"Time Travel" in Gödel Spacetime: Why It Doesn't Pay to Work Out All the Kinks*

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Abstract

Here we provide a proof that there exist closed timelike curves in Gödel spacetime with total acceleration less than $2\pi(9+6\sqrt{3})^{1/2}$. This answers a question posed by David Malament.

1 Introduction

Most of the known solutions to Einstein's equation have the properties we have come to expect from models of our own universe. However, Kurt Gödel's contribution to relativity theory showed there are solutions that allow for "time travel" in some sense (one may visit the same spacetime point more than once via a closed timelike curve (CTC)).[1] A number of questions have been posed concerning these CTCs in Gödel spacetime.[2] We know that such a journey could not be accomplished without accelerating to some degree.[3] But, one question was whether such a trip could be accomplished with an arbitrarily small amount of acceleration. A bit of formalism will help to illustrate what exactly was at issue.

Let γ be a closed timelike curve with tangent field ξ^b.[4] Let the acceleration vector field be $\alpha^b = \xi^a \nabla_a \xi^b$ and the magnitude of acceleration be

*I am grateful to David Malament and Bob Geroch for many helpful discussions on this topic.

[1]See Gödel 447.

[2]See Earman Chapter 6.

[3]See Hawking and Ellis 168-170.

[4]Timelike curves are those that are "smooth everywhere unless they are closed, in which case smoothness will be allowed to fail at initial (=terminal) points." See Malament (1985) 777 and (1987) 2430.

$a = (-\alpha^b \alpha_b)^{1/2}$. The *total (integrated) acceleration* of γ (a quantity without units[5]) is given by

$$TA(\gamma) = \int_\gamma a \, ds$$

where s is elapsed proper time along γ. The question, then, was the following: (Q1) Was there some number $k > 0$ such that, for all CTCs γ in Gödel spacetime, $TA(\gamma) \geq k$?[6] David Malament showed there was indeed such a number (the value in his proof was $\ln(2 + \sqrt{5})$).[7]

To give some idea of what this value means, we consider a connection between total acceleration and "fuel consumption".[8] Even if a traveler (in any spacetime) has a perfectly efficient rocket, the following inequality must be satisfied (here m_r is the mass of the rocket and m_f is the mass of the fuel):

$$\frac{m_r}{m_r + m_f} \leq e^{-TA(\gamma)}.$$

Inserting $\ln(2 + \sqrt{5})$ for $TA(\gamma)$, we find that at least 76% of the rocket's mass must be fuel. This may or may not be "physically reasonable" but one thing is for sure: we now know that one cannot "time travel" in Gödel spacetime by accelerating an arbitrarily small amount.

This brings us to our next question. Let GLB be the largest k such that every CTC has a total acceleration greater than or equal to k. Our next question is this: (Q2) What is GLB?[9] This is still an open question. We can (and will in this paper) calculate the total acceleration for very simple CTCs, but the smallest value of total acceleration known up until now is $2\pi(9+6\sqrt{3})^{1/2}$.[10] Using the inequality above, this value of total acceleration implies that for every two grams of payload, a rocket will have to carry *at least* 10^{12} grams of fuel! Surely this is not "physically reasonable" if the term is interpreted as meaning a *practical* possibility.

So, we know that $\ln(2 + \sqrt{5}) \leq$GLB$\leq 2\pi(9 + 6\sqrt{3})^{1/2}$. But, although we know that GLB falls within this range, pinning down where it lies exactly proves to be a formidable task. We can, however, ask a less difficult question:

[5] Total acceleration is invariant under rescaling of the metric and does not depend on our choice of units for space and time. See Malament (1985) 774, 777.

[6] This question was posed by Robert Geroch. See Chakrabarti, Geroch, and Liang 597.

[7] See Malament (1985) 775-776. This number is approximately 1.44.

[8] For a derivation of the following inequality, see Chakrabarti, Geroch, and Liang 597.

[9] This question, as well as (Q3), was posed by Malament. See Malament (1985) 776.

[10] This number is approximately 27.67.

