive.

Keywords: Abelardian axiom; Boolean connexive logics; hyper-connexivity; relating semantics; totally connexive logics

Introduction

There is a common agreement that a connexive logic is based on the following theses of Aristotle and Boethius:

(A1) $\neg(A \to \neg A)$
(A2) $\neg(\neg A \to A)$
(B1) $(A \to B) \to \neg(A \to \neg B)$
(B2) $(A \to \neg B) \to \neg(A \to B)$.

However, there is also common agreement that a logic defined only by (A1), (A2), (B1), (B2) is very weak and should be strengthened in some way or other. If we interpret $\to$ as material implication and $\neg$ as classical negation then Aristotle's and Boethius' theses may be false, thus none of formulas (A1), (A2), (B1), (B2) is a tautology of classical logic. On the other hand, Aristotle's and Boethius' theses are true if we interpret $\to$ as material implication and $\neg$ as an assertion; i.e., $v(A) = v(\neg A)$. To

Special Issue: Relating Logic and Relating Semantics. II. Guest Editors: T. Jarmużek & F. Paoli

exclude such a strange interpretation, Jarmużek and Malinowski defined in [6] a class of Boolean connexive logics — that is, connexive logics where conjunction, disjunction and negation behave in the standard classical way. In this paper we will follow this line of research. As a consequence, throughout this paper, the connectives $\neg$, $\vee$ and $\wedge$ are Boolean.

A hyper-connexive logic [see 15] is a connexive logic where the following converses of Boethius' theses are also valid:

(B1′) $\neg(A \to \neg B) \to (A \to B)$
(B2′) $\neg(A \to B) \to (A \to \neg B)$.

Even if there have been criticisms about hyper-connexivity [see 12, pp. 446–447], beyond mentioning some motivations for supporting it, we will not in this paper discuss the correctness or incorrectness of hyper-connexive principles from a philosophical point of view. We consider them as one of possible ways of how to strengthen Aristotle's and Boethius' theses.

There is also a counterexample based on another strange interpretation. Aristotle's and Boethius' theses are true if we interpret $\neg$ as classical negation and $\to$ as material equivalence, but it makes $\to$ symmetric. It would seem to be natural to exclude also interpretations where $\to$ is symmetric. For this reason we hold that symmetry of implication, represented by the following schema should be invalid:

(SI) $(A \to B) \to (B \to A)$

Estrada-González and Ramírez-Cámara [1] have considered totally connexive logics defined by means of the schemas (A1)–(B2) and some other widely considered formulas defining the following logics: Abelardian logic, anti-paradox logic, simplificative logic, conjunction-idempotent logic, (in)consistent logic and Kapsner-strong logic. As with hyper-connexivity, we will not here be concerned with either philosophical motivations for totally connexive logics or even the formal consequences of the conditions that define them. In this paper we will concentrate only on characterizing the conditions in terms of relating semantics for Boolean connexive logics.

The paper is structured as follows. In section 1, we present the rudiments of relating semantics for Boolean connexive logics, following [4, 6]. In section 2, we evaluate some connexive-related principles found in the literature of connexivity and collected under the heading of 'totally connexive logics' introduced in [1] and further studied in [14]. In section 3, we study the adjustments that need to be made to the relating semantics to model hyper-connexive logics.

1. Relating semantics

1.1. Conceptual motivations

Relating semantics is important for the study of intensional logic, that is, logic that focuses on valid arguments whose validity relies on more than their truth values. We will explain it by way of an example. Consider the following argument (in slightly-stilted English to avoid issues relating to tense):

$$\frac{\text{If I get shot, then if I die, I am buried}}{\text{If I die, then if I get shot, I am buried}} \qquad (*)$$

In classical propositional logic, one formalizes this argument as follows:

$$\text{(Permutation)} \qquad \frac{p \to (q \to r)}{q \to (p \to r)}$$

but (Permutation) is valid according to classical propositional logic. However, from an *intuitive* point of view, there seems to be something wrong with the argument (∗), and so, it could be argued, one would expect that its formalization is not valid.

One might think that the validity of (Permutation) is given by the values of the propositional variables that appear in the argument, whereas the invalidity of (∗) is given by the fact that one is not only considering the values of the components of the premises and the conclusion, but also the *relations* between them. Presumably, these relations have to do with the sequential order of sentences, an order that does not let to interchange the antecedents of the implications at one's will.

