Is Prediction Possible in General Relativity?*

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Abstract

Here we briefly review the concept of "prediction" within the context of classical relativity theory. We prove a theorem asserting that one may predict one's own future only in a closed universe. We then question whether prediction is possible at all (even in closed universes). We note that interest in prediction has stemmed from considering the epistemological predicament of the observer. We argue that the definitions of prediction found thus far in the literature do not fully appreciate this predicament. We propose a more adequate alternative and show that, under this definition, prediction is essentially impossible in general relativity.

In general relativity, "prediction" and "determinism" are two very different concepts [2, p. 93]. The difference can be easily noted by considering Minkowski spacetime $(\mathbb{R}^4, \eta)$.[1] Of course, in this spacetime, any set $S = \{(t, x, y, z) \in \mathbb{R}^4 : t = \text{constant}\}$ is a Cauchy surface. So, for such a surface, $D(S) = \mathbb{R}^4$ where $D(S)$ is the domain of dependence of S.[2] It is a theorem that, given the physical situation on any achronal surface S, the physical situation in all of $D(S)$ is uniquely determined.[3] So, any $t =$ constant surface S in Minkowski spacetime *determines* the physical situation on the entire manifold.

However, consider an actual observer in Minkowski spacetime at some point $q \in \mathbb{R}^4$. The observer at q is not able to make a *prediction* about the physical situation on all of $\mathbb{R}^4$ because there is no $t =$ constant surface S contained in $J^-(q)$ where $J^-(q)$ is the causal past of the point q.[4]

Additionally, the observer at q cannot even make a local prediction. It is a simple result that, for *any* point q and *any* achronal surface S in Minkowski

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[1]For details on Minkowski spacetime, see [5, p. 118-124].

[2]The *future domain of dependence* $D^+(S)$ of an achronal surface S is the set of points p such that every past inextendible causal curve through p intersects S. The *past domain of dependence* $D^-(S)$ of an achronal surface S is defined analogously. The (total) *domain of dependence* $D(S)$ of achronal surface S is just the set $D^-(S) \cup D^+(S)$. See [8, p. 200-201].

[3]See chapter 10 in [8].

[4]The *causal past* $J^-(q)$ of a point q is the set of points p such that there exists a past-directed causal curve from q to p. The *causal future* $J^+(q)$ of a point q is defined analogously. See [8, p. 190].

spacetime, if $S \subset J^-(q)$ then $D(S) \subset J^-(q)$ [2, p. 128]. This means that by the time the observer is able to gather the data on S, the region of spacetime determined by this data is in the causal past of the observer (a retrodiction is possible but a *prediction* is not).

There are, however, spacetimes that seem to allow for predictions. For example, consider the (two-dimensional) spacetime (M, η) where $M = \{(t, x) : t \in \mathbb{R} \ \& \ x \in \mathbb{S}\}$ and η is the Minkowskian metric. Here, each $t =$ constant surface S is a Cauchy surface. For any point $q \in M$, there exists some $t =$ constant surface S such that $S \subset J^-(q)$. So, in this spacetime, it seems that one *can* make a genuine prediction at q concerning any point in $M - J^-(q)$ [4, p. 89-91].

Examples of such spacetimes have motivated various definitions of prediction.[5] Here is one that seems fairly intuitive due to Geroch:[6]

Definition: Let (M, g) be a spacetime. Let q be any point in M. We say a point $p \in M$ is in the *domain of prediction* of q (written $P(q)$) iff (i) $p \notin J^-(q)$ and (ii) there exists an achronal, closed, spacelike surface S in $J^-(q)$ such that $p \in D(S)$.

The physical intuition behind this definition is the following. The set $J^-(q)$ represents the region of spacetime from which information can be gathered at point q. Condition (i) requires that, whatever else is the case, any knowledge concerning the state of affairs at p is a prediction and not a retrodiction. Condition (ii) requires that every causal influence that could affect the point p must have been registered on some achronal, spacelike surface S in $J^-(q)$.

An interesting subset of the domain of prediction is the *directly verifiable domain of prediction* [4, p. 89]. Given a point q, this is the set $P(q) \cap I^+(q)$ where $I^+(q)$ is the chronological future of q.[7] It represents the events in one's own future that are predictable. It has been conjectured by Geroch[8] that, if a spacetime has a non-empty directly verifiable domain of prediction, then the spacetime is closed in the sense that it admits a compact, spacelike slice.[9] Here we provide a proof of this conjecture.