(Q3) Are there any CTCs in Gödel spacetime with total acceleration less than $2\pi(9+6\sqrt{3})^{1/2}$? Malament believed that there were not. To him, it seemed "overwhelmingly likely" that GBL=$2\pi(9+6\sqrt{3})^{1/2}$ but he was unable to prove the claim.[11] In this paper, I provide a proof that Malament's conjecture is false. My result turns on the fact that CTCs are not required to be smooth everywhere. At the initial (which is the same as the final) point of a CTC, there may be a "kink" in the curve.[12] It turns out that the total acceleration of some of these kinked CTCs is less than $2\pi(9+6\sqrt{3})^{1/2}$.

2 Preliminaries

Here we review some basic facts concerning Gödel spacetime. Let (M, g_{ab}) be Gödel spacetime. Here the manifold M is just $\mathbb{R}^4$. The metric g_{ab} is such that for any point $p \in M$, there is a global adapted (cylindrical) coordinate system t, r, φ, y in which $t(p) = r(p) = y(p) = 0$ and

$$\begin{aligned} g_{ab} &= (\nabla_a t)(\nabla_b t) - (\nabla_a r)(\nabla_b r) - (\nabla_a y)(\nabla_b y) \\ &\quad + j(r)(\nabla_a \varphi)(\nabla_b \varphi) + 2k(r)(\nabla_{(a}\varphi)(\nabla_{b)} t) \end{aligned}$$

where $j(r) = \sinh^4 r - \sinh^2 r$ and $k(r) = \sqrt{2}\sinh^2 r$. Here $-\infty < t < \infty$, $-\infty < y < \infty$, $0 \le r < \infty$, and $0 \le \varphi \le 2\pi$ with $\varphi = 0$ identified with $\varphi = 2\pi$.

The vector field $(\frac{\partial}{\partial\varphi})^a$ is a rotational Killing field with squared norm $j(r)$. The closed integral curves of $(\frac{\partial}{\partial\varphi})^a$ (curves with constant t, r, and y values) will be called *Gödel circles.* Let r_c be such that $\sinh r_c = 1$ (so $j(r_c) = 0$). Gödel circles with radius less than r_c are closed spacelike curves. If the radius is larger than r_c, the Gödel circles are CTCs. Gödel circles with radius r_c are closed null curves. Because of the simple nature of these curves, it is fairly straightforward to calculate the total acceleration of Gödel circles as a function of r. Because these curves play a central role in our argument, we carry out the calculation here.

Lemma 1: A Gödel circle γ with radius $r > r_c$ has a total acceleration of $\pi \sinh 2r(2\sinh^2 r - 1)j(r)^{-1/2}$.

[11] See Malament (1987), 2429-2430.

[12] We know that Malament took the possibility of kinked CTCs very seriously. At one point, he devotes a paragraph to explaining that only the possibility of kinked CTCs kept him from doubling his minimal acceleration requirements. See Malament (1985) 776.

Proof: The unit timelike vector field for a Gödel circle or radius r is $\xi^a = j(r)^{-1/2}(\frac{\partial}{\partial\varphi})^a$. We know that $\xi^a\nabla_a j(r)^{-1/2} = 0$. So the acceleration vector $\alpha_b = \xi^a\nabla_a\xi_b$ is $j(r)^{-1}(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial\varphi})_b$. But because $(\frac{\partial}{\partial\varphi})^a$ is a Killing field, this is just $-j(r)^{-1}(\frac{\partial}{\partial\varphi})^a\nabla_b(\frac{\partial}{\partial\varphi})_a = -\frac{1}{2}j(r)^{-1}\nabla_b j(r)$. Differentiating, we have $\alpha_b = -\frac{1}{2}j(r)^{-1}\sinh 2r(2\sinh^2 r - 1)\nabla_b r$. Thus, $a(r) = (-\alpha^b\alpha_b)^{1/2} = \frac{1}{2}j(r)^{-1}\sinh 2r(2\sinh^2 r - 1)$. Next we compute $\frac{d\varphi}{ds} = \xi^a\nabla_a\varphi = j(r)^{-1/2}$. So, integrating, we have

$$TA(\gamma) = \int_\gamma a(r)\ ds = \int_0^{2\pi} a(r)j(r)^{1/2}\ d\varphi = 2\pi a(r)j(r)^{1/2}$$

So the total acceleration is $\pi\sinh 2r(2\sinh^2 r - 1)j(r)^{-1/2}$ as claimed. □

Note that the total acceleration of a Gödel circle approaches infinity as $r \to r_c$ *and* as $r \to \infty$. The total acceleration is minimized when r is such that $\sinh^2 r = (1+\sqrt{3})/2$ (for reference later on, call this optimal radius r_o). The total acceleration of this optimal Gödel circle is $2\pi(9+6\sqrt{3})^{1/2}$.