In a relating semantics, one tries to capture these and other kinds of relations in a formal language. For more examples on the kind of relations that can be represented in a relating semantics, as well as its history, its philosophical motivations, and recent achievements in this field, one can consult [4, 5, 6, 9, 11, 13].

1.2. Technical preliminaries

Let Form be a propositional language defined in a usual way by a set of propositional variables $\mathrm{Var} = \{p_1, \ldots, p_n\}$ an unary connective $\neg$ and binary connectives $\wedge$, $\vee$, $\to$.

A *relating model* $\mathfrak{M}$ for Form is a pair $\langle v, R\rangle$, where $v\colon \mathrm{Var} \longrightarrow \{1, 0\}$ and $R \subseteq \mathrm{Form} \times \mathrm{Form}$. A variable A is *satisfied* in a relating model $\mathfrak{M}$ (denoted by $\langle v, R\rangle \models A$) if and only if $v(A) = 1$, and in the case of complex formulas, particularly of the conditional, it is required that R meets specific conditions (given below). As a notational convention, in this text $\langle A, B\rangle \in R$ is denoted by $R(A, B)$, and $\langle A, B\rangle \notin R$ is denoted by $\tilde{R}(A, B)$. A formula A that is not satisfied in a relating model is denoted by $\langle v, R\rangle \not\models A$.

Let $\mathcal{M}$ be the set of all the relating models for Form. For any model $\langle v, R\rangle \in \mathcal{M}$, formulas have the following truth conditions:

Definition 1.1. (Truth conditions for Form)

- $\langle v, R\rangle \models A$ if and only if $v(A) = 1$, if $A \in \mathsf{Var}$
- $\langle v, R\rangle \models \neg A$ if and only if $\langle v, R\rangle \not\models A$
- $\langle v, R\rangle \models A \wedge B$ if and only if $\langle v, R\rangle \models A$ and $\langle v, R\rangle \models B$
- $\langle v, R\rangle \models A \vee B$ if and only if $\langle v, R\rangle \models A$ or $\langle v, R\rangle \models B$
- $\langle v, R\rangle \models A \to B$ if and only if $[\langle v, R\rangle \not\models A$ or $\langle v, R\rangle \models B]$ and $R(A, B)$.

The truth conditions for most of the connectives are the classical ones, the only exception being implication, which besides having its usual truth conditions, requires that antecedents and consequents are related by R.

In this paper we will write $R \models A$ if and only if for all v, $\langle v, R\rangle \models A$. To obtain suitable models for connexive logics in $\mathcal{M}$, we recall here some results from [6].

Definition 1.2. For any $R \subseteq \mathrm{Form} \times \mathrm{Form}$:

- R satisfies (a1) if and only if for any $A \in \mathrm{Form}$, $\tilde{R}(A, \neg A)$.
- R satisfies (a2) if and only if for any $A \in \mathrm{Form}$, $\tilde{R}(\neg A, A)$.
- R satisfies (b1) if and only if for any $A, B \in \mathrm{Form}$,
  - If $R(A, B)$ then $\tilde{R}(A, \neg B)$
  - $R((A \to B), \neg(A \to \neg B))$.
- R satisfies (b2) if and only if for any $A, B \in \mathrm{Form}$,
  - If $R(A, B)$ then $\tilde{R}(A, \neg B)$
  - $R((A \to \neg B), \neg(A \to B))$.

Theorem 1.1 ([6], p. 435). *For any* $R \subseteq \mathrm{Form} \times \mathrm{Form}$ *and* $A, B \in \mathrm{Form}$*:*

- *If R satisfies* (a1) *then* $R \models \neg(A \to \neg A)$
- *If R satisfies* (a2) *then* $R \models \neg(\neg A \to A)$
- *If R satisfies* (b1) *then* $R \models (A \to B) \to \neg(A \to \neg B)$
- *If R satisfies* (b2) *then* $R \models (A \to \neg B) \to \neg(A \to B)$.

To obtain the converses of the implications of the previous theorem, we need that the relation R be closed under negation, that is:

DEFINITION 1.3. (Closure under negation)

(c1) R is *closed under negation* if and only if for any $A, B \in$ Form: if $R(A, B)$ then $R(\neg A, \neg B)$.