Theorem: Let (M, g) be a spacetime. If there are points $q, p \in M$ such that $p \in P(q) \cap I^+(q)$, then (M, g) admits a compact, spacelike slice.

Proof. Let (M, g) be a spacetime. Let $q, p \in M$ be such that $p \in P(q) \cap I^+(q)$.

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[5] See, for example, [1], [4], [6], and [7].

[6] Private communication. This definition differs slightly from the one given in [4]. Here we make the physically reasonable requirement that the surface S in the definition be spacelike. We also follow standard practice and require S to be closed for mathematical convenience [8, p. 200].

[7] The *chronological future* $I^+(q)$ of a point q is the set of points p such that there exists a future-directed timelike curve from q to p. The *chronological past* $I^-(q)$ of a point q is defined analogously. See [8, p. 190].

[8] Private communication. The corresponding claim made by Geroch in [4, p. 92] was later shown to be false [6, p. 726-727].

[9] A *slice* is a closed, achronal set without edge. See [8, p. 200].

By definition, there is a closed, spacelike, achronal set S such that $S \subseteq J^-(q)$ and $p \in D(S)$. If $p \in D^-(S)$, then $p \in J^-(q)$ which is, by definition of $P(q)$, impossible. So, $p \in D^+(S)$. It can be easily verified that $q \in I^-[D^+(S)]$ (becuase $p \in I^+(q)$ and $p \in D^+(S)$) and $q \in I^+(S)$ (because $S \subseteq J^-(q)$ and S is spacelike). So by [8, prop. 8.3.3], $q \in \text{int}[D^+(S)]$. This implies, by [5, prop. 6.6.6], that $J^+[S] \cap J^-(q)$ is compact. Clearly $S \subset J^+[S] \cap J^-(q)$. Because S is closed, it follows that S is compact.

Finally we show that S is edgeless. Assume not and let r be any point in edge(S). We show a contradiction. So, there is a neighborhood of r in $I^-(q)$ containing a point $u \in I^+(r)$, a point $v \in I^-(r)$, and a past directed timelike curve from u to v which does not intersect S.[10] Let γ be any past inextendible timelike curve from v. If γ were to intersect S (say at point w), we could find a past inextendible timelike curve γ' from r to v to w. But this cannot be given that S is achronal and $r, w \in S$. So, there is a past inextendible timelike curve from v which does not intersect S. Now, because $r \in J^-(q)$ it follows from [5, prop. 6.5.2-3] that either $r \in I^-(q)$ or $q \in H^+(S)$ where $H^+(S)$ is the future Cauchy horizon of S.[11] If $r \in I^-(q)$, then it follows from the argument above that there exists a past inextendible timelike curve from q which fails to meet S. But this contradicts the fact that $q \in D^+(S)$. If $q \in H^+(S)$, this contradicts the fact that $q \in \text{int}[D^+(S)]$. So, we are done. $\Box$

One may wonder if the definition of the domain of prediction given above accurately reflects the physical notion of "making predictions in general relativity"? It is our position that it does not (and that the other definitions given in [1], [4], [6], and [7] are also insufficient). Recall that interest in prediction within the context of general relativity stemmed from considering the epistemological predicament of the *observer*. As we have noted, even in spacetimes possessing Cauchy surfaces, there does not always exist an observer with a Cauchy surface in her causal past. But here is a question: Does an observer who *has* a Cauchy surface in her causal past *know* that she has a Cauchy surface in her causal past?

Consider the following example. As before, let (M, η) be such that $M = \{(t, x) : t \in \mathbb{R} \ \& \ x \in \mathbb{S}\}$ and η is the Minkowskian metric. Now let p be any point in M and let $(M', \eta_{|M'})$ be a second spacetime such that $M' = M - \{p\}$. Now consider any point $q \in M$ such that $p \notin J^-(q)$. Of course, by definition $p \in P(q)$. But can an observer at q really make a sure prediction about the point p? How does an observer at q know that she does not inhabit the spacetime $(M', \eta_{|M'})$ where the point p is "missing" from the manifold? Of course, one could require that spacetime be inextendible. However, more complicated examples could be constructed to show that, even under this assumption, one cannot make a genuine prediction.[12] It seems that knowledge about one's domain of prediction requires knowledge not only of one's causal past but also of the spacetime in

[10] See [8, p. 200].