Next, for ease of presentation in the next lemma, we give a list of identities that are true in Gödel spacetime.

Lemma 2: Let (M, g_{ab}) be Gödel spacetime. The following are true:

(i) $(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial r})_b = 0$

(ii) $(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial r})_b = (\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial\varphi})_b$

(iii) $(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial\varphi})_b = -\frac{1}{2}(\frac{\partial j}{\partial r})\nabla_b r$

(iv) $(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial\varphi})_b = (\frac{dj}{dr})\nabla_b\varphi + (\frac{dk}{dr})\nabla_b t$

Proof: We know (i) is true because $(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial r})_b = -(\frac{\partial}{\partial r})^a\nabla_a\nabla_b r$. But because r is a scalar field, this is just $-(\frac{\partial}{\partial r})^a\nabla_b\nabla_a r$. This becomes $(\frac{\partial}{\partial r})^a\nabla_b(\frac{\partial}{\partial r})_a$ which is the zero vector.

To see why (ii) holds, note that $(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial\varphi})_b = -(\frac{\partial}{\partial r})^a\nabla_b(\frac{\partial}{\partial\varphi})_a$ because $(\frac{\partial}{\partial\varphi})_b$ is a Killing field. But this is just $(\frac{\partial}{\partial\varphi})^a\nabla_b(\frac{\partial}{\partial r})_a$. We rewrite this as $-(\frac{\partial}{\partial\varphi})^a\nabla_b\nabla_a r$, switch the differential operators because r is a scalar field, and wind up with $-(\frac{\partial}{\partial\varphi})^a\nabla_a\nabla_b r$ which is just $(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial r})_b$ as claimed.

Because $(\frac{\partial}{\partial\varphi})^a$ is a Killing field, $(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial\varphi})_b = -(\frac{\partial}{\partial\varphi})^a\nabla_b(\frac{\partial}{\partial\varphi})_a$. But this is just $-\frac{1}{2}\nabla_b j(r) = -\frac{1}{2}(\frac{\partial j}{\partial r})\nabla_b r$ as claimed. So (iii) is true.

To see why (iv) holds, consider the following. $\nabla_a(\frac{\partial}{\partial\varphi})_b = \nabla_{[a}(\frac{\partial}{\partial\varphi})_{b]}$ because $(\frac{\partial}{\partial\varphi})^a$ is a Killing field. So we can rewrite this with the exterior derivative operator as $d_a(\frac{\partial}{\partial\varphi})_b$. This is the same as $d_a(j\nabla_b\varphi + k\nabla_b t)$. But this is just $\nabla_a j\nabla_b\varphi + \nabla_a k\nabla_b t$. Differentiating, we have $(\frac{dj}{dr})\nabla_a r\nabla_b\varphi + (\frac{dk}{dr})\nabla_a r\nabla_b t$. So $(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial\varphi})_b = (\frac{dj}{dr})\nabla_b\varphi + (\frac{dk}{dr})\nabla_b t$ as claimed. □

Let S be any submanifold of M on which t =const and y =const. In this paper, we will be concerned only with CTCs that are contained entirely within S. We now find an expression for the magnitude of acceleration of this limited class of curves.