Imposing this condition on R results in the validity of the formula $\neg((A \to B) \wedge \neg B \wedge \neg(\neg A \to \neg B)))$, a formula that is not valid with the adoption of the conditions (a1)–(b2) on R. A classically equivalent schema was used in [10] for an axiomatization of Boolean connexive logic with closure under negation. This condition is also independent of (a1), (a2), (b1) and (b2) and lets us pass from the appropriate models for connexive logics to axiomatics and vice versa.

The relating semantics presented above, and constructed in [6], determines Boolean connexive logics. However, the corresponding theorem above does not provide by itself the right completeness result. The full completeness theorem has been proven by Klonowski in [10]. We will formulate it here in a way appropriate for our aims. We use $\supset$ to denote material implication. Formally, $A \supset B$ could be considered as a shorthand for $\neg(A \wedge \neg B)$.

By *minimal Boolean connexive logic* we mean the least set of sentences of the language Form containing:

- all classical tautologies expressed by means of $\neg, \wedge, \vee$ within the language Form
- (A1), (A2), (B1), (B2)
- $(A \to B) \supset (A \supset B)$

and closed under modus ponens with respect to $\supset$. Let TJ denote a class of all relations R such that R satisfies (a1), (a2), (b1), (b2). Let $TJ^{\neg}$ denote a class of all relations from TJ satisfying (c1).

In [10] is proved the following completeness theorem:

THEOREM 1.2. *The class TJ determines the minimal Boolean connexive logic. The class $TJ^{\neg}$ determines the least Boolean connexive logic satisfying the following two axioms*:

- $(A \to B) \supset (\neg\neg A \to \neg\neg B)$
- $(A \to B) \supset ((\neg A \to \neg B) \vee (\neg A \wedge B))$.

2. Totally connexive logics

In [1, pp. 5–6] there are identified and named several desiderata for a connexive logic, besides the satisfaction of the schemas (A1)–(B2). These desiderata are the following:

Definition 2.1 (Desiderata for a connexive logic).

- An *Abelardian logic* is a logic that validates either of the following schemas:
  - $\neg((A \to B) \wedge (\neg A \to B))$
  - $\neg((A \to B) \wedge (A \to \neg B))$
- An *anti-paradox logic* is a logic that does not validate the following schemas:
  - $A \to (B \to A)$
  - $A \to (\neg A \to B)$
  - $A \to (B \to C)$

    (where A is a contingency and $(B \to C)$ is a logical truth).
- A *simplificative logic* is a logic that validates the following schemas:
  - $(A \wedge B) \to A$
  - $(A \wedge B) \to B$
- A *conjunction-idempotent logic* is a logic that validates the following schemas:
  - $(A \wedge A) \to A$
  - $A \to (A \wedge A)$
- A *weakly consistent logic* is a logic that does not validate any formula and its negation.
- A *weakly inconsistent logic* is a logic where any formula and its negation are both satisfiable.
- A *strongly consistent logic* is a logic where no formula and its negation are both satisfiable.
- A *strongly inconsistent logic* is a logic that validates at least one formula and its negation.
- A *Kapsner-strong logic* is a logic where:
  - $\neg A \to A$ is unsatisfiable,
  - $A \to \neg A$ is unsatisfiable,
  - $A \to B$ and $A \to \neg B$ are not simultaneously satisfiable.
- A *totally connexive logic* is a connexive logic that is also Abelardian, anti-paradox, simplificative, conjunction-idempotent and Kapsner-strong.

In [1] it is remarked that there are some difficulties in obtaining a totally connexive logic. For example, a consistent connexive logic cannot be simplificative if it satisfies contraposition for implication, modus ponens and validates $(A \to B) \to ((B \to C) \to (A \to C))$.[1] As a consequence, they pose the open problem of whether there are totally connexive logics, and if any, which is the minimal one. In [14], using contrapossible implication and a refined version of Kapsner strength, Omori and Wansing identify three logics that are expansions of Wansing's logic **C** [16] as possible candidates for a totally connexive logic.

In the following, we will examine whether the desiderata for totally connexive logics hold in the models for connexive logics introduced in [6].