[11] The *future Cauchy horizon* $H^+(S)$ of an achronal surface S is the set $\overline{D^+(S)} - I^-[D^+(S)]$. See [8, p. 203].

[12] For such an example, see the theorem below.

which one's causal past is embedded.[13]

One may naturally wonder if genuine prediction is possible *at all* in general relativity. In order to answer the question we must give a definition of the domain of prediction which requires that the observer not only have the resources to make a prediction but also the resources to *know* that she can make a prediction. Let (M, g) be a spacetime. More precisely, a genuine prediction at $q \in M$ about some point $p \in P(q)$ requires that, for all spacetimes (M', g'), if there is an isometric embedding $\phi : J^-(q) \to M'$, then it must also be possible to extend the domain of this embedding to the set $J^-(q) \cup J^-(p)$ (there must be an isometric embedding $\phi' : J^-(q) \cup J^-(p) \to M'$ such that $\phi = \phi'_{|J^-(q)}$). This would ensure that $\phi'(p) \in P(\phi'(q))$. In this sense, the observer at q can be sure that genuine prediction about p is possible (regardless of whether or not she knows which spacetime she inhabits). This suggests the following definition:

Definition: Let (M, g) be a spacetime. Let q be any point in M. We say a point $p \in M$ is in the *domain of genuine prediction* of q (written $\mathcal{P}(q)$) iff $p \in P(q)$ and, for all inextendible spacetimes (M', g'), if there is an isometric embedding $\phi : J^-(q) \to M'$, then there is an isometric embedding $\phi' : J^-(q) \cup J^-(p) \to M'$ such that $\phi = \phi'_{|J^-(q)}$.

What is interesting about genuine prediction is that it is essentially impossible. We have:

Theorem: Let (M, g) be any spacetime and let q be any point in M. Then $\mathcal{P}(q) \subseteq \partial J^-(q)$.

Proof. Let (M, g) be any (four-dimensional) spacetime and let q be any point in M. Let p be a point in $\mathcal{P}(q)$. By definition, $p \in P(q)$ and therefore $p \notin J^-(q)$. We assume $p \notin \partial J^-(q)$ and show a contradiction. Let U be a neighborhood of p such that $U \cap J^-(q) = \varnothing$ (such a neighborhood must exist because $p \notin \overline{J^-(q)}$ and M is open). Let S be any three-dimensional, closed, achronal surface in U such that p is in ∂S. Let $(M', g_{|M'})$ be the spacetime where $M' = M - S$. Consider two copies of the spacetime $(M', g_{|M'})$. Excluding boundary points ∂S, identify the upper edge of S in the first copy with the lower edge of S is the second copy. Similarly, identify the lower edge of S in the first copy with the upper edge of S in the second copy.[14] Call the resulting (inextendible) spacetime (M'', g''). Because $S \cap J^-(q) = \varnothing$, there is an isometric embedding $\phi : J^-(q) \to M''$. Yet, because the boundary points ∂S are "missing" from M'' and because $p \in \partial S$, there is no isometric embedding $\phi' : J^-(q) \cup J^-(p) \to M''$ such that $\phi = \phi'_{|J^-(q)}$. So, $p \notin \mathcal{P}(q)$ and we are done. $\square$

The theorem shows that one cannot make a genuine prediction outside the boundary of one's observational past. In other words, the only possible predic-

[13] See [4, p. 85-86].
[14] See [4, p. 89-90] and [5, p. 58-59] for other examples of this type of construction.

tions are those “on the verge” of being retrodictions. One might wonder about the role of matter fields and Einstein’s equation in our discussion of genuine prediction. It has been argued, for example, that Maxwell’s equations constrain some spacetimes in such a way as to allow for prediction.[15] But, notice that the spacetime (M'', g'') constructed in the proof above is locally isometric to the spacetime (M, g). So, the theorem goes through even under the imposition of any local conditions (e.g. the energy conditions, Maxwell’s equations, or Einstein equation). Thus, if the epistemological predicament of the observer is fully considered, there seems to be an interesting and robust sense in which genuine prediction is not possible in general relativity.

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[15] See [2, p. 125-130] and [3, p. 41-42].