Lemma 3: Let $\xi^a = f(r,\varphi)(\frac{\partial}{\partial\varphi})^a + g(r,\varphi)(\frac{\partial}{\partial r})^a$ be the unit tangent to some curve $\gamma : I \to S$. Then the acceleration $a(r,\varphi)$ at a point on $ran[\gamma]$ is

$$[-f^2(\frac{\partial f}{\partial\varphi})^2 j - 2f(\frac{\partial f}{\partial\varphi})g(\frac{\partial f}{\partial r})j - 4f^2(\frac{\partial f}{\partial\varphi})g(\frac{dj}{dr}) + \frac{1}{4}f^4(\frac{dj}{dr})^2 + f^3(\frac{\partial g}{\partial\varphi})(\frac{dj}{dr})$$
$$+f^2 g(\frac{\partial g}{\partial r})(\frac{dj}{dr}) + f^2(\frac{\partial g}{\partial\varphi})^2 + 2fg(\frac{\partial g}{\partial r})(\frac{\partial g}{\partial\varphi}) - g^2(\frac{\partial f}{\partial r})^2 j - 4g^2(\frac{\partial f}{\partial r})^2(\frac{dj}{dr})f$$
$$+g^2(\frac{\partial g}{\partial r})^2 + 4(\frac{dj}{dr})^2 g^2 f^2 m - 8(\frac{dj}{dr})g^2 f^2(\frac{dk}{dr})km + 4(\frac{dk}{dr})^2 g^2 f^2 jm]^{1/2}$$

where $m(r) = 1/(\sinh^4 r + \sinh^2 r)$.

Proof: Let ξ^a be as above. Consider the acceleration vector $\alpha_b = \xi^a\nabla_a\xi_b$:

$$\alpha_b = [f(\frac{\partial}{\partial\varphi})^a + g(\frac{\partial}{\partial r})^a][(\nabla_a f)(\frac{\partial}{\partial\varphi})_b + f\nabla_a(\frac{\partial}{\partial\varphi})_b$$
$$+(\nabla_a g)(\frac{\partial}{\partial r})_b + g\nabla_a(\frac{\partial}{\partial r})_b].$$

By (i) and (ii) of Lemma 2 and direct computation, we know that α_b becomes

$$f\frac{\partial f}{\partial\varphi}(\frac{\partial}{\partial\varphi})_b + f^2(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial\varphi})_b + f\frac{dg}{d\varphi}(\frac{\partial}{\partial r})_b + g\frac{\partial f}{\partial r}(\frac{\partial}{\partial\varphi})_b$$
$$+2fg(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial\varphi})_b + g(\frac{\partial g}{\partial r})(\frac{\partial}{\partial r})_b \qquad .$$

Let $m(r) = 1/(\sinh^4 r + \sinh^2 r)$. Now we compute $a = (-\alpha_b\alpha^b)^{1/2}$. By (iii) and (iv) of Lemma 2 and direct computation, we have our result.[13] □

[13] It is helpful during the calculation to have the inverse to g_{ab}. It is given by $g^{ab} = -j(r)m(r)(\frac{\partial}{\partial t})^a(\frac{\partial}{\partial t})^b - (\frac{\partial}{\partial r})^a(\frac{\partial}{\partial r})^b - (\frac{\partial}{\partial y})^a(\frac{\partial}{\partial y})^b - m(r)(\frac{\partial}{\partial\varphi})^a(\frac{\partial}{\partial\varphi})^b + 2k(r)m(r)(\frac{\partial}{\partial\varphi})^{(a}(\frac{\partial}{\partial t})^{b)}$. See Malament (1985) 777.

3 A Theorem

In this section we present our result. It will be useful to have a general idea of how we will go about proving our claim. Eventually, we seek to answer (Q3) by showing there exists a curve in Gödel spacetime with total acceleration less than $2\pi(9+6\sqrt{3})^{1/2}$. We will do this by considering the behavior of a particular curve $\gamma : I \to S$ contained entirely in the submanifold S.

We can think of γ as three separate curves joined smoothly together. From $0 \le \varphi \le \epsilon$ for some ϵ the curve γ makes its way from the point $(r_o, 0)$ to (r_ϵ, ϵ) where $r_c < r_\epsilon < r_o$. We will call this portion of the curve γ_1. From $\epsilon \le \varphi \le 2\epsilon$, γ makes its way from the point (r_ϵ, ϵ) to $(r_o, 2\epsilon)$. This portion of the curve we will call γ_2. From $2\epsilon < \varphi < 2\pi$, γ is simply a Gödel circle of radius r_o. We call this portion of the curve γ_3. We are careful to make the three portions of γ join together smoothly except at the point (r_ϵ, ϵ). Thus, at this point, there will be a "kink" in the curve and so we stipulate that this will be the initial (and therefore the final) point of the CTC.