THEOREM 2.1. *Minimal Boolean Connexive Logic* (*or alternatively*: *the logic determined by TJ*) *is Abelardian, strongly consistent, Kapsner strong and antiparadox.*

PROOF. To prove the validity of $\neg((A \to B) \land (A \to \neg B))$ suppose that $\langle v, R\rangle \nvDash \neg((A \to B) \land (A \to \neg B))$. From truth conditions of the formulas in Form, $\langle v, R\rangle \vDash (A \to B) \land (A \to \neg B)$. Then $\langle v, R\rangle \vDash A \to B$ and $\langle v, R\rangle \vDash A \to \neg B$. That is to say, $R(A, B)$ and $R(A, \neg B)$. Consider $R(A, B)$. As R satisfies (b1) and (b2), one has $\tilde{R}(A, \neg B)$, but this contradicts $R(A, \neg B)$.[2]

To prove that it is consistent, we consider a relation R as the union of the following sets:

1. $\{\langle A \to B, \neg(A \to \neg B)\rangle : A, B \in \text{Form}\}$
2. $\{\langle A \to \neg B, \neg(A \to B)\rangle : A, B \in \text{Form}\}$

[1] The proof has been known since Alberic of Paris [see 2]. Just consider as instances of simplification the following formulas $(A \land \neg A) \to A$ and $(A \land \neg A) \to \neg A$. Next contrapose the first one, $\neg A \to \neg(A \land \neg A)$, instantiate the schema $(A \to B) \to ((B \to C) \to (A \to C))$ with $((A \land \neg A) \to \neg A) \to ((\neg A \to \neg(A \land \neg A)) \to ((A \land \neg A) \to \neg(A \land \neg A)))$ and use modus ponens twice. One thus obtains $((A \land \neg A) \to \neg(A \land \neg A))$, but as the logic is connexive, one also obtains $\neg((A \land \neg A) \to \neg(A \land \neg A))$. Note that the proof only shows that one would need to add $((A \land \neg A) \to \neg(A \land \neg A))$ to the set of theses, going against the spirit of connexive logics.

[2] Note that $\neg((A \to B) \land (\neg A \to B))$ can be validated if R is also closed under negation: to prove the validity of $\neg((A \to B) \land (\neg A \to B))$ suppose that $\langle v, R\rangle \nvDash \neg((A \to B) \land (\neg A \to B))$. From the truth conditions of the formulas in Form, $\langle v, R\rangle \vDash (A \to B) \land (\neg A \to B)$. Then $\langle v, R\rangle \vDash A \to B$ and $\langle v, R\rangle \vDash \neg A \to B$. That is to say, $R(A, B)$ and $R(\neg A, B)$. If R were closed under negation, one would have that $R(\neg A, \neg B)$. Moreover, R satisfies (b1) and (b2), so considering $R(\neg A, B)$ one has $\tilde{R}(\neg A, \neg B)$, but this would contradict $R(\neg A, \neg B)$.

It is clear that this relation defines a model that satisfies all Jarmużek and Malinowski's conditions, that is, it is a model for connexive logic. However this model is not a model for both a formula and its negation.

To prove the unsatisfiability of $A \to \neg A$ consider condition (a1).

To prove the non-simultaneous satisfiability of $A \to B$ and $A \to \neg B$ suppose that $A \to B$ is satisfiable. Then $R(A, B)$ and using (b1), one obtains $\tilde{R}(A, \neg B)$, but this makes $A \to \neg B$ unsatisfiable. Conversely, suppose that $A \to \neg B$ is satisfiable. Then $R(A, \neg B)$ and using contraposition on the first condition given in (b1), one obtains $\tilde{R}(A, B)$. But if $\tilde{R}(A, B)$ then $A \to B$ is unsatisfiable.

To prove the invalidity of $A \to (B \to A)$ and $A \to (\neg A \to B)$ consider again the relation R as defined above. There is no pair of formulas $\{\langle A, B \to A\rangle \colon A, B \in \mathrm{Form}\}$ and $\{\langle A, \neg A \to B\rangle \colon A,\ B \in \mathrm{Form}\}$ that fall under R.