The basic structure of our proof is simple. We show that along γ, (a) the acceleration of γ_1 is always decreasing (from the constant acceleration of the optimal Gödel circle) and (b) the acceleration of γ_2 is always increasing (up to the constant acceleration of the optimal Gödel circle). With this information we can integrate along γ to show that the total acceleration from $0 \le \varphi \le 2\epsilon$ is less than the total acceleration of the optimal Gödel circle from $0 \le \varphi \le 2\epsilon$. Because the total acceleration of γ_3 just is that of the optimal Gödel circle from $\varphi = 2\epsilon$ to $\varphi = 2\pi$, we have our result.

Theorem: There exists a CTC in Gödel spacetime with total acceleration less than $2\pi(9+6\sqrt{3})^{1/2}$.

Proof: The first step is to define our curve. Consider the vector field $\xi^a(r,\varphi) = f(r,\varphi)(\frac{\partial}{\partial\varphi})^a + g(\varphi)(\frac{\partial}{\partial r})^a$ defined for all values of $r > r_c$ and on some interval $[0,\epsilon]$ of φ. Let $f(r,\varphi) = j(r)^{-1/2}h(\varphi)$ where $h(\varphi) = (1 + e^{-2/\varphi})^{1/2}$. Let $g(\varphi) = -e^{-1/\varphi}$ For continuity considerations later, let $h(0) = 1$ and $g(0) = 0$. Clearly, ξ^a is a unit timelike vector field. Now, for some interval $I \subseteq \mathbb{R}$, let $\gamma_1 : I \to S$ be such that its tangent vector at each point is ξ^a and $(r_o, 0) \in ran[\gamma_1]$ (i.e. γ_1 is an integral curve of ξ^a).

We have also chosen ξ^a to be such that at $\varphi = 0$, it joins smoothly with $j(r)^{-1/2}(\frac{\partial}{\partial\varphi})^a$ (the unit tangent field associated with Gödel circles). Finally, we note two important facts concerning our functions f and g. The first is a relationship between f and g and their partial derivatives with respect to φ. The second states that as φ approaches zero, g and $dg/g\varphi$ both go to zero more quickly than $d^2g/d\varphi^2$. These facts will play a crucial role in our proof. They are easily verifiable and so we present them here without any proof:

(1) $f\frac{\partial f}{\partial \varphi} = j^{-1}g\frac{dg}{d\varphi}$

(2) $\lim_{\varphi\to 0^+} g/\frac{d^2g}{d\varphi^2} = \lim_{\varphi\to 0^+} \frac{dg}{d\varphi}/\frac{d^2g}{d\varphi^2} = 0$

Let a_1 be the magnitude of acceleration at any point on $ran[\gamma_1]$. Next, consider the expression $\xi^b\nabla_b a_1 = f\frac{\partial a_1}{\partial \varphi} + g\frac{\partial a_1}{\partial r}$. This is the rate of change of the magnitude of the acceleration in the direction of ξ^b. The claim is that this quantity will be negative when evaluated at points $(r, \varphi) \in ran[\gamma_1]$ very close to $(r_o, 0)$. We can differentiate the expression for a given in Lemma 3 to find that $f\frac{\partial a_1}{\partial \varphi} + g\frac{\partial a_1}{\partial r}$ is a (very long) string of terms with a peculiar characteristic. Using the identity (1) we can rewrite the string of terms such that all the terms *except for one* contain, as a factor, either g or $\frac{dg}{d\varphi}$. The one exception is the term $\frac{1}{2}a_1^{-1/2}f^4(\frac{d^2g}{d\varphi^2})(\frac{dj}{dr})$ (call this term ω). The following can also be verified:

(3) *All* of the terms in $f\frac{\partial a_1}{\partial \varphi} + g\frac{\partial a_1}{\partial r}$ approach zero as the point $(r_o, 0)$ is approached.

(4) *All* of the various factors of the terms approach a real number as the point $(r_o, 0)$ is approached (none of them "blow up").

(5) ω goes to zero as $\frac{d^2g}{d\varphi^2}$ does.

We know that (3), (4), and (5), combined with (2), all imply that, as the point $(r_o, 0)$ is approached, ω becomes the dominate term (it goes to zero slower than any term containing g or $\frac{dg}{d\varphi}$). To illustrate this, we can pick any term in $\xi^b\nabla_b a_1$ (other than ω) and show that it must go to zero faster than ω as the point $(r_o, 0)$ is approached. Take, for example, the term $-a_1^{-1}f^2(\frac{\partial f}{\partial \varphi})^3 j$ (this is one of the terms that results in taking the derivative of the first term in the expression for a_1 in Lemma 3 and multiplying by f). Using (1) we can rewrite this term as $-a_1^{-1}f(\frac{\partial f}{\partial \varphi})^2 g(\frac{\partial g}{\partial \varphi})$. We know that as $(r_o, 0)$ is approached, a_1 goes to some positive real number (the acceleration of the optimal Gödel circle). Similarly, f approaches some positive real number (the number is $j(r_o)^{-1/2}$). The remaining three factors all go to zero as $(r_o, 0)$ is approached. So, the entire term approaches zero. How fast does it go? We know it must go at least as fast as any one of the factors. So, it must go at least as fast as g. But now consider ω. By (3), we know that it goes to zero as $\frac{d^2g}{d\varphi^2}$ does. Now by (2) we know that ω must go to zero slower than the term that we picked (it dominates the term that we picked as the point (r_o) is approached.) The claim is that if we repeated this process and compared all the terms in $\xi^b\nabla_b a_1$, ω would dominate them all.

What is the behavior of ω near $(r_o, 0)$? It is negative. So, there exists an ϵ_1 such that for all $\varphi \in (0, \epsilon_1]$, $\xi^b\nabla_b a_1 < 0$ (moving along γ_1 away from $(r_o, 0)$ the value of acceleration decreases).

Now we define another curve γ_2. Pick any point (r_o, δ) in the optimal Gödel circle. Let $f'(r, \varphi) = j(r)^{-1/2}h'(\varphi)$ where $h'(\varphi) = (1 + e^{-2/(\delta-\varphi)})^{1/2}$ and $g'(\varphi) = e^{-1/(\delta-\varphi)}$ (for continuity considerations, let $h'(\delta) = 1$ and $g'(\delta) = 0$). Let $\eta^a = f'(\frac{\partial}{\partial\varphi})^a + g'(\frac{\partial}{\partial r})^a$ and let $\gamma_2 : I' \to M$ be such that its tangent vector at each point is η^a and $(r_o, \delta) \in ran[\gamma_2]$. Note that for all points (r, φ) where $0 \geq \varphi \geq \delta$ we have $f'(r, \delta - \varphi) = f(r, \varphi)$ and $g'(\delta - \varphi) = -g(\varphi)$. Thus, under that same interval of φ, it is the case that $ran[\gamma_1]$ is the mirror image of $ran[\gamma_2]$ across the line of symmetry $\varphi = \delta/2$.

Let a_2 be the magnitude of acceleration for any point on γ_2. By an argument very similar to the one made above for γ_1 we can establish that there exists some ϵ_2 such that for all $\varphi \in [\epsilon_2, \delta)$, $\eta^b\nabla_b a_2 > 0$ (moving along γ_2 toward (r_o, δ) the value of acceleration increases). Let $\epsilon = min\{\epsilon_1, \delta - \epsilon_2\}$. Because δ was arbitrarily chosen and because $\epsilon \leq \delta - \epsilon_2$, we know (if we let $\delta = 2\epsilon$) that for all $\varphi \in [\epsilon, 2\epsilon)$, $\eta^b\nabla_b a_2 > 0$. Of course, because $\epsilon \leq \epsilon_1$ we know that for all $\varphi \in (0, \epsilon]$, $\xi^b\nabla_b a_1 < 0$.