Likewise, for the invalidity of $A \to (B \to C)$, where A is a contingency and $(B \to C)$ is a logical truth, consider R as above. ⊣

However, the relating models just considered are not models for a simplificative logic or for conjunction-idempotent or for strongly inconsistent logics. Consider again the relation R as above. In the model defined by this relation, there is no pair of formulas $\{\langle A \wedge B, A\rangle \colon A, B \in \mathrm{Form}\}$ or $\{\langle A \wedge B, B\rangle \colon A, B \in \mathrm{Form}\}$ that falls under R. In addition, there is no pair of formulas $\{\langle A \wedge A, A\rangle \colon A \in \mathrm{Form}\}$ or $\{\langle A, A \wedge A\rangle \colon A \in \mathrm{Form}\}$ that falls under R. That a model for a connexive logic is not necessarily a model for a (weakly/strongly) inconsistent logic is a corollary of the fact that those models are precisely models for a strongly consistent logic.

Moreover those models do not validate the schemas $\neg(A \to \neg B) \to (A \to B)$ and $\neg(A \to B) \to (A \to \neg B)$, and thus they are not models for a hyper-connexive logic, either. The above model on R gives a countermodel.

3. Hyper-connexivity

Hyper-connexive principles have appeared in applications in categorial grammar [see 17], in theories of counterfactual conditionals [see 8] but also in the simplest semantics for connexive logics [see 16].

Sylvan [15] ascribes to Boethius himself a view on which logical connectives are not merely truth-functional but presuppose an association between the formulas that are connected by the connectives, such an association indicating understanding, nature or sense. For example, according to Sylvan, in Boethius' view implication requires the following analysis (where '$\Box$' is a necessity connective):

$$A \to B \text{ iff } \{\Box(A \supset B) \text{ and } A \sim B\},$$

where '$A \sim B$' represents the association between A and B. If the association and the modality are somehow encapsulated in the R of relating semantics, one could expect an easy and straightforward model for Boethius's views in relating semantics.

It would probably be expected that to validate one of those formulas and thus obtain hyper-connexivity, one would just need the converses of the implications that appear in (b1) and (b2) together with an appropriate relation between the antecedent and the consequent, that is:

Definition 3.1. For any $R \subseteq \text{Form} \times \text{Form}$:

- R satisfies (b1*) if and only if for any $A, B \in \text{Form}$,
  - if $\tilde{R}(A, \neg B)$ then $R(A, B)$
  - $R(\neg(A \to \neg B), (A \to B))$
- R satisfies (b2*) if and only if for any $A, B \in \text{Form}$,
  - if $\tilde{R}(A, \neg B)$ then $R(A, B)$
  - $R(\neg(A \to B), (A \to \neg B))$.

But in fact, this is not enough (nor necessary for that matter). For assume R satisfies (b1*), and assume for reductio that $\langle v, R\rangle \nvDash \neg(A \to \neg B) \to (A \to B)$. That is $\langle v, R\rangle \vDash \neg(A \to \neg B)$ and $\langle v, R\rangle \nvDash A \to B$, because one has that $R(\neg(A \to \neg B), (A \to B))$. Thus $\langle v, R\rangle \nvDash A \to \neg B$ and $\langle v, R\rangle \nvDash A \to B$. Now the implications in those formulas fail due to one of the following situations:

1. $R(A, B)$, $R(A, \neg B)$; $\langle v, R\rangle \vDash A$, $\langle v, R\rangle \nvDash B$, $\langle v, R\rangle \nvDash \neg B$, but this is impossible
2. $R(A, B)$, $\tilde{R}(A, \neg B)$; $\langle v, R\rangle \vDash A$ and $\langle v, R\rangle \nvDash B$
3. $R(A, \neg B)$, $\tilde{R}(A, B)$; $\langle v, R\rangle \vDash A$ and $\langle v, R\rangle \nvDash \neg B$
4. $\tilde{R}(A, B)$ and $\tilde{R}(A, \neg B)$, this is impossible because by $\tilde{R}(A, \neg B)$ and (b1*) one has $R(A, B)$, and this contradicts $\tilde{R}(A, B)$.

In other words, when assuming (b1*) alongside conditions (a1)–(b2) as a restriction on R, one obtains that R cannot fail to relate to both any formula and its negation, as it is indicated in the fourth case. But one still has two counterexamples to $\neg(A \to \neg B) \to (A \to B)$ — just consider the second and the third cases. Something similar happens with (b2*) and $\neg(A \to B) \to (A \to \neg B)$.

To avoid the previous counterexamples, we propose to add on top of (b1*) and (b2*) that a relation $R \subseteq \mathrm{Form} \times \mathrm{Form}$ satisfies (jm) if and only if $\exists_v \langle v, R\rangle \vDash A \wedge B \Rightarrow R(A, B)$. This proposal blocks the counterexamples to $\neg(A \to \neg B) \to (A \to B)$ and $\neg(A \to B) \to (A \to \neg B)$.

Theorem 3.1. *For any* $R \subseteq \mathrm{Form} \times \mathrm{Form}$ *and* $A, B \in \mathrm{Form}$:

If R *satisfies the conditions that define* TJ *together with* (b1*), (b2*) *and* (jm), *then*:

- $R \vDash \neg(A \to \neg B) \to (A \to B)$
- $R \vDash \neg(A \to B) \to (A \to \neg B)$.

Proof. The proof proceeds by cases.

Case 1. Assume R satisfies the conditions that define TJ together with (b1*), (b2*) and (jm). Then $R(\neg(A \to \neg B), (A \to B))$, and thus $\tilde{R}(\neg(A \to \neg B), \neg(A \to B))$ by (b1). Take any valuation v. Suppose that $\langle v, R\rangle \models \neg(A \to \neg B)$. By $\tilde{R}(\neg(A \to \neg B), \neg(A \to B))$ and (jm), $\langle v, R\rangle \not\models \neg(A \to \neg B) \wedge \neg(A \to B)$. But then $\langle v, R\rangle \not\models \neg(A \to B)$ in virtue of $\langle v, R\rangle \models \neg(A \to \neg B)$. Thus $\langle v, R\rangle \models (A \to B)$, and then $\langle v, R\rangle \vDash \neg(A \to \neg B) \to (A \to B)$. As v was arbitrary, $R \vDash \neg(A \to \neg B) \to (A \to B)$.

Case 2. Assume R satisfies the conditions that define TJ together with (b1*), (b2*) and (jm). Then $R(\neg(A \to B), (A \to \neg B))$, and thus $\tilde{R}(\neg(A \to B), \neg(A \to \neg B))$ by (b1). Take any valuation v. Suppose that $\langle v, R\rangle \models \neg(A \to B)$. By $\tilde{R}(\neg(A \to B), \neg(A \to \neg B))$ and (jm), $\langle v, R\rangle \not\models \neg(A \to B) \wedge \neg(A \to \neg B)$. But then $\langle v, R\rangle \not\models \neg(A \to \neg B)$ in virtue of $\langle v, R\rangle \models \neg(A \to B)$. Thus $\langle v, R\rangle \models (A \to \neg B)$, and then $\langle v, R\rangle \vDash \neg(A \to B) \to (A \to \neg B)$. As v was arbitrary, $R \vDash \neg(A \to B) \to (A \to \neg B)$. ⊣

Now, the relating models defined by the conditions given in TJ together with (b1*), (b2*) and (jm) are not empty, as is proved in the following theorem:

Theorem 3.2. *The class of models determined by* TJ *and the conditions* (b1*), (b2*), *and* (jm) *is not empty.*

Proof. Take a classical valuation v defined on Var. We define a valuation w on $\mathsf{Var} \cup \{A \to B \colon A, B \in \mathrm{Form}\}$ in the following way:

- $w(A) = v(A)$, if $A \in \mathsf{Var}$,
- $w(A \to B) = 0$, if there is no $C \in$ Form such that $B = \neg C$,
- $w(A \to B) = 1$, if there is a $C \in$ Form such that $B = \neg C$.

The valuation w is extended to the rest of Form by the standard classical conditions.

We then define a relation R such that: $R(A, B)$ iff $w(A) = w(B)$. It is clear that $w(A) \neq w(\neg A)$, $w(A \to B) = w(\neg(A \to \neg B))$, $w(A \to \neg B) = w(\neg(A \to B))$, and for any A and B, if $w(A) = w(B)$ then $w(A) \neq w(\neg B)$. Since all the equalities are symmetric, R is (a1), (a2), (b1), (b2), (b1*) and (b2*). Now, we take a valuation v' and the set $\{A \wedge B \colon v'(A \wedge B) = 1 \;\&\; w(A) \neq w(B)\}$. Then, $\langle v', R\rangle \models A \wedge B$, but $\tilde{R}(A, B)$, so R is not (jm).