Let the curve σ be the optimal Gödel circle. Let $\gamma_3 : I'' \to S$ be that portion of σ from $\varphi = 2\epsilon$ to $\varphi = 2\pi$. Let γ be such that $ran[\gamma] = ran[\gamma_1] \cup ran[\gamma_2] \cup ran[\gamma_3]$.

Now we integrate. We reparametrize a_1 along γ_1 so that it is only a function of φ. Next, note that $\frac{d\varphi}{ds}$ for σ is $j(r)^{-1/2}$ while $\frac{d\varphi}{ds}$ for γ_1 is $j(r)^{-1/2}h(\varphi)$. We also reparametrize $j(r)$ along γ_1 so that it is a function of φ. Since along γ_1, $j(\varphi)^{1/2} \leq j(0)^{1/2}$ and $h(\varphi) \geq 1$ we may conclude that

$$\int_0^\epsilon a_1(\varphi)j(\varphi)^{1/2}h(\varphi)^{-1}d\varphi \leq j(0)^{1/2}\int_0^\epsilon a_1(\varphi)d\varphi.$$

Let $a_\sigma(\varphi)$ be the acceleration at any point in σ. Because $\xi^b\nabla_b a_1 < 0$ along γ_1, for all $0 < \varphi \leq \epsilon$, we know that $a_\sigma(\varphi) > a_1(\varphi)$ over that same interval and, of course, $a_1(0) = a_\sigma(0)$. From Lemma 1 we know that the total acceleration of the optimal Gödel circle over this interval is $\epsilon(9 + 6\sqrt{3})^{1/2}$. So, we have

$$j(0)^{1/2}\int_0^\epsilon a_1(\varphi)d\varphi < j(0)^{1/2}\int_0^\epsilon a_\sigma(\varphi)d\varphi = \epsilon(9 + 6\sqrt{3})^{1/2}.$$

So, we have

$$TA(\gamma_1) = \int_0^{\epsilon} a_1(\varphi) j(\varphi)^{1/2} h(\varphi)^{-1} d\varphi < \epsilon(9+6\sqrt{3})^{1/2}.$$

A similar argument establishes that for γ_2, we have

$$TA(\gamma_2) = \int_{\epsilon}^{2\epsilon} a_2(\varphi) j(\varphi)^{1/2} h(\varphi)^{-1} d\varphi < \epsilon(9+6\sqrt{3})^{1/2}.$$

Finally, for γ_3 we have

$$TA(\gamma_3) = \int_{2\epsilon}^{2\pi} a_\sigma(0) j(0)^{1/2} d\varphi = (2\pi - 2\epsilon)(9+6\sqrt{3})^{1/2}.$$

So we may conclude that

$$TA(\gamma) = TA(\gamma_1) + TA(\gamma_2) + TA(\gamma_3) < 2\pi(9+6\sqrt{3})^{1/2}.$$

Thus, there exists a closed timelike curve in Gödel spacetime with total acceleration less that $2\pi(9+6\sqrt{3})^{1/2}$. □

4 Conclusion

So, we have answered question (Q3) concerning CTCs in Gödel spacetime. We have shown there exists a curve (and therefore a family of curves) with total acceleration less than $2\pi(9+6\sqrt{3})^{1/2}$. It is uncertain if a curve of the type we have proposed will actually approach GLB (or if there is another type of curve entirely that approaches it). As previously mentioned, (Q2) or "What is GLB?" remains open.

Because there is a family of curves determined by varying the value of ϵ, a natural next step in the development of this project would be to try and optimize ϵ so as to find the curve that minimizes the total acceleration within that class. This we have been unable to do. But by extending the methods developed by Malament, we can show that *any* CTC contained entirely within S (and therefore this particular class of "kinked" Gödel circles) must have a total acceleration of at least $4\ln(4+\sqrt{17})$.[14] This implies that for every 2 grams of payload, a rocket traversing one of these kinked Gödel circles will have to carry at least 10,000 grams of fuel. So, although these kinked curves are more efficient than the optimal Gödel circle, it is clear that they still don't allow for any "physically reasonable" time travel in the Gödel universe.

[14] This number is approximately 8.38.

References

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