Let $R_1 = R$. For any $n > 1$, we define R_n to be the least relation Q such that: $R_{n-1} \subseteq Q$, and if $\langle v, R_{n-1}\rangle \models A \wedge B$, then $Q(A, B)$. As it was shown above, if $Q = R_1$, it might be the case that $\langle v', R_1\rangle \models A \wedge B$ but $\tilde{R}_1(A, B)$. Suppose that this holds for any $R_k, R_k < R_n$. In particular, it holds for R_{n-1}. Then, R_{n-1} is (a1), (a2), (b1), (b2), (b1*) and (b2*) but still (jm) is not satisfied. Given that $R_{n-1} \subseteq R_n$, R_n is (a1), (a2), (b1), (b2), (b1*) and (b2*). Consider again the valuation v' defined above. Then $\langle v', R_n\rangle \models A \wedge B$ but $\tilde{R}_n(A, B)$, so R_n is not (jm).

Take $\mathfrak{M} = \langle v, \bigcup_{m\in\mathbb{N}} R_m\rangle$. It is the case that: $\mathfrak{M} \models A$ iff for some n, $\langle v, R_n\rangle \models A$. We need to show that if $\mathfrak{M} \models A \wedge B$ then $\langle A, B\rangle \in \bigcup_{m\in\mathbb{N}} R_m$. Suppose that $\langle v, R_{n-1}\rangle \models A \wedge B$. Then $R_n(A, B)$, and thus $R_n \subseteq \bigcup_{m\in\mathbb{N}} R_m$. As all R_n's are (al), (a2), (b1), (b2), (b1*) and (b2*), $\bigcup_{m\in\mathbb{N}} R_m$ is (a1), (a2), (b1), (b2), (b*), (b2*) and (jm). We would like to note that a similar construction was used in [7]. ⊣

Adopting the conditions (b1*), (b2*) and (jm) on top of TJ also preserves all the results from Theorem 2.1. That is, this adoption amounts to a preservative class of models for minimal Boolean connexive logic with respect to the properties mentioned in section 2.

It is worth noting that with the new conditions adopted, there also will be new theses. Consider again a relation R as the union of the following sets:

1. $\{\langle A \to B, \neg(A \to \neg B)\rangle \colon A, B \in \text{Form}\}$
2. $\{\langle A \to \neg B, \neg(A \to B)\rangle \colon A, B \in \text{Form}\}$

As previously remarked, this relation defines a model for connexive logic satisfying all Jarmużek and Malinowski's conditions. However, it is possible that $\tilde{R}(\neg(A \to \neg A), ((A \to B) \to \neg(A \to \neg B)))$ and thus $\langle v, R\rangle \not\models \neg(A \to \neg A) \to ((A \to B) \to \neg(A \to \neg B))$. But it is true that $\langle v, R\rangle \models \neg(A \to \neg A)$ and $\langle v, R\rangle \models (A \to B) \to \neg(A \to \neg B)$, and thus $\langle v, R\rangle \models \neg(A \to \neg A) \wedge ((A \to B) \to \neg(A \to \neg B))$. By expanding R with (jm) one obtains that $\tilde{R}(\neg(A \to \neg A), ((A \to B) \to \neg(A \to \neg B)))$, and thus $\langle v, R\rangle \models \neg(A \to \neg A) \to ((A \to B) \to \neg(A \to \neg B))$.

4. Conclusions

In this paper, we have proposed one way to model hyper-connexive logics using a relating semantics for Boolean connexive logics. It remains for future work to discuss some doubts about the correction of hyper-connexivity and probe whether alternative semantics within the relating setting are more adequate to model hyper-connexive logics or even totally connexive logics.

Acknowledgments. We would want to express our gratitude to Tomasz Jarmużek and Mateusz Klonowski for their attentive reading and invaluable assistance on previous versions of this paper. The second author also thanks Luis Estrada-González for directing his attention to the field of relating semantics, as well as the PAPIIT project IN403719: "Intensionality all the way down: a new plan for logical relevance".